Research Article

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Defective functions of meromorphic functions in the unit disc

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Abstract: Let f be a meromorphic function in the unit disc and $(a_\nu)_{\nu=1}^k$ a set of distinct meromorphic functions small with respect to f. An analogue of the second main theorem for f and $(a_\nu)_{\nu=1}^k$ is given. Upper limits for the sum of defects of an admissible meromorphic function and an admissible holomorphic function follow. For meromorphic and holomorphic functions in the unit disc and their small functions the analogues of Ullrich's theorem are presented.

Keywords: Meromorphic function, holomorphic function, small function, defect

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1 Introduction

We apply the standard notations of value distribution theory of meromorphic functions: $N(r, a, f)$, $N(r, f)$ and $\overline{N}(r, f)$ for the functions counting a-points, poles with and without multiplicity, respectively, $m(r, a, f)$ and $m(r, f)$ for the mean proximity functions, $T(r, f)$ for the characteristic function. We also write $\delta(a, f)$ for the Nevanlinna's defect of f at a value a, $\Delta(a, f)$ for the Valiron's defect and $\theta(a, f)$ for the index of multiplicity ([4, 6]).

In the paper we concentrate on meromorphic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. If we replace constants with functions of relatively slow growth, it is still possible to obtain estimates similar to the one in the second main theorem of Nevanlinna. The main purpose of this paper is to present a generalisation of the second main theorem and a defect relation for meromorphic functions in the unit disc and their small functions. We also show parallel theorems for holomorphic functions in the unit disc. Then we move on to a few Ullrich-type results following from the defect relations.

The characteristic $T(r, f)$ of a meromorphic function f is a non-decreasing function of r. In case of meromorphic functions in $\mathbb{C}$ the speed of growth is usually compared with $\log r$ in terms of the order of growth. For various purposes it is also convenient to compare the growth of a function with the characteristic of a fixed meromorphic function.

Definition 1.1. Let f be a meromorphic function in the complex plane. We say that a function $s : [0, +\infty) \to \mathbb{R}$ is an $S(r, f)$ if

$$s(r) = o(T(r, f)), \quad r \to \infty,\ r \notin E,$$

for a set E of finite linear measure.

If f and a are meromorphic functions, we say that *a is a small function with respect to f* if $T(r, a) = S(r, f)$. The set of all small functions with respect to f will be denoted $\mathcal{S}(f)$. As can easily be shown the sum and product of a finite number of functions belonging to $\mathcal{S}(f)$ are also small functions with respect to f.

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In 1986 Steinmetz proved the following theorem concerning small defective functions of meromorphic functions in the complex plane.

Theorem 1.2 ([10]). *Let f be a nonconstant meromorphic function and let $\{a_\nu\}_{\nu=1}^k$ be a set of pairwise distinct meromorphic functions such that for $1 \le \nu \le k$ we have $T(r, a_\nu) = o(T(r, f))$ $(r \to \infty)$. Then for any $\varepsilon > 0$,*

$$m(r, f) + \sum_{\nu=1}^{k} m(r, a_\nu, f) \le (2 + \varepsilon) T(r, f) + S(r, f).$$

Let us remind that a value $a \in \overline{\mathbb{C}}$ is called a defective value of a meromorphic function f if

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, a, f)}{T(r, f)} > 0.$$

If a is a meromorphic function, we say that it is *a defective function of f* if

$$\delta(a, f) := \delta(0, 1/(f - a)) > 0.$$

It follows from Theorem 1.2 that for meromorphic functions in the complex plane the set of their defective small functions is at most countable and

$$\delta(\infty, f) + \sum_{a\in S(f)} \delta(a, f) \le 2.$$

2 Main results

In case of meromorphic functions in the unit disc we determine the speed of growth of the characteristic in comparison with $\log 1/(1 - r)$. We say that f is *admissible* if

$$\limsup_{r\to 1^-} \frac{T(r, f)}{-\log(1 - r)} = +\infty.$$

The order and lower order of a meromorphic in the unit disc function f are defined by

$$\varrho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r, f)}{-\log(1 - r)}, \quad \lambda(f) := \liminf_{r\to 1^-} \frac{\log^+ T(r, f)}{-\log(1 - r)}.$$

An easy observation is that $0 \le \lambda(f) \le \varrho(f) \le +\infty$.

We also define $S(r, f)$ in the following way.

Definition 2.1. Let f be a meromorphic function in the unit disc. We say that a function $s : [0, 1) \to \mathbb{R}$ is an $S(r, f)$ if

$$s(r) = o(T(r, f)), \quad r \to 1^-,\ r \notin E,$$

for a set E such that $\int_E \frac{dr}{1-r} < +\infty$.

We can now formulate an extension of the second main theorem in the unit disc.

Theorem 2.2. *Let f be a meromorphic function in the unit disc and let $a_1, \dots, a_k$ be distinct meromorphic small functions with respect to f. For any $\varepsilon > 0$ the following inequality holds:*

$$m(r, f) + \sum_{\nu=1}^{k} m(r, a_\nu, f) \le (2 + \varepsilon) T(r, f) + O\Big(\log \frac{1}{1 - r}\Big) + S(r, f).$$

Let us remind that a value $a \in \overline{\mathbb{C}}$ is called *a (Nevanlinna's) defective value* of a meromorphic function f in the unit disc if

$$\delta(a, f) = \liminf_{r\to 1^-} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r\to 1^-} \frac{N(r, a, f)}{T(r, f)} > 0.$$

We define a defective function similarly as in the case of functions meromorphic in the plane.

Corollary 2.3. *Let f be an admissible meromorphic function in the unit disc. The set of small defective functions of f is at most countable and*

$$\delta(\infty, f) + \sum_{a \in S(f)} \delta(a, f) \leq 2.$$

The first sharp estimate of the number of defective functions of an entire function belongs to Chuang [1]. Here we present a similar estimate for functions holomorphic in the unit disc.

Corollary 2.4. *Let f be a holomorphic function in the unit disc and let $a_1, \ldots, a_k$ be distinct meromorphic small functions with respect to f. For any $\varepsilon > 0$ we have the inequality*

$$\sum_{\nu=1}^{k} m(r, a_\nu, f) \leq (1 + \varepsilon) T(r, f) + O\Big(\log \frac{1}{1 - r}\Big) + S(r, f).$$

A suitable defect relation follows.

Corollary 2.5. *Let f be an admissible holomorphic function in the unit disc. The set of small defective functions with respect to f is at most countable and*

$$\sum_{a \in S(f)} \delta(a, f) \leq 1.$$

Corollary 2.5 leads to the following analogue of Picard's theorem for small functions in $\mathbb{D}$.

Corollary 2.6. *An admissible holomorphic function f in the unit disc omits at most one function a such that $T(r, a) = S(r, f)$.*

The term 'omit' in Corollary 2.6 is used in the meaning that f omits a function a if $f - a$ has no zeros in the unit disc. The estimate in Theorem 2.2 is sharp. Indeed, the function $f(z) = \exp(\frac{1}{(1-z)^2})$ is an admissible holomorphic function in the unit disc of order $\varrho(f) = 1$ omitting 0 and ∞. If a function meromorphic in $\mathbb{D}$ is not admissible and therefore its characteristic does not grow faster than $O(\log \frac{1}{1-r})$, then the term $O(\log \frac{1}{1-r})$ ceases to be a remainder in Theorem 2.2. In this case Picard's theorem is not valid. For instance, the function $f(z) = z$ omits all values a such that $|a| \geq 1$. Suitable examples of functions with unbounded characteristic $T(r, f)$ in the unit disc and three or more (even infinite number of) omitted values can be found among automorphic functions [6].

The second main theorem of Nevanlinna leads to the following result shown by Ullrich.

Theorem 2.7 ([12]). *If f is an entire function of finite order, then*

$$\sum_{a \in \mathbb{C}} \delta(a, f) \leq \delta(0, f').$$

Let $a_1, \ldots, a_k$ and f be meromorphic functions and let $\{b_1, \ldots, b_n\}$ be the basis of $\operatorname{lin}(\{a_1, \ldots, a_k\})$. We put

$$L(f) := W(b_1, \ldots, b_n, f),$$

where $W(b_1, \ldots, b_n, f)$ denotes the Wronskian of functions $b_1, \ldots, b_n, f$. It is easy to see that if we take constants $a_1, \ldots, a_k$, the basis of $\operatorname{lin}(\{a_1, \ldots, a_k\})$ consists of a single non-zero constant and we get $L(f) = f'$. On the other hand, if $a_1, \ldots, a_k$ are small functions with respect to f, it is possible to obtain results similar to Theorem 2.7 with $L(f)$ in the place of f'. We present here an analogue of Ullrich's theorem for small functions with respect to a meromorphic function in the unit disc.

Let f be a meromorphic function in $\mathbb{D}$ and let $a \in \overline{\mathbb{C}}$. We call

$$\Delta(a, f) = \limsup_{r \to 1^-} \frac{m(r, a, f)}{T(r, f)} = 1 - \liminf_{r \to 1^-} \frac{N(r, a, f)}{T(r, f)}.$$

Valiron's defect of f at a. Let us also remind that the multiplicity index of a meromorphic function f in the unit disc at a value a is defined by

$$\theta(a, f) = \liminf_{r \to 1^-} \frac{N(r, a, f) - \overline{N}(r, a, f)}{T(r, f)}.$$

Theorem 2.8. *Let f be a meromorphic admissible function in the unit disc and $a_1, \dots, a_k$ meromorphic functions small with respect to f. Let $\{b_1, \dots, b_n\}$ be the basis of* $\operatorname{lin}(\{a_1, \dots, a_k\})$ *and put $L(f) = W(b_1, \dots, b_n, f)$. Then the inequality*

$$\sum_{\nu=1}^{k} \delta(a_\nu, f) \le \Delta(0, L(f))(n + 1 - n\theta(\infty, f))$$

holds. In particular, if f is holomorphic, we have the inequality

$$\sum_{\nu=1}^{k} \delta(a_\nu, f) \le \Delta(0, L(f)).$$

Remark 2.9. By similar reasoning as in [10] or [3] it is possible to show that for meromorphic functions in $\mathbb{C}$,

$$\sum_{\nu=1}^{k} \delta(a_\nu, f) \le \Delta(0, L(f))(n + 1 - n\theta(\infty, f)),$$

for entire functions,

$$\sum_{\nu=1}^{k} \delta(a_\nu, f) \le \Delta(0, L(f)),$$

and for entire functions of finite order,

$$\sum_{\nu=1}^{k} \delta(a_\nu, f) \le \delta(0, L(f)),$$

where $L(f)$ is defined in a similar way as in Theorem 2.8.

It is also possible to formulate a separate result of this kind concerning admissible meromorphic functions and polynomials.

Theorem 2.10. *Let f be a meromorphic admissible function in the unit disc and let $p_1, \dots, p_k$ be polynomials, $d := \max_{1\le\nu\le k} \deg p_\nu$. The inequality*

$$\sum_{\nu=1}^{k} \delta(p_\nu, f) \le \Delta(0, f^{(n)})(n + 1 - n\theta(\infty, f))$$

holds for $n = d + 1, d + 2, \dots$. In particular, if f is holomorphic, the inequality

$$\sum_{\nu=1}^{k} \delta(p_\nu, f) \le \Delta(0, f^{(n)})$$

holds for all integers $n > d$.

Theorem 2.11. *Let d be a non-negative integer and let $\mathcal{P}_d$ denote the set of all polynomials of degree at most d. If f is an admissible holomorphic function in $\mathbb{D}$, then for all integers $n > d$ we have*

$$\sum_{p\in\mathcal{P}_d} \delta(p, f) \le \Delta(0, f^{(n)}).$$

3 Basic notions of Nevanlinna theory. Extensions of the second main theorem

Let us now give some background for the results presented in the previous section. We begin with the first and the second fundamental theorems of Nevanlinna for meromorphic functions in the unit disc with fixed values.

Theorem 3.1 ([6]). *For any meromorphic function f in the unit disc the equality*

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(1)$$

holds for any value $a \in \mathbb{C}$ and $r \to 1^-$.

Put

$$N_1(r, f) = N(r, 1/f') + 2N(r, f) - N(r, f').$$

Theorem 3.2 ([6]). *Let $\{a_k\}_{\nu=1}^k$ be a finite set of pairwise distinct complex numbers and let f be a meromorphic function in the unit disc. The inequality*

$$m(r, f) + \sum_{\nu=1}^{k} m(r, a_\nu, f) \le 2T(r, f) - N_1(r, f) + O\Big(\log^+ T(r, f) + \log \frac{1}{1-r}\Big)$$

is true for all $r \to 1^-$, possibly except for r in a set E such that $\int_E \frac{dr}{1-r} < +\infty$.

Let us recall that the defect relation in the unit disc is

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \le 2 + \frac{1}{A}, \tag{3.1}$$

where

$$A := \limsup_{r \to 1^-} \frac{T(r, f)}{-\log(1-r)},$$

provided that $0 < A < +\infty$ (see [6, 9]). The examples in [6] show that the equality in the defect relation (3.1) is possible for a positive value A.

If f is an admissible function, then

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \le 2.$$

For admissible functions we also have the following extension of Picard's theorem [11]. If f and a are meromorphic functions, f is admissible and $T(r, a) = O(1)$ $(r \to 1^-)$, then, apart from the possible two exceptional functions a, $f - a$ has infinitely many zeros in the unit disc. Note here that the functions which are meromorphic in the whole complex plane are of bounded characteristic in the unit disc and therefore not admissible.

In 1964 Chuang obtained a generalization of the second main theorem of Nevanlinna for the case of entire functions and their small functions [1]. For meromorphic functions in the plane, but without the ramification term such a generalization was first proved by Osgood, who used number-theoretic methods (see [7, 8]). In 1986 Frank and Weissenborn, applying the Wronski determinant managed to prove a similar result involving rational functions.

Theorem 3.3 ([2]). *Let f be a transcendental meromorphic function in the complex plane. Then for any $\varepsilon > 0$ and for distinct rational functions $q_1, \dots, q_k$ we have*

$$m(r, f) + \sum_{\nu=1}^{k} m(r, q_\nu, f) \le (2 + \varepsilon)T(r, f) + S(r, f).$$

In the same year, developing the ideas of Frank and Weissenborn, Steinmetz proved his result for small functions in general. The exact analogue of the second main theorem, including the ramification term, was finally obtained by Yamanoi in 2004.

Theorem 3.4 ([13]). *Let f be a nonconstant meromorphic function on $\mathbb{C}$ and let $a_1, \dots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_ν are small functions with respect to f for all $\nu = 1, \dots, k$. Then we have the second main theorem,*

$$(k - 2 - \varepsilon)T(r, f) \le \sum_{\nu=1}^{k} \overline{N}(r, a_\nu, f)$$

for all $\varepsilon > 0$, for $r \to \infty$, $r \notin E$, $\int_E d \log r < \infty$, and the defect relation,

$$\sum_{a \in S(f)} (\delta(a, f) + \theta(a, f)) \le 2.$$

4 Auxiliary results

We start with a formulation of a classical theorem of Milloux.

Lemma 4.1 ([4]). *Let f be an admissible meromorphic function in the unit disc and let a_ν ($\nu = 0, \dots, k$, $k \in \mathbb{N}$) be meromorphic functions in the unit disc small with respect to f. Put $\Psi(f)(z) = \sum_{\nu=0}^{k} a_\nu(z) f^{(\nu)}(z)$. Then*

$$m\Big(r, \frac{\Psi(f)}{f}\Big) = S(r, f) \quad (r \to 1^-)$$

and

$$T(r, \Psi(f)) \le (k+1)T(r, f) + S(r, f) \quad (r \to 1^-).$$

We should note here that in general for meromorphic functions in $\mathbb{D}$ the relationships would be

$$m\Big(r, \frac{\Psi(f)}{f}\Big) = O\Big(\log \frac{1}{1-r}\Big) + S(r, f) \quad (r \to 1^-) \tag{4.1}$$

and

$$T(r, \Psi(f)) \le (k+1)T(r, f) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f) \quad (r \to 1^-). \tag{4.2}$$

Let $f_1, \dots, f_k$ be meromorphic functions. We define the Wronskian of functions $f_1, \dots, f_k$ in the following way:

$$W(f_1, \dots, f_k) := \begin{vmatrix} f_1 & f_2 & \dots & f_k \\ f_1' & f_2' & \dots & f_k' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \dots & f_k^{(k-1)} \end{vmatrix}.$$

Lemma 4.2 ([5]). *Let $D \subseteq \mathbb{C}$ be a domain and let $f_1, \dots, f_k \in \mathcal{M}(D)$, where $\mathcal{M}(D)$ denotes the space of meromorphic functions in D. The Wronskian $W(f_1, \dots, f_k)$ vanishes identically if and only if $f_1, \dots, f_k$ are linearly dependent.*

Other useful properties of Wronskians are as follows (see, for instance, [5, Section 1.4]):

$$W(gf_1, \dots, gf_k) = g^k W(f_1, \dots, f_k) \quad \text{for a meromorphic function } g, \tag{4.3}$$

$$W(f_1, \dots, f_k, g_1, \dots, g_n) = \frac{W(W(f_1, \dots, f_k, g_1), \dots, W(f_1, \dots, f_k, g_n))}{(W(f_1, \dots, f_k))^{n-1}}. \tag{4.4}$$

The following result concerning poles of meromorphic functions in the unit disc is an analogue of the lemma proved by Frank and Weissenborn for meromorphic functions in $\mathbb{C}$.

Lemma 4.3. *Let f be a meromorphic function in the unit disc and let $a_1, \dots, a_n$ be linearly independent (as vectors in the linear space of meromorphic functions in the unit disc over the field $\mathbb{C}$) functions small with respect to f. Put $L(f) := W(a_1, \dots, a_n, f)$. Then for every $\varepsilon > 0$ and $r \to 1^-$,*

$$n\overline{N}(r, f) \le N\Big(r, \frac{1}{L(f)}\Big) + (1+\varepsilon)N(r, f) + O\Big(\frac{1}{1-r}\Big) + S(r, f).$$

Proof. The proof follows basically the same lines as in [3]. Let $a_1, \dots, a_n$ be linearly independent meromorphic functions in the unit disc. For a fixed integer $s \ge 0$ consider a vector space

$$H_s := \operatorname{lin}\{a_1^{k_1} \dots a_n^{k_n} : k_1, \dots, k_n \ge 0,\ k_1 + \dots + k_n = s\}.$$

Let now $B = \{b_1, \dots, b_m\}$ be a basis of H_s. By definition, the vectors $a_1 b_1, \dots, a_1 b_m$ belong to H_{s+1} and it can be easily seen that they are linearly independent. It follows that the basis of H_{s+1} must contain at least m vectors. Thus $\dim H_s \le \dim H_{s+1}$. However, the following result is true ([10]):

$$\inf_{s \in \mathbb{N}} \frac{\dim H_{s+1}}{\dim H_s} = 1.$$

Let then $\varepsilon > 0$ be a fixed number and we choose s_0 such that

$$\frac{\dim H_{s_0+1}}{\dim H_{s_0}} \leq 1 + \varepsilon.$$

Let $\alpha_1, \ldots, \alpha_j$ be the basis of H_{s_0} and $\beta_1, \ldots, \beta_{j+k}$ the basis of H_{s_0+1}, so that $\frac{k}{j} \leq \varepsilon$. For a meromorphic function f in the unit disc, such that $a_1, \ldots, a_n$ are small with respect to f, we consider the Wronskians $L(f) = W(a_1, \ldots, a_n, f)$ and

$$W_f := W(\beta_1, \ldots, \beta_{j+k}, \alpha_1 f, \ldots, \alpha_j f).$$

We put

$$A(z) := \frac{W_f(z)}{(L(f)(z))^j}. \tag{4.5}$$

We now apply property (4.4) to W_f. We get

$$W_f = W(\beta_1, \ldots, \beta_{j+k}, \alpha_1 f, \ldots, \alpha_j f) = \frac{W(W(\beta_1, \ldots, \beta_{j+k}, \alpha_1 f), \ldots, W(\beta_1, \ldots, \beta_{j+k}, \alpha_j f))}{(W(\beta_1, \ldots, \beta_{j+k}))^{j-1}}. \tag{4.6}$$

Applying property (4.3) to the enumerator in (4.5) we have from (4.6),

$$A = \frac{W\Big(\frac{W(\beta_1,\ldots,\beta_{j+k},\alpha_1 f)}{L(f)}, \ldots, \frac{W(\beta_1,\ldots,\beta_{j+k},\alpha_j f)}{L(f)}\Big)}{(W(\beta_1, \ldots, \beta_{j+k}))^{j-1}}. \tag{4.7}$$

For $\nu = 1, \ldots, j$ we can see that

$$W(\beta_1, \ldots, \beta_{j+k}, \alpha_\nu f) = \begin{vmatrix} \beta_1 & \ldots & \beta_{j+k} & \alpha_\nu f \\ \beta_1' & \ldots & \beta_{j+k}' & \alpha_\nu' f + \alpha_\nu f' \\ \vdots & \ddots & \vdots & \vdots \\ \beta_1^{(j+k)} & \ldots & \beta_{j+k}^{(j+k)} & \sum_{i=0}^{j+k} \binom{j+k}{s} \alpha_\nu^{(j+k-i)} f^{(i)} \end{vmatrix} = \sum_{i=0}^{j+k} W_i^\nu \cdot f^{(i)},$$

where each W_i^ν $(i = 0, \ldots, j+k)$ depends on $\beta_1, \ldots, \beta_{j+k}, \alpha_\nu$ and their derivatives only. On the other hand,

$$L(f) = \begin{vmatrix} a_1 & \ldots & a_n & f \\ a_1' & \ldots & a_n' & f' \\ \vdots & \ddots & \vdots & \vdots \\ a_1^{(n)} & \ldots & a_n^{(n)} & f^{(n)} \end{vmatrix} = \sum_{i=0}^{n} \tilde{W}_i \cdot f^{(i)}, \tag{4.8}$$

where each $\tilde{W}_i$ $(i = 0, \ldots, n)$ depends on $a_1, \ldots, a_n$ and their derivatives only. For $1 \leq i \leq n$, $1 \leq \nu \leq j$, the products $\alpha_\nu a_i$ belong to H_{s_0+1}, so

$$W(\beta_1, \ldots, \beta_{j+k}, \alpha_\nu a_i) = 0.$$

As $a_1, \ldots, a_n$ are linearly independent and $L(a_i) = 0$ for $1 \leq i \leq n$, it follows that

$$W(\beta_1, \ldots, \beta_{j+k}, \alpha_\nu f) = \Psi_\nu(L(f)),$$

where each Ψ_ν $(1 \leq \nu \leq j)$ is a linear differential polynomial with coefficients depending on $\alpha_\nu, \beta_1, \ldots, \beta_{j+k}$, $a_1, \ldots, a_n$ and their derivatives only.

For $\nu = 1, \ldots, j$ we now consider the fractions $\frac{W(\beta_1,\ldots,\beta_{j+k},\alpha_\nu f)}{L(f)}$. Applying (4.1) we get for $r \to 1^-$,

$$m\Big(r, \frac{W(\beta_1, \ldots, \beta_{j+k}, \alpha_\nu f)}{L(f)}\Big) = m\Big(r, \frac{\Psi_\nu(L(f))}{L(f)}\Big) = O\Big(\log \frac{1}{1-r}\Big) + S(r, L(f)) = O\Big(\log \frac{1}{1-r}\Big) + S(r, f). \tag{4.9}$$

By Theorem 3.1,

$$m\Big(r, \frac{1}{(W(\beta_1, \ldots, \beta_{j+k}))^{j-1}}\Big) \leq T(r, (W(\beta_1, \ldots, \beta_{j+k}))^{j-1}) + O(1) = S(r, f) + O(1), \tag{4.10}$$

so it follows from (4.7), (4.9) and (4.10) that

$$m(r, A) = O\Big(\log \frac{1}{1-r}\Big) + S(r, f). \tag{4.11}$$

From Theorem 3.1 and (4.11), for $r \to 1^-$,

$$N\Big(r, \frac{1}{A}\Big) \le T(r, A) + O(1) \le N(r, A) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f). \tag{4.12}$$

Let now z_0 be a pole of f of order p, which is a pole of neither of the functions $\alpha_1, \dots, \alpha_j$ nor $\beta_1, \dots, \beta_{j+k}$. Applying (4.3) to W_f, we obtain

$$W_f = W(\beta_1, \dots, \beta_{j+k}, \alpha_1 f, \dots, \alpha_j f) = f^{2j+k} W\Big(\frac{\beta_1}{f}, \dots, \frac{\beta_{j+k}}{f}, \alpha_1, \dots, \alpha_j\Big),$$

so we can see that for $z \to z_0$,

$$W_f(z) = O((z - z_0)^{-p(2j+k)})$$

and

$$A(z) = O((z - z_0)^{j(n+p)-p(2j+k)}) = O((z - z_0)^{jn-p(j+k)}). \tag{4.13}$$

We put $\overline{N}_p^0(r, f)$, $\overline{N}_p^\infty(r, f)$, $\overline{N}_p^*(r, f)$ for the functions counting once each of the poles of f of order p, where A has a zero, a pole, or a finite non-zero value, respectively. It follows from (4.13) and (4.12) that for $r \to 1^-$,

$$\sum_{p=1}^{\infty} (jn - p(j+k))\overline{N}_p^0(r, f) \le N\Big(r, \frac{1}{A}\Big) \le \sum_{p=1}^{\infty} (p(j+k) - jn)\overline{N}_p^\infty(r, f) + jN\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f)$$

and

$$jn \sum_{p=1}^{\infty} (\overline{N}_p^0(r, f) + \overline{N}_p^\infty(r, f)) \le (j+k) \sum_{p=1}^{\infty} p(\overline{N}_p^0(r, f) + \overline{N}_p^\infty(r, f)) + jN\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f). \tag{4.14}$$

From (4.13) we also get

$$jn\overline{N}_p^*(r, f) \le p(j+k)\overline{N}_p^*(r, f), \tag{4.15}$$

so (4.14) and (4.15) give

$$jn\overline{N}(r, f) \le (j+k)N(r, f) + jN\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f).$$

It follows that

$$n\overline{N}(r, f) \le \Big(1 + \frac{k}{j}\Big)N(r, f) + N\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log \frac{1}{1-r}\Big) + S(r, f).$$

As $\frac{k}{j} \le \varepsilon$, this completes the proof. □

Lemma 4.4. *If f is a meromorphic function in the unit disc and $a_1, \dots, a_k$ are meromorphic functions small with respect to f, then for $r \to 1^-$ we have*

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f - a_\nu}\Big) = m\Big(r, \prod_{\nu=1}^{k} \frac{1}{f - a_\nu}\Big) + S(r, f) + O(1).$$

Proof. Let f be a meromorphic function in $\mathbb{D}$ and let $a_1, \dots, a_k$ be meromorphic functions small with respect to f. From Theorem 3.1 and the fact that each a_ν is small with respect to f we have for $r \to 1^-$,

$$T\Big(r, \frac{1}{f - a_\nu}\Big) = T(r, f - a_\nu) + O(1) = T(r, f) + S(r, f) + O(1) \quad (1 \le \nu \le k),$$

so

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) = kT(r,f) - \sum_{\nu=1}^{k} N\Big(r, \frac{1}{f-a_\nu}\Big) + S(r,f) + O(1). \tag{4.16}$$

Also

$$T\Big(r, \prod_{\nu=1}^{k} \frac{1}{f-a_\nu}\Big) = T\Big(r, \prod_{\nu=1}^{k} (f-a_\nu)\Big) + O(1) = \sum_{\nu=1}^{k} T(r, f-a_\nu) + O(1) = kT(r,f) + S(r,f) + O(1),$$

so by (4.16) we get

$$\begin{aligned}\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) &= T\Big(r, \prod_{\nu=1}^{k} \frac{1}{f-a_\nu}\Big) - \sum_{\nu=1}^{k} N\Big(r, \frac{1}{f-a_\nu}\Big) + S(r,f) + O(1)\\ &= m\Big(r, \prod_{\nu=1}^{k} \frac{1}{f-a_\nu}\Big) + S(r,f) + O(1),\end{aligned} \tag{4.17}$$

as desired. □

Lemma 4.5. *If f is a meromorphic function in the unit disc and $a_1, \dots, a_k$ are meromorphic functions small with respect to f, then for $r \to 1^-$ we have*

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) \le T(r,f) + n\overline{N}(r,f) - N\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log \frac{1}{1-r}\Big) + S(r,f),$$

where $L(f) = W(b_1, \dots, b_n, f)$ and $\{b_1, \dots, b_n\}$ is a basis of $\operatorname{lin}(\{a_1, \dots, a_k\})$.

Proof. Let f be a meromorphic function in the unit disc and let $a_1, \dots, a_k$ be meromorphic functions small with respect to f. We consider the product $\prod_{\nu=1}^{k} \frac{1}{f-a_\nu}$. By decomposing it into fractions we obtain

$$\prod_{\nu=1}^{k} \frac{1}{f-a_\nu} = \sum_{\nu=1}^{k} \frac{\tilde{A}_\nu}{f-a_\nu},$$

where $\tilde{A}_\nu$ $(1 \le \nu \le k)$ are small functions with respect to f. Now let $\{b_1, \dots, b_n\}$ be a basis of $\operatorname{lin}(\{a_1, \dots, a_k\})$ and put

$$L(f) := W(b_1, \dots, b_n, f).$$

As $b_1, \dots, b_n, f$ are linearly independent, by Lemma 4.2 we have $L(f) \neq 0$. Also for $\nu = 1, \dots, k$ we have

$$W(b_1, \dots, b_n, a_\nu) = 0,$$

which leads to the equality

$$W(b_1, \dots, b_n, f) = W(b_1, \dots, b_n, f) - W(b_1, \dots, b_n, a_\nu) = W(b_1, \dots, b_n, f-a_\nu)$$

for $1 \le \nu \le k$. Like in (4.8), for each fixed a_ν we have

$$L(f) = \sum_{i=0}^{n} B_i f^{(i)} = \sum_{i=0}^{n} B_i (f-a_\nu)^{(i)}, \tag{4.18}$$

where each B_i $(i = 0, \dots, n)$ is a small function with respect to f and therefore also with respect to $f - a_\nu$. For $i = 0, \dots, n$, $\nu = 1, \dots, k$ we put $C_{i,\nu} := \tilde{A}_\nu B_i$, which means that each $C_{i,\nu}$ is also a small function with respect to $f - a_\nu$. This way we have

$$\tilde{A}_\nu L(f) = \tilde{A}_\nu L(f-a_\nu) = \sum_{i=0}^{n} C_{i,\nu} (f-a_\nu)^{(i)}.$$

From (4.1) we get

$$m\Big(r, \frac{\tilde{A}_\nu L(f)}{f-a_\nu}\Big) = O\Big(\log \frac{1}{1-r}\Big) + S(r, f-a_\nu) = O\Big(\log \frac{1}{1-r}\Big) + S(r,f). \tag{4.19}$$

Applying (4.18) and (4.1) we obtain

$$m(r, L(f)) = m\Big(r, f\sum_{i=0}^{n} B_i\frac{f^{(i)}}{f}\Big) \le m(r,f) + m\Big(r, \sum_{i=0}^{n} B_i\frac{f^{(i)}}{f}\Big) = m(r,f) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f).$$

Also, if at a point z_0, which is not a pole of $b_1, \dots, b_n$, the function f has a pole of order p, then $L(f)$ has a pole of order $p+n$ at this point. Thus

$$N(r, L(f)) \le N(r,f) + n\overline{N}(r,f) + S(r,f).$$

It follows that

$$T(r, L(f)) \le T(r,f) + n\overline{N}(r,f) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f). \tag{4.20}$$

Applying (4.19) and (4.20) into (4.17), we obtain

$$\begin{aligned}
\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) &= m\Big(r, \sum_{\nu=1}^{k}\frac{\tilde{A}_\nu}{f-a_\nu}\Big) + S(r,f) + O(1)\\
&\le m\Big(r, \sum_{\nu=1}^{k}\frac{\tilde{A}_\nu L(f)}{f-a_\nu}\Big) + m\Big(r, \frac{1}{L(f)}\Big) + S(r,f) + O(1)\\
&\le \sum_{\nu=1}^{k} m\Big(r, \frac{\tilde{A}_\nu L(f)}{f-a_\nu}\Big) + m\Big(r, \frac{1}{L(f)}\Big) + S(r,f) + O(1)\\
&= T(r, L(f)) - N\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f)\\
&\le T(r,f) + n\overline{N}(r,f) - N\Big(r, \frac{1}{L(f)}\Big) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f). \qquad \square
\end{aligned}$$

5 Proofs of the main results

5.1 Proof of Theorem 2.2

The theorem follows directly from Lemma 4.3 and Lemma 4.5.

By Lemma 4.3 we get the estimate

$$-N\Big(r, \frac{1}{L(f)}\Big) \le -n\overline{N}(r,f) + (1+\varepsilon)N(r,f) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f),$$

which together with Lemma 4.5 gives

$$\begin{aligned}
m(r,f) + \sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) &\le 2T(r,f) + \varepsilon N(r,f) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f)\\
&\le (2+\varepsilon)T(r,f) + O\Big(\log\frac{1}{1-r}\Big) + S(r,f).
\end{aligned}$$

5.2 Proof of Theorem 2.8

Let f and $a_1, \dots, a_k$ be like in the statement of the theorem. Put $\{b_1, \dots, b_n\}$ for the basis of $\operatorname{lin}(\{a_1, \dots, a_k\})$ and denote $L(f) = W(b_1, \dots, b_n, f)$. In a similar way as in the proof of Lemma 4.5 we can estimate for $r \to 1^-$,

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) \le T(r, L(f)) - N\Big(r, \frac{1}{L(f)}\Big) + S(r,f) \le m\Big(r, \frac{1}{L(f)}\Big) + S(r,f). \tag{5.1}$$

Also, by (4.20),

$$T(r, L(f)) \le T(r,f) + n\overline{N}(r,f) + S(r,f). \tag{5.2}$$

Inequalities (5.1) and (5.2) give

$$\begin{aligned}\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-a_\nu}\Big) &\le m\Big(r, \frac{1}{L(f)}\Big) + S(r,f)\\ &\le \frac{m(r, \frac{1}{L(f)})}{T(r, L(f))}\big(T(r,f) + n\overline{N}(r,f) + S(r,f)\big) + S(r,f)\\ &\le \frac{m(r, \frac{1}{L(f)})}{T(r, L(f))}\big((n+1)T(r,f) - n(N(r,f) - \overline{N}(r,f))\big) + S(r,f).\end{aligned}$$

Dividing both sides of the inequality by $T(r,f)$ and passing to the limit with $r \to 1^-$, we get the statement.

5.3 Proof of Theorem 2.10

Let f be an admissible meromorphic function in the unit disc and let $p_1, \dots, p_k$ $(k \ge 2)$ be distinct polynomials. By Lemma 4.4, for $r \to 1^-$ we have

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-p_\nu}\Big) = m\Big(r, \prod_{\nu=1}^{k} \frac{1}{f-p_\nu}\Big) + O(1). \tag{5.3}$$

Since in this case small functions are polynomials, the term $S(r,f)$ appearing in the estimate in Lemma 4.4 can be replaced in (5.3) by $O(1)$. Decomposing the product into fractions, we obtain

$$\prod_{\nu=1}^{k} \frac{1}{f-p_\nu} = \sum_{\nu=1}^{k} \frac{q_\nu}{f-p_\nu}, \tag{5.4}$$

where $q_1, \dots, q_k$ are rational functions. Let us take $n > d = \max_{1\le\nu\le k} \deg p_\nu$. Then $(f-p_\nu)^{(n)} = f^{(n)}$. It follows that

$$\begin{aligned}\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-p_\nu}\Big) &= m\Big(r, \sum_{\nu=1}^{k} \frac{q_\nu}{f-p_\nu}\Big) + O(1)\\ &\le \sum_{\nu=1}^{k} m\Big(r, \frac{f^{(n)}}{f-p_\nu}\Big) + m\Big(r, \frac{1}{f^{(n)}}\Big) + \sum_{\nu=1}^{k} m(r, q_\nu) + O(1)\\ &= \sum_{\nu=1}^{k} m\Big(r, \frac{f^{(n)}}{f-p_\nu}\Big) + m\Big(r, \frac{1}{f^{(n)}}\Big) + O(1).\end{aligned}$$

Applying (4.1) to $\frac{f^{(n)}}{f-p_\nu}$, we get for $r \to 1^-$,

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-p_\nu}\Big) \le m\Big(r, \frac{1}{f^{(n)}}\Big) + S(r,f). \tag{5.5}$$

Moreover,

$$T(r, f^{(n)}) \le T(r,f) + n\overline{N}(r,f) + S(r,f). \tag{5.6}$$

From (5.5) and (5.6), similarly as in the proof of Theorem 2.8, we obtain

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-p_\nu}\Big) \le \frac{m(r, \frac{1}{f^{(n)}})}{T(r, f^{(n)})}\big((n+1)T(r,f) - n(N(r,f) - \overline{N}(r,f))\big) + S(r,f).$$

Dividing both sides of the inequality by $T(r,f)$ and passing to the limit with $r \to 1^-$, we get the statement for meromorphic functions.

For an admissible holomorphic function f we have $T(r,f) = m(r,f)$, so (5.6) gives

$$m(r, f^{(n)}) \le m(r,f) + S(r,f).$$

Applying (5.5), we obtain

$$\sum_{\nu=1}^{k} m\Big(r, \frac{1}{f-p_\nu}\Big) \le \frac{m(r, \frac{1}{f^{(n)}})}{T(r, f^{(n)})}(T(r,f)+S(r,f))+S(r,f).$$

We divide both sides of the inequality by $T(r,f)$, pass to the limit with $r \to 1^-$ and get the statement in this case.

5.4 Proof of Theorem 2.11

The statement follows from Theorem 2.10 by the standard argument. Let f be an admissible holomorphic function in $\mathbb{D}$. Let also a non-negative integer d, an integer $n > d$ and a positive integer s be fixed. By $\mathcal{P}_{d,s}$ we denote the set of all polynomials in $\mathcal{P}_d$ such that if $p \in \mathcal{P}_{d,s}$, then $\delta(p,f) \ge \frac{\Delta(0,f^{(n)})}{s}$. It follows from Theorem 2.10 that for $p_1, \dots, p_k \in \mathcal{P}_{d,s}$ the inequality

$$k\frac{\Delta(0,f^{(n)})}{s} \le \sum_{\nu=1}^{k} \delta(p_\nu, f) \le \Delta(0, f^{(n)})$$

holds. Thus $k \le s < \infty$, so each of the sets $\mathcal{P}_{d,s}$ is finite. Moreover, we have the equality $\Omega_d(f) = \bigcup_{s=1}^{\infty} \mathcal{P}_{d,s}$, where $\Omega_d(f)$ denotes the set of all defective polynomials of f of degree at most d. It follows that the set $\Omega_d(f)$ is at most countable, which leads to the desired inequality.

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