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On the extremum dissipation for steady state incompressible flow past a sphere at low Reynolds number

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Abstract: A methodology based on sound non-equilibrium thermodynamics principles is developed to estimate the extremum dissipation point for steady-state incompressible flow past a sphere at low Reynolds numbers. It is shown, that the extremum dissipation point appears at the point when both the shear stress and the pressure at the surface of the sphere are equal to zero. The Reynolds number and the position of the extremum dissipation flow past a sphere were further estimated with the aid of a mathematical model for pressure distribution on the sphere surface, accounting for both creeping and ideal flow. The parameters of the model were determined by comparison of the calculated pressure distribution at the surface with the available literature data. The conditions at which the separation angle and the extremum dissipation angle coincide were also investigated. It is believed that this work could be used to further elucidate the flow past a sphere.

Keywords: creeping flow; dissipation; extremum; flow past sphere; ideal flow; low Reynolds number

1 Introduction

Flow past a sphere attained significant scientific interest since the 19th century due to its importance in many

phenomena in science and technology. In particular, the flow past a sphere at a low Reynolds number (1–400) due to its wide applications in many industrial processes, such as chemical reactor engineering and process designing, was the subject of extensive investigation during the past years.

For example, Taneda (1956) used photographic techniques to experimentally investigate the wake behind a sphere moving in a water tank at Reynolds numbers from 5 to 300. Jenson (1959) utilized relaxation methods to study the viscous incompressible flow around a sphere at low Reynolds numbers (<40). Moreover, the pressure distribution results calculated at the surface of the sphere for $Re = 5, 10, 20$, and 40 were tabulated.

Chester, Breach, and Proudman (1969) considered the flow of an incompressible, viscous fluid past a sphere for small values of the Reynolds number. They also proposed a novel correlation for the drag force. Rimon and Cheng (1969) studied theoretically the transient uniform flow around a sphere by solving the complete Navier–Stokes equations. The considered fluid was incompressible and homogeneous, while the range of Reynolds number studied was from 1 to 1000.

Abraham (1970) proposed a correlation for the drag coefficient of a sphere in Newtonian incompressible flow valid for a wide range of Reynolds numbers: $0 \leq Re \leq 5000$ by using simple but effective arguments. Lin and Lee (1973) studied the transient state solutions of the Navier–Stokes equations for incompressible flow around a sphere by solving a set of implicit finite difference equations. Seeley, Hummel, and Smith (1975) studied the velocity profiles in laminar flow around spheres experimentally using a non-disturbing flow visualization technique for uniform laminar flow around a sphere at Reynolds numbers 290, 750, 1300, and 3000.

Lee, Downie, and Bettess (1991) presented a procedure for calculating the starting flow around a sphere in a uniform stream. In particular, the flow field was simulated by a flow of ideal fluid with embedded vorticity. With the assumption that the flow remains symmetric, the vorticity field was approximated by a number of discrete circular line vortices. Later, Lee (2000) solved the discretized Navier–Stokes equations in three-dimensional space using Galerkin's finite element method. Numerical simulations of uniform flow past a sphere were presented for Reynolds numbers from 100 to 500.

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The flow of an incompressible viscous fluid past a sphere was investigated numerically and experimentally over flow regimes, including steady and unsteady laminar flow at Reynolds numbers of up to 300 by Johnson and Patel (2000). The unsteady three-dimensional flow around the sphere was also studied by Sadikin et al. (2014) by using numerical simulation computational fluid dynamics for moderate Reynolds number between $20 \leq \text{Re} \leq 500$.

The three-dimensional structures of flow around a rigid sphere at moderate Reynolds number (Re) between 20 and 400 were also investigated by Li and Zhou (2021) by using the finite volume method with adaptive mesh refinement. For a recent detailed review of the published literature for flow past a single stationary sphere, the interested reader must resort to Tiwari et al. (2020a, b).

Strongly related to the flow problems are the dissipation energy (Ψ) and entropy production (σ). These fundamental quantities of non-equilibrium thermodynamics are related to each other as $\Psi = \sigma T$. T stands for the absolute temperature. The entropy production inflows was the subject of intensive research during past years, as reviewed by Sadiki, Agrebi, and Ries (2022) and Hussien et al. (2021).

The recent advance in the area of complex fluid flow (Christov 2022), biofluids (Sanal Kumar et al. 2022), and microfluidics (Hafemann and Fröhlich 2023) lead to a re-examination of flows in terms of the entropy production rate (Gandhi et al. 2023; Mallikarjuna et al. 2022; Shahsavari et al. 2021). In our previous work (Verros 2020), we developed comprehensive criteria for the extrema in entropy production rate for heat transfer at the limit of the linear region of the extended thermodynamics framework.

This communication aims to investigate the flow past a sphere in terms of energy dissipation. In particular, in the following sections, a detailed methodology for estimating the extremum point of energy dissipation for flow past a sphere is proposed, and the results of this work are presented. Finally, conclusions are drawn, and ideas for future work are proposed.

2 Theoretical part

The starting point of this analysis is the definition of total dissipation energy Ψ in terms of drag force F_k for flow past a sphere (Bird, Stewart, and Lightfoot 2002):

$$\Psi = - \int_0^{2\pi} \int_0^\pi \int_R^\infty (\tau : \nabla v) r^2 dr \sin \theta d\theta d\varphi = F_k V_0 \quad (1)$$

where V_0 is a constant velocity of a fluid approaching the sphere, τ is the shear stress tensor (see Figure 1). The rest

symbols are explained in the notation section. Alternatively, the drag force acting on the interface between the fluid and a particle is defined as the component of the fluid force in the fluid direction. The fluid drag comprises skin friction drag and pressure drag. Therefore, the fluid drag force F_k is defined as (Dey, Ali, and Padhi 2019):

$$F_k = - \int_S \tau_w \sin \theta ds - \int_S P \cos \theta ds \quad (2)$$

where τ_w is the wall shear stress, P is the pressure intensity, and S is the surface area of the particle (see Figure 1).

Eq. (1) relates a fundamental quantity of non-equilibrium thermodynamics, such as the total dissipation energy Ψ with an important quantity of fluid mechanics, such as drag force F_k . Eq. (2) is nothing else than a force balance at the surface of the sphere.

The following equation holds true for a sphere having a diameter equal to D by taking into account its symmetry (Dey, Ali, and Padhi 2019):

$$dS = (\pi D^2 \sin \theta d\theta) / 2 \quad (3)$$

By further using Eqs. (1–3), the following equation holds true for the derivative of total dissipation energy with respect to angle θ :

$$\frac{\partial \Psi}{\partial \theta} = F(\theta) = -\tau_w (\pi D^2 \sin^2 \theta) / 2 - P (\pi D^2 \sin \theta \cos \theta) / 2 \quad (4)$$

Therefore, the extremum point for total dissipation energy appears when $F(\theta) = 0$.

This extremum point is at a maximum if $\frac{\partial F(\theta)}{\partial \theta} < 0$ and vice versa for a minimum point. The extremum point condition is automatically satisfied if $\tau_w = 0$ and $P = 0$. Clearly, the

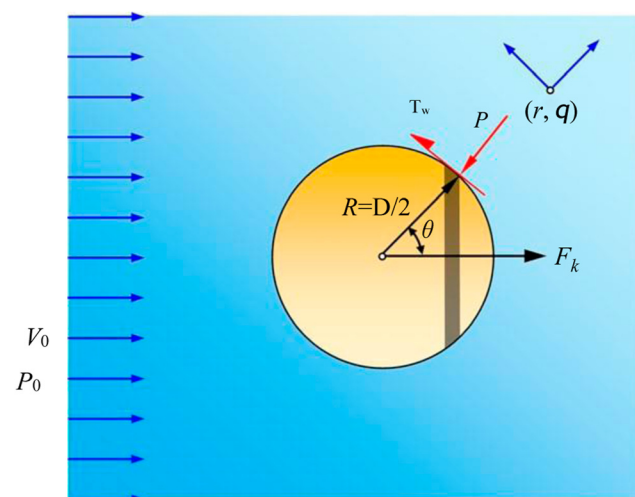


Figure 1: Schematic of fluid drag on a sphere.

first condition implies a point of separation of wake, and the angle at which the second condition is satisfied has to be determined.

To calculate the condition for $P = 0$ at the surface of the sphere, one has to resort to dimensional analysis. According to the dimensional analysis, the pressure P of a Newtonian fluid at the surface of the sphere could be written as:

$$k = \frac{P - P_0}{1/2\rho V_0^2} = C(\theta) \left(\frac{\mu}{\rho V_0 D} \right)^\xi \text{ or in the most general case.}$$

$$k = \frac{P - P_0}{1/2\rho V_0^2} = C(\theta) \left(\frac{1}{f(\text{Re})} \right)^\xi, \text{Re} = \left(\frac{\rho V_0 D}{\mu} \right) \quad (5)$$

where P_0 is the pressure of the fluid as approaching the sphere, μ is the viscosity, ρ stands for density, and D stands for sphere diameter. $C(\theta)$ is a function of angle θ .

Outside the boundary layer, the flow could be approximated as ideal flow; in other words, the flow is independent of viscosity ($\xi = 0$) (Landau and Lifshitz 1959). By setting $\xi = 0$ and transforming the area outside the boundary ($R + \delta$) to the surface of the sphere, the pressure distribution at the surface of a sphere having radius R could be written as:

$$k = \frac{P - P_0}{1/2\rho V_0^2} = C(\theta) \left(\frac{R + \delta}{R} \right)^2 = C(\theta) \left(1 + \frac{\delta}{R} \right)^2 \quad (6)$$

δ is the boundary layer thickness as proposed by Abraham (1970): $\delta/R = \delta_0/\text{Re}^{0.5}$, $\delta_0 \approx 9.06$.

The above Eq. (6) is compatible with Eq. (5). This analysis for the transformation of ideal flow with the aid of the boundary layer theory is not a new idea; the origin of this idea could be found in the pioneering work of Abraham (1970) for the functional dependence of drag force of a sphere on Reynolds number. Besides its simplicity, this methodology has been proven very effective in developing correlations for the drag force valid over a wide range of Reynolds numbers: $0 \leq \text{Re} \leq 5000$.

According to Bird, Stewart, and Lightfoot (2002), and Landau and Lifshitz (1959), the pressure distribution for ideal flow past a sphere is written as:

$$k = \frac{P - P_0}{1/2\rho V_0^2} = \left(1 - \frac{9}{4} \sin^2 \theta \right) \quad (7)$$

The pressure distribution at the surface of the sphere for creeping flow is written as (Dey, Ali, and Padhi 2019):

$$k = \frac{P - P_0}{1/2\rho V_0^2} = - \left(\frac{6}{\text{Re}} \right) \cos \theta \quad (8)$$

Finally, in this work, the function $C(\theta)$ was approached as follows:

$$C(\theta) = \left(-\frac{6}{\delta_0^2} \cos \theta \right) (1 + c_1 \text{Re}) e^{-c_2 \text{Re}} + \left(1 - \frac{9}{4} \sin^2 \theta \right) \times (1 - e^{-c_3 \text{Re}}) \quad (9)$$

c_i are adjustable parameters estimated by fitting to Eq. (6) literature data for the k parameter. In this way, the limit of pure creeping flow is reached for Eq. (6) for $\text{Re} \approx 0$, while for $\text{Re} \rightarrow \infty$ the limit of ideal flow is approximated.

3 Results & discussion

The first step is to estimate the c_i parameters (see Eq. (9)) by fitting predictions obtained by using Eq. (6) to the data reported by Jenson (1959) for Reynolds numbers equal to 5 and 40, respectively. The resulting graph is illustrated in Figure 2.

The selected Reynolds numbers are the lower and upper limits of the tabulated data of Jenson (1959) for pressure distribution at the surface of the sphere. The estimated c_i parameters have the values -0.4967 , 6.3×10^{-3} , and 2.97×10^{-3} , respectively. For a detailed analysis of the results shown in Figure 2, the interested reader has to resort to the original work of Jenson (1959).

To further validate the developed model, the calculated pressure distribution by using Eq. (6) and Eq. (9) was compared with the limit of creeping flow ($\text{Re} = 10^{-3}$) and the limit of ideal flow ($\text{Re} = 10^5$).

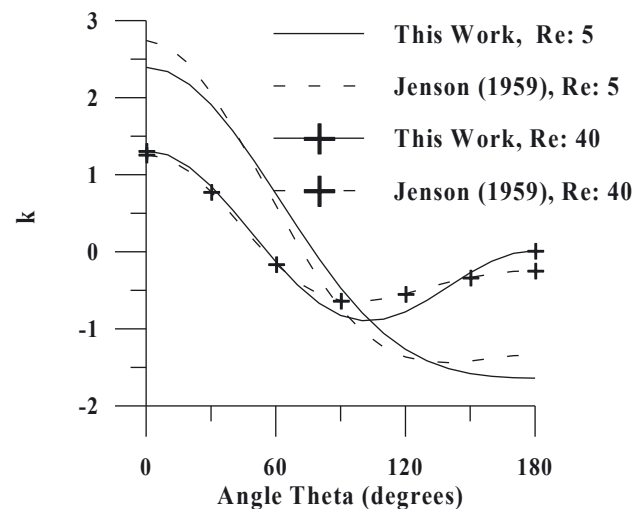


Figure 2: Comparison of calculated pressure distribution at the surface of the sphere with literature results (Jenson 1959).

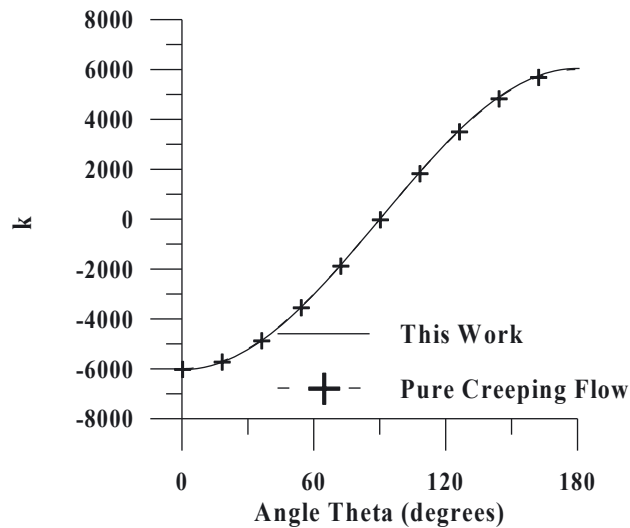


Figure 3: Comparison of calculated pressure distribution at the surface of the sphere at the limit of creeping flow (Dey, Ali, and Padhi 2019).

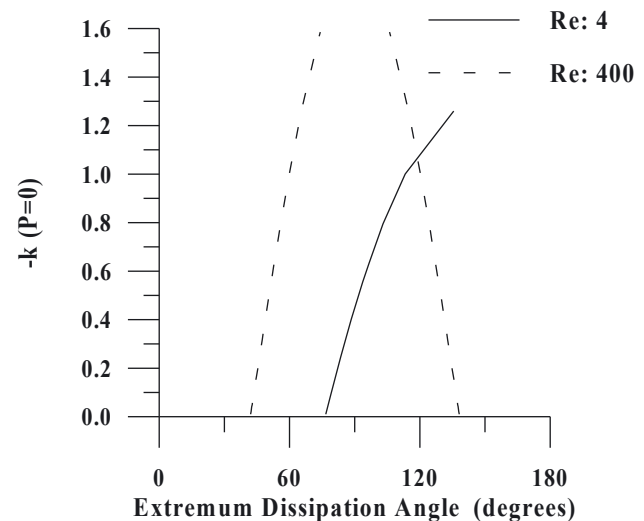


Figure 5: Parameter k versus extremum dissipation angle for various Reynolds Numbers.

The excellent fitting depicted in Figures 3 and 4 could be attributed to the fact that Eq. (9) is reduced to the limit of creeping flow and ideal flow as the Reynolds number tends to zero and infinite, respectively.

The calculated extremum dissipation angles as a function of k parameters for various Reynolds numbers are depicted in Figure 5.

The different shapes of curves shown in Figure 5 for Reynolds numbers equal to 4 and 400 could be attributed to the different flow phenomena observed at the above

numbers. In particular, according to the experimental results of Taneda (1956), there is no flow separation or vortices formation at Reynolds number equal to 4. However, according to Li and Zhou (2021) vortices exist at Reynolds number equal to 400 while the separation phenomena are dominant above Reynolds 40.

A more profound question arises from the above analysis: What are the conditions under which the extremum dissipation value, as calculated in this work, coincides with the separation angle? In Table 1, the separation angles for various Reynolds numbers, as determined experimentally by Taneda (1956), are summarized with the k values as calculated in this work for the same angle.

Strongly related to this work is the Prigogine theorem (Kondepudi and Prigogine 2015) that states, for systems in the linear regime (such as a Newtonian fluid), the total internal entropy production reaches a minimum value at non-equilibrium steady states. Therefore, one could anticipate that the described extremum of entropy production (σ) in an isothermal flow corresponds to a minimum of energy dissipation ($\Psi = \sigma T$) if a non-equilibrium steady state was assumed. Regarding the results shown in Table 1, one could

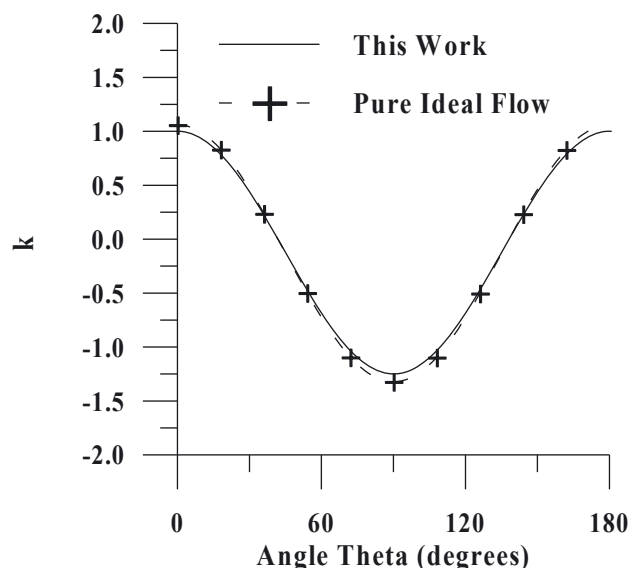


Figure 4: Comparison of calculated pressure distribution at the surface of the sphere at the limit of pure ideal flow (Bird, Stewart, and Lightfoot 2002; Landau and Lifshitz 1959).

Table 1: Separation angle (Taneda 1956) and extremum dissipation angle as a function of parameter k .

Reynolds number	Re: 40	Re: 100	Re: 200	Re: 300
Separation angle (Taneda 1956)- extremum dissipation angle	34°	53°	63°	67°
$-k (P=0)$	–	0.39	0.94	1.23

anticipate highly convective flows in the case that the separation angle coincides with the extremum dissipation angle and minimum energy dissipation for the system was assumed according to the Prigogine theorem (Kondepudi and Prigogine 2015).

4 Conclusions

In this work, it was shown using sound non-equilibrium principles that the extremum dissipation point appears at the point where both the shear stress and the pressure at the surface of the sphere are equal to zero. Furthermore, a novel mathematical model for pressure distribution on the sphere surface accounting for creeping and ideal flow was developed. The parameters of this model were determined by fitting literature data. The conditions at which the separation angle and the extremum dissipation angle coincide were also calculated to give values here. Moreover, the extremum dissipation point was related to a minimum point according to the Prigogine theorem if a non-equilibrium steady state was assumed. Limitations of this work include the empirical methodology involved in the model development for pressure distribution at the sphere surface that limits the application to a low Reynolds number. The authors are hopeful that this work will attract the interest of both experimentalists and theoreticians to investigate further fluid mechanics phenomena related to the extremum dissipation angle.

Notation

$C(\theta)$	dimensionless auxiliary parameter
c_i	adjustable dimensionless parameter
D	diameter of the sphere
F_k	drag force
k	dimensionless pressure coefficient
P	pressure function at the surface of the sphere
P_0	static pressure in an undisturbed stream
Re	Reynolds number
R	radius of the sphere
r	radial spherical polar co-ordinate
S	surface of the particle
V_0	undisturbed stream velocity
$\mathbf{v}$	velocity vector

Greek letters

δ_0	dimensionless parameter
δ	boundary layer thickness

θ	angular spherical polar co-ordinate
μ	coefficient of viscosity
ρ	density
σ	entropy production rate
τ	shear tensor
τ_w	shear stress at the surface of the sphere
φ	angular spherical polar co-ordinate
ψ	energy dissipation

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