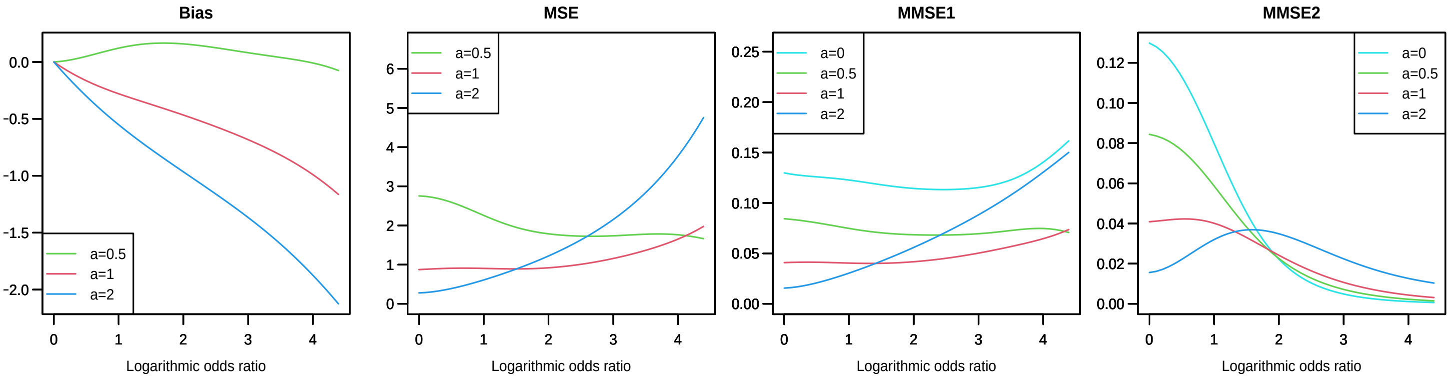
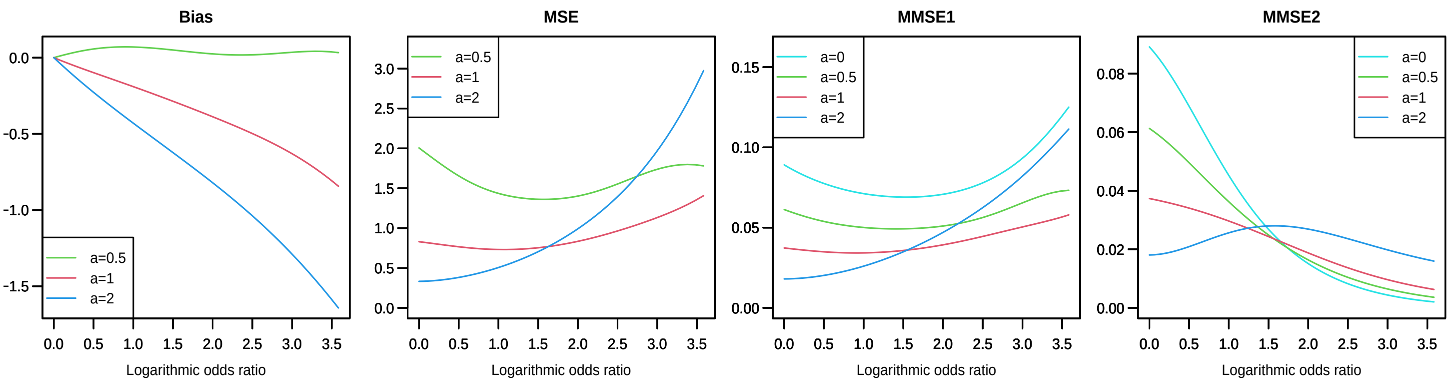
$(n, m) = (10, 10)$, $q = 0.1$



$(n, m) = (10, 10)$, $q = 0.2$



$(n, m) = (10, 10)$, $q = 0.3$

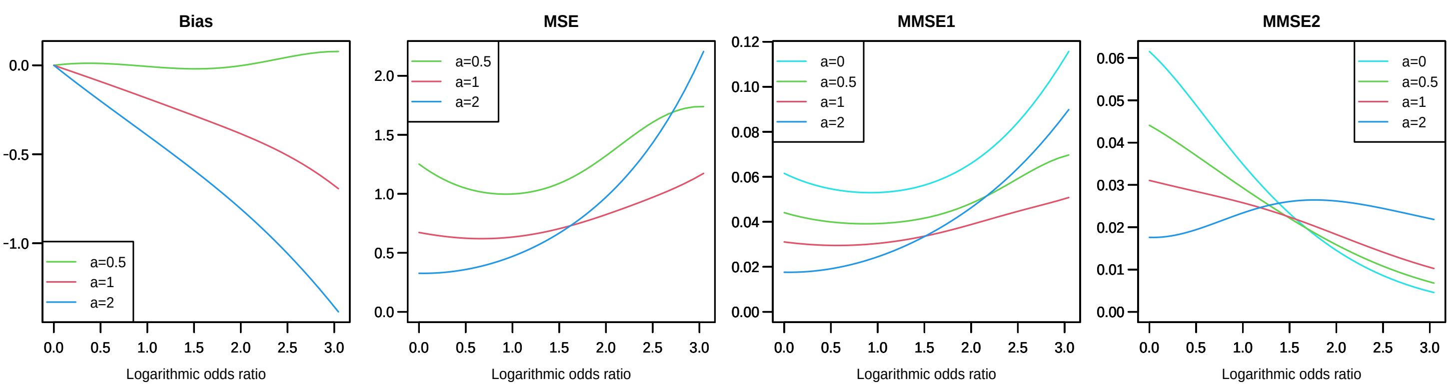


Figure S1 The choice of prior by graphical comparison of three risks and bias numerically calculated; the case of $(n, m) = (10, 10)$.

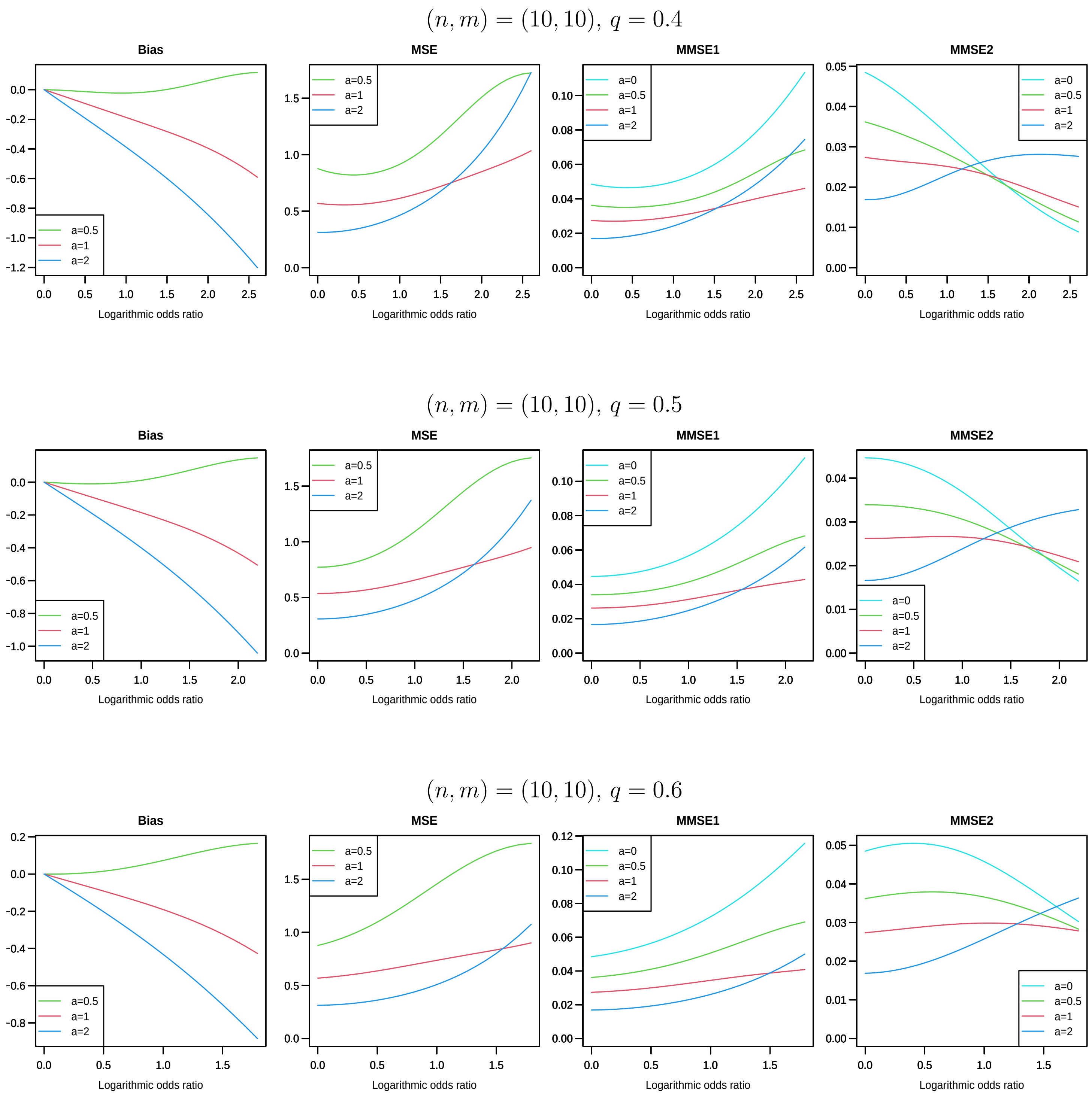
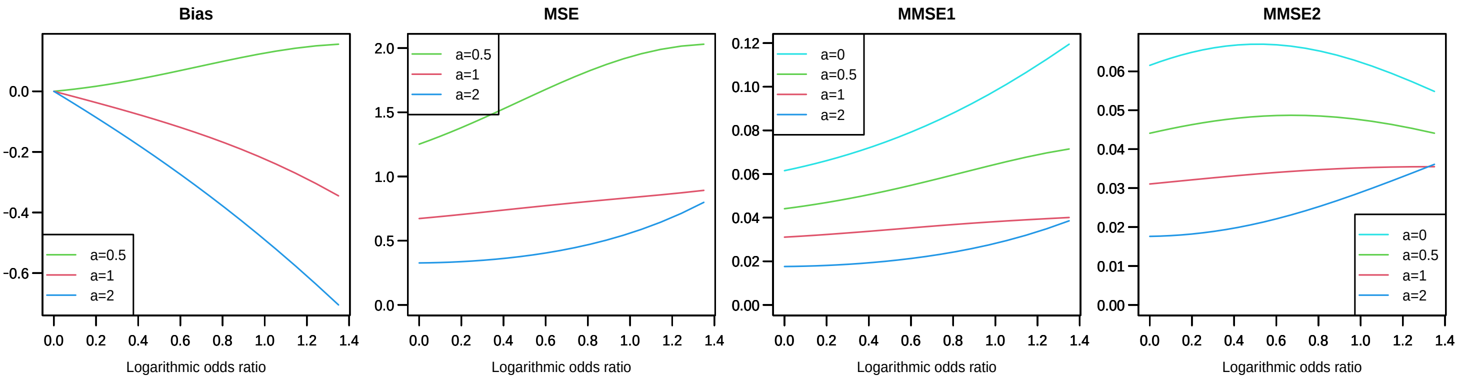
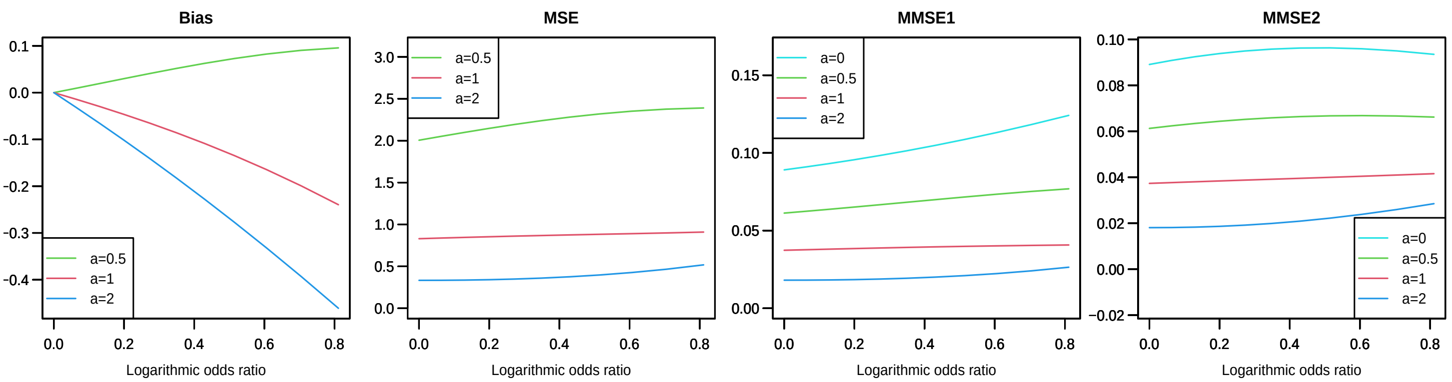


Figure S1 (continued).

$$(n, m) = (10, 10),\ q = 0.7$$



$$(n, m) = (10, 10),\ q = 0.8$$



$$(n, m) = (10, 10),\ q = 0.9$$

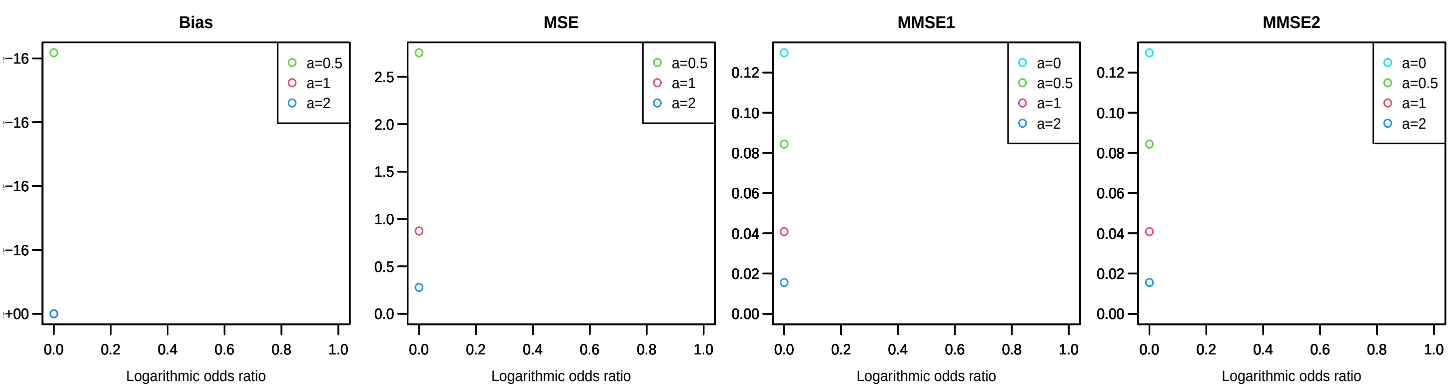
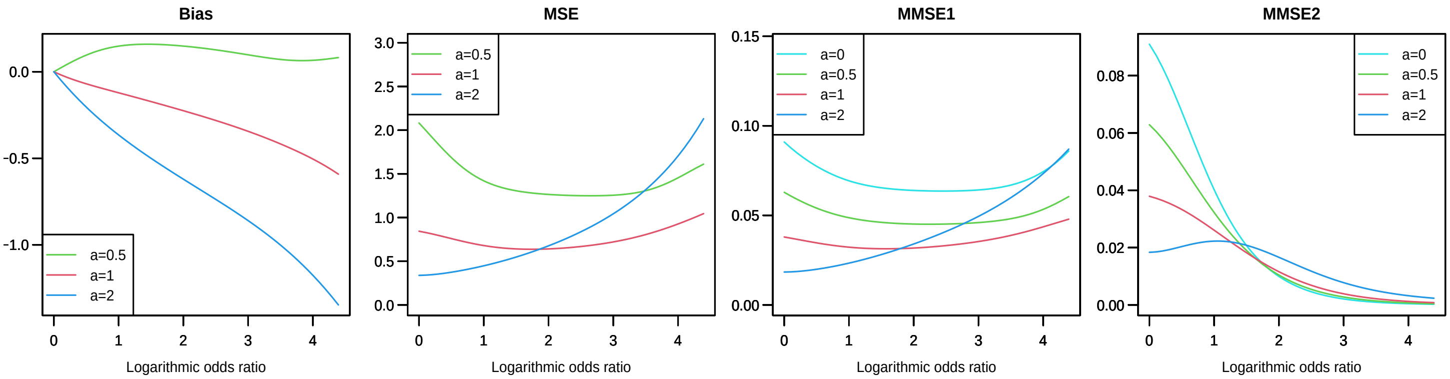
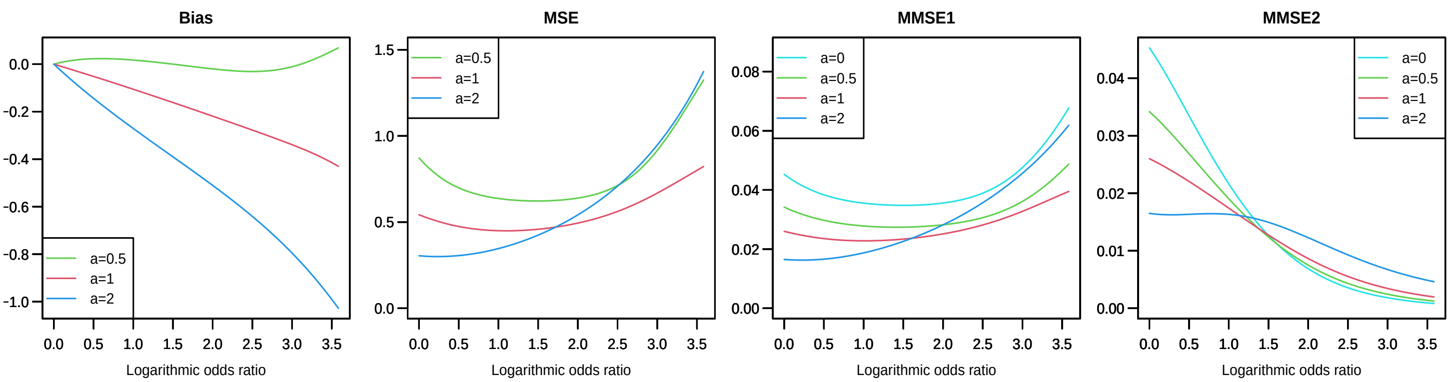


Figure S1 (continued).

$(n, m) = (20, 20),\ q = 0.1$



$(n, m) = (20, 20),\ q = 0.2$



$(n, m) = (20, 20),\ q = 0.3$

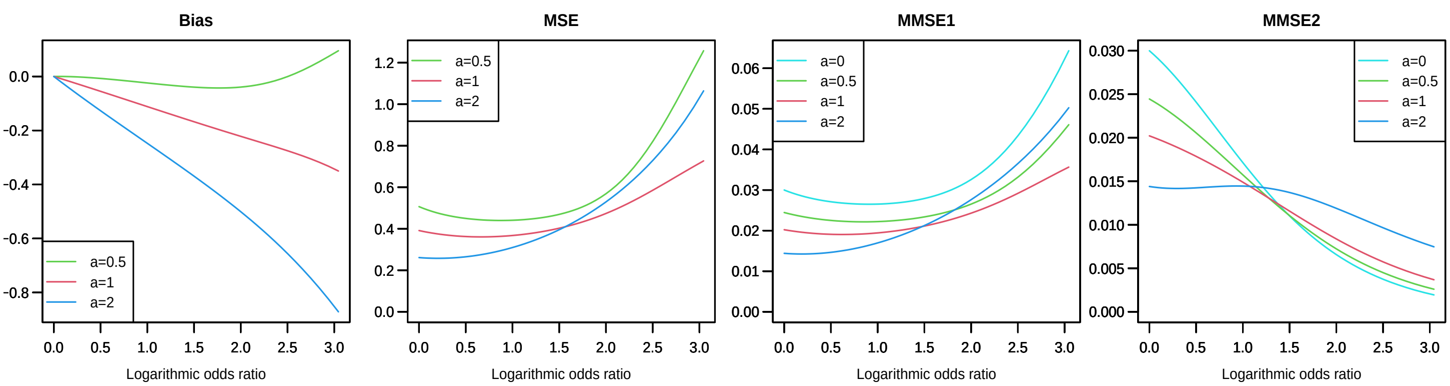
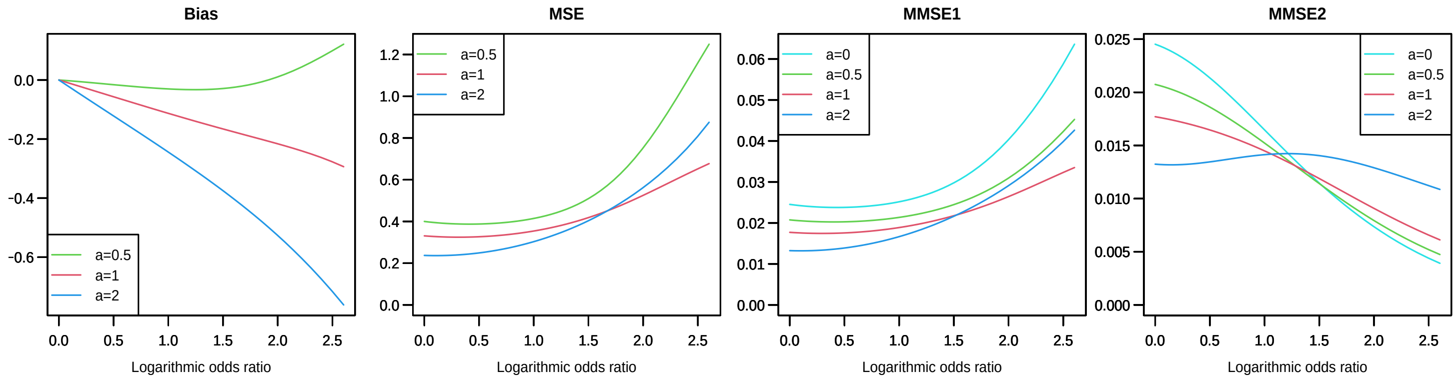
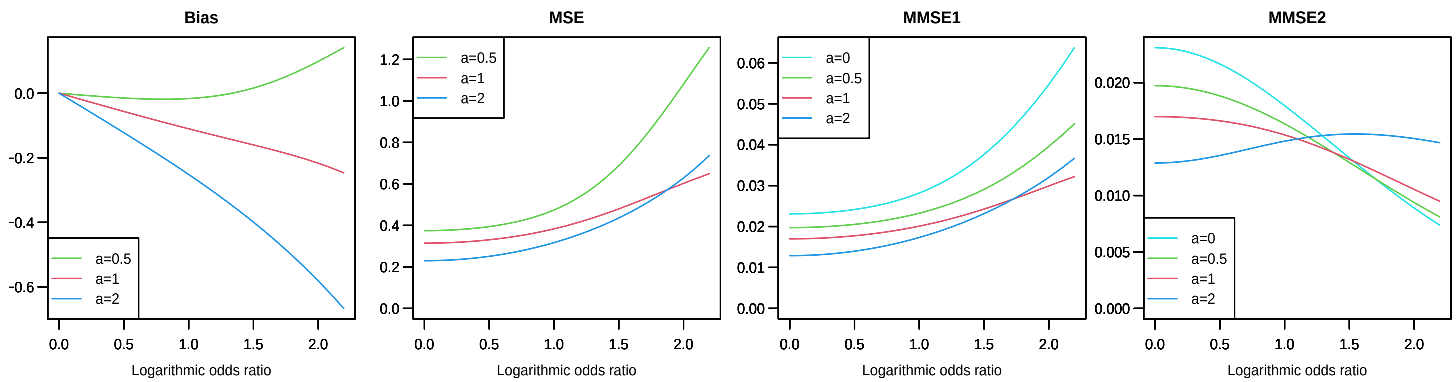


Figure S2 The choice of prior by graphical comparison of three risks and bias numerically calculated; the case of $(n, m) = (20, 20)$.

$(n, m) = (20, 20),\ q = 0.4$



$(n, m) = (20, 20),\ q = 0.5$



$(n, m) = (20, 20),\ q = 0.6$

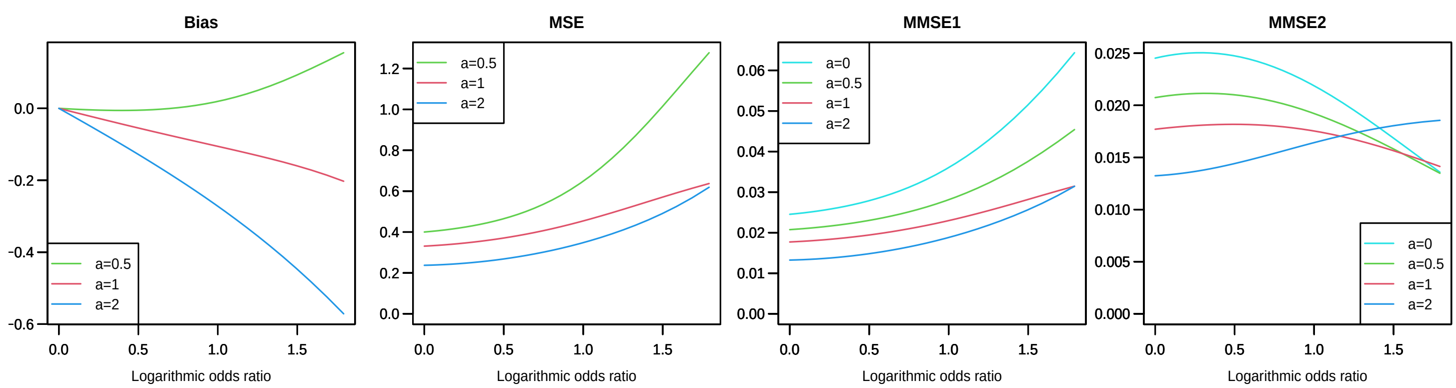
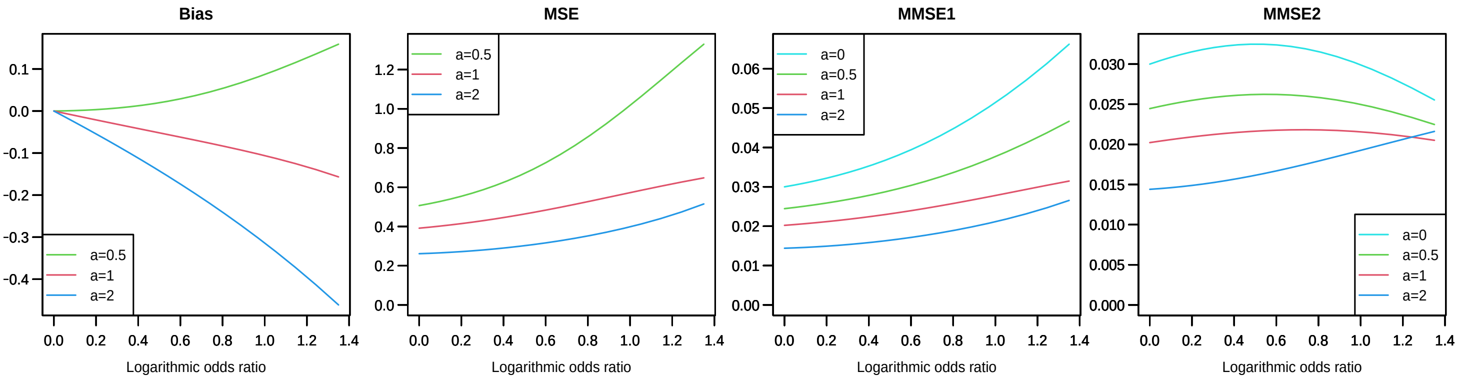
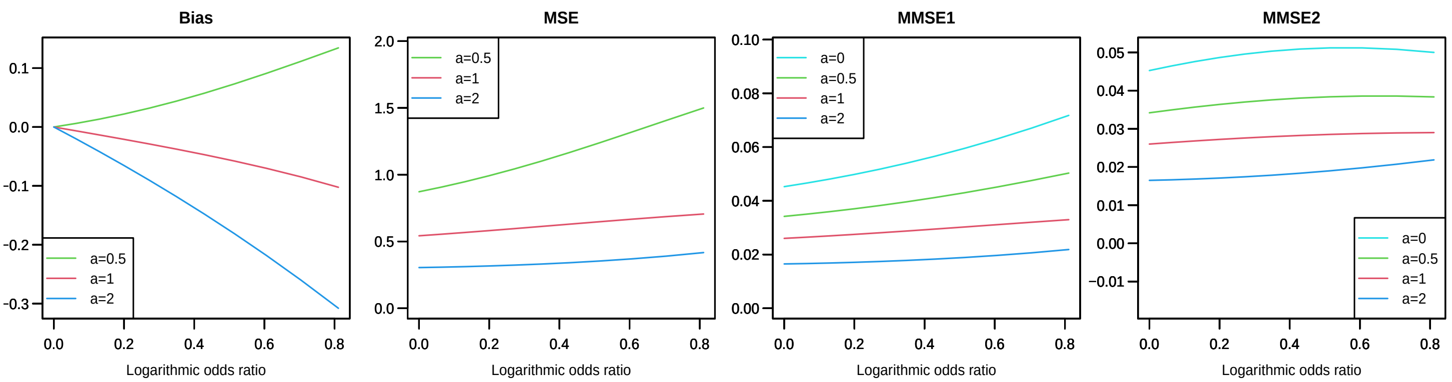


Figure S2 (continued).

$(n, m) = (20, 20),\ q = 0.7$



$(n, m) = (20, 20),\ q = 0.8$



$(n, m) = (20, 20),\ q = 0.9$

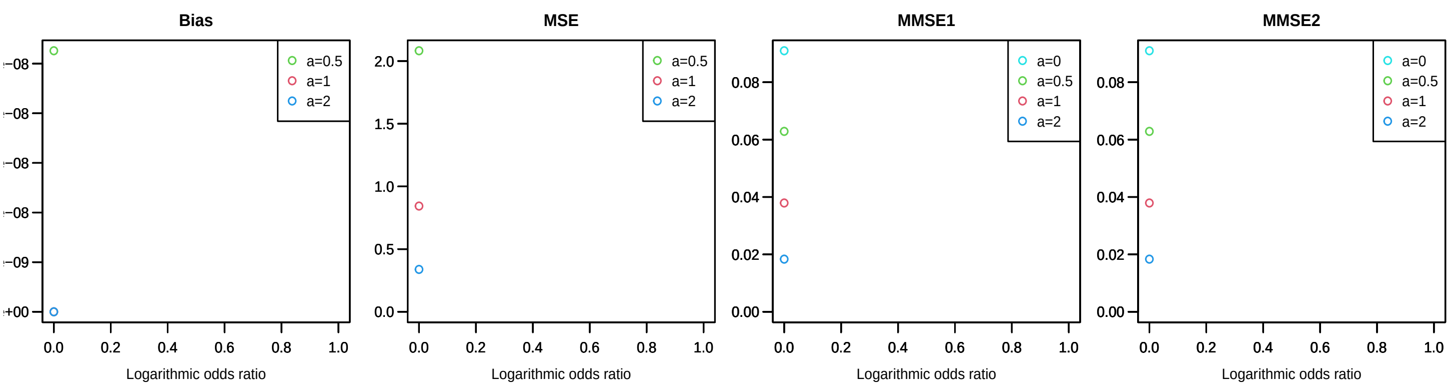


Figure S2 (continued).

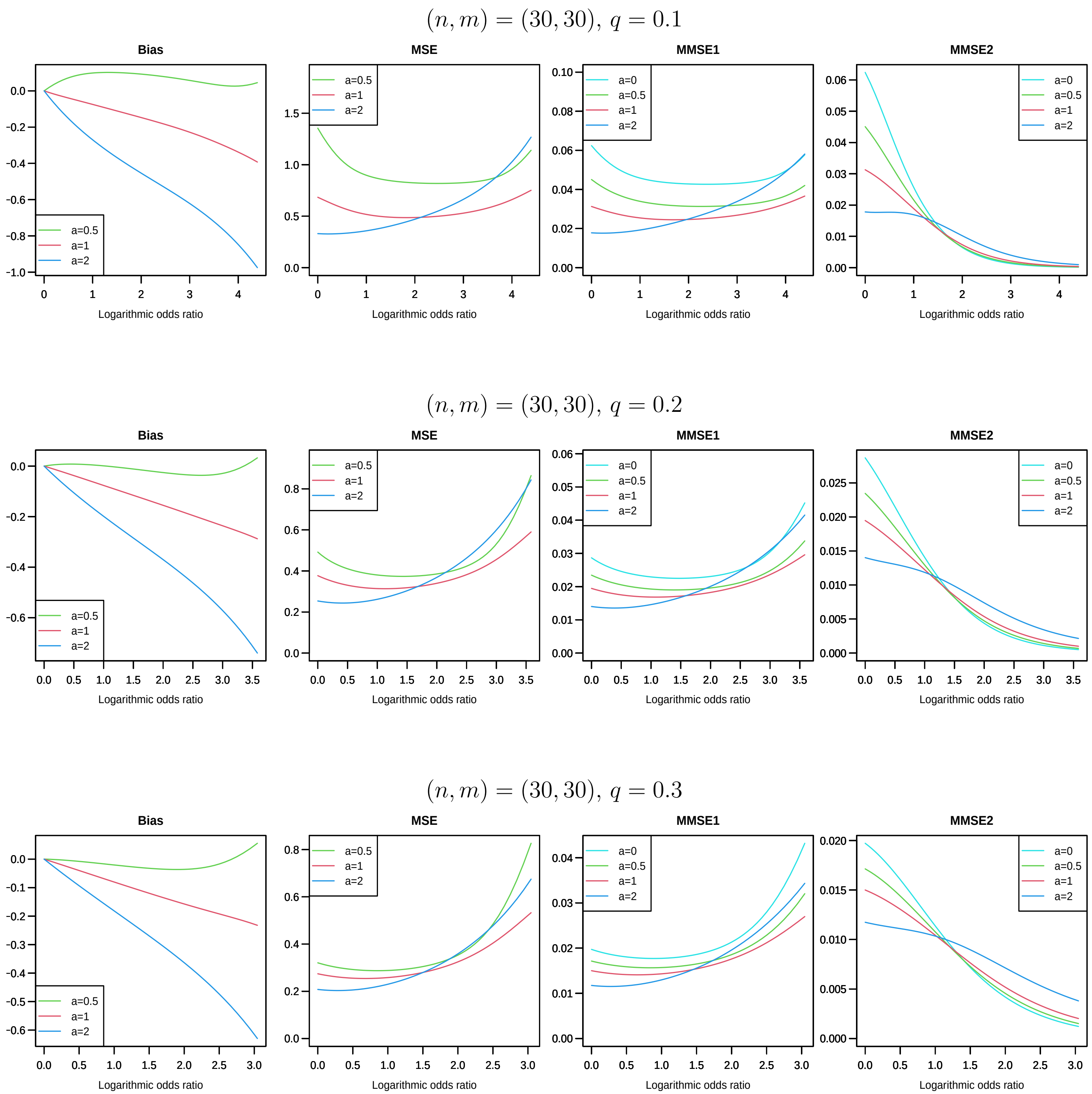
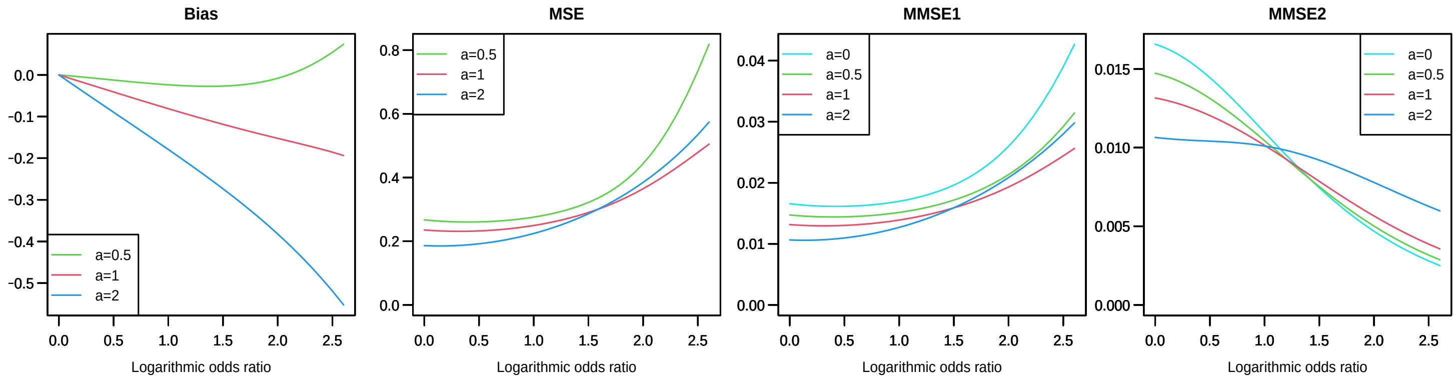
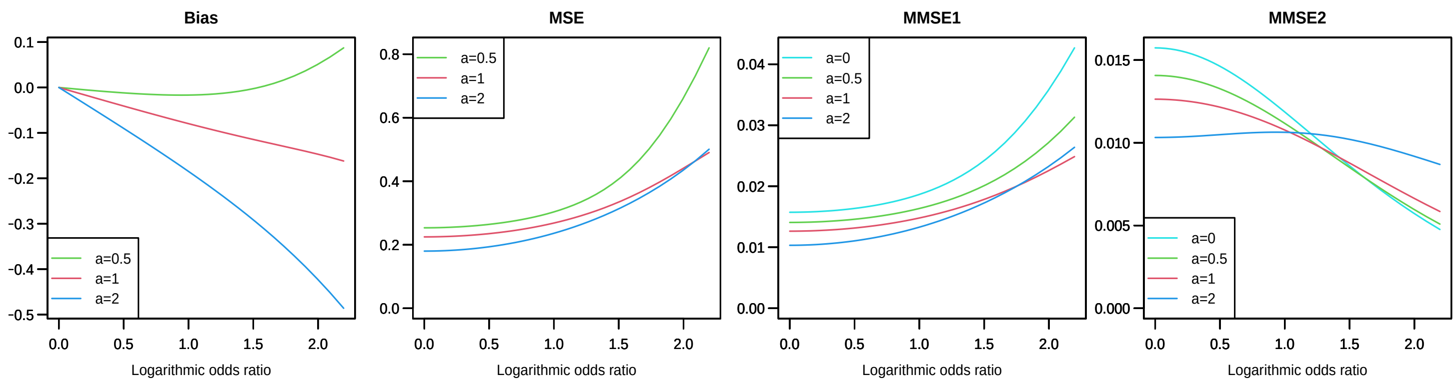


Figure S3 The choice of prior by graphical comparison of three risks and bias numerically calculated; the case of $(n, m) = (30, 30)$.

$(n, m) = (30, 30),\ q = 0.4$



$(n, m) = (30, 30),\ q = 0.5$



$(n, m) = (30, 30),\ q = 0.6$

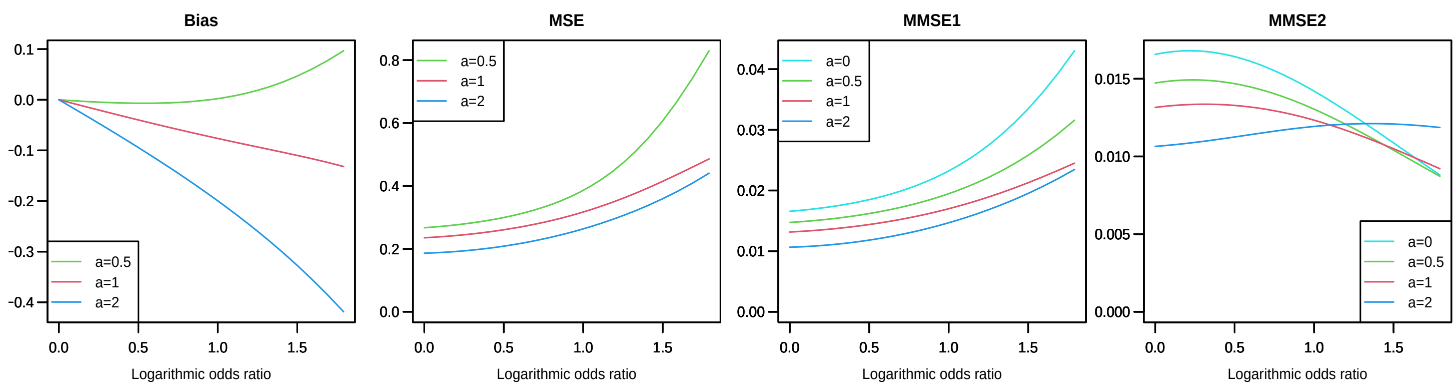
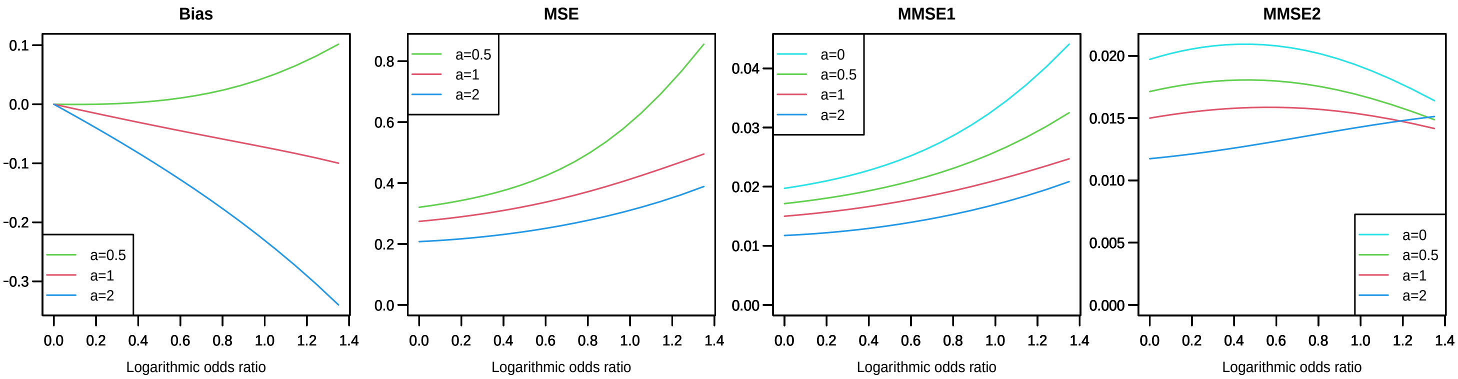
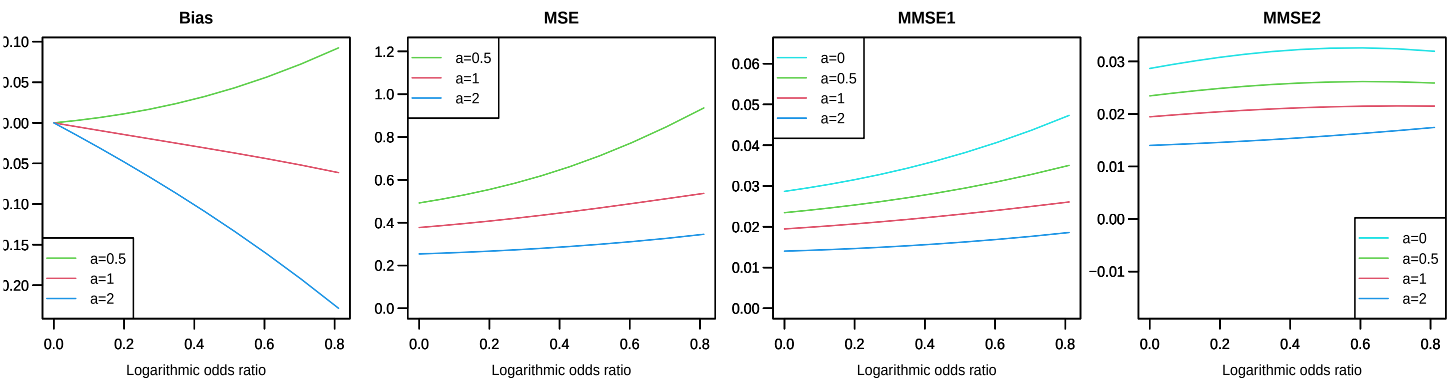


Figure S3 (continued).

$(n,m) = (30,30),\ q = 0.7$



$(n,m) = (30,30),\ q = 0.8$



$(n,m) = (30,30),\ q = 0.9$

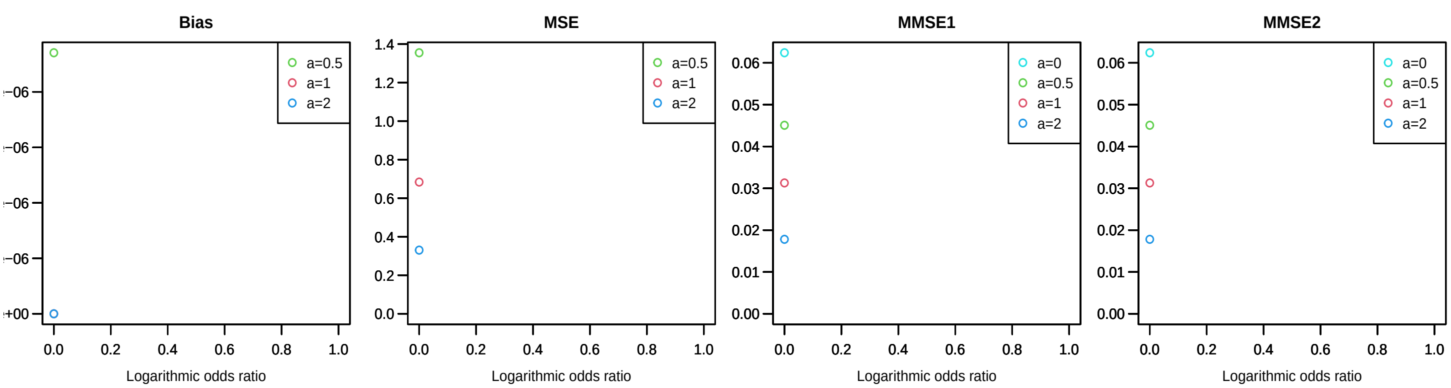


Figure S3 (continued).

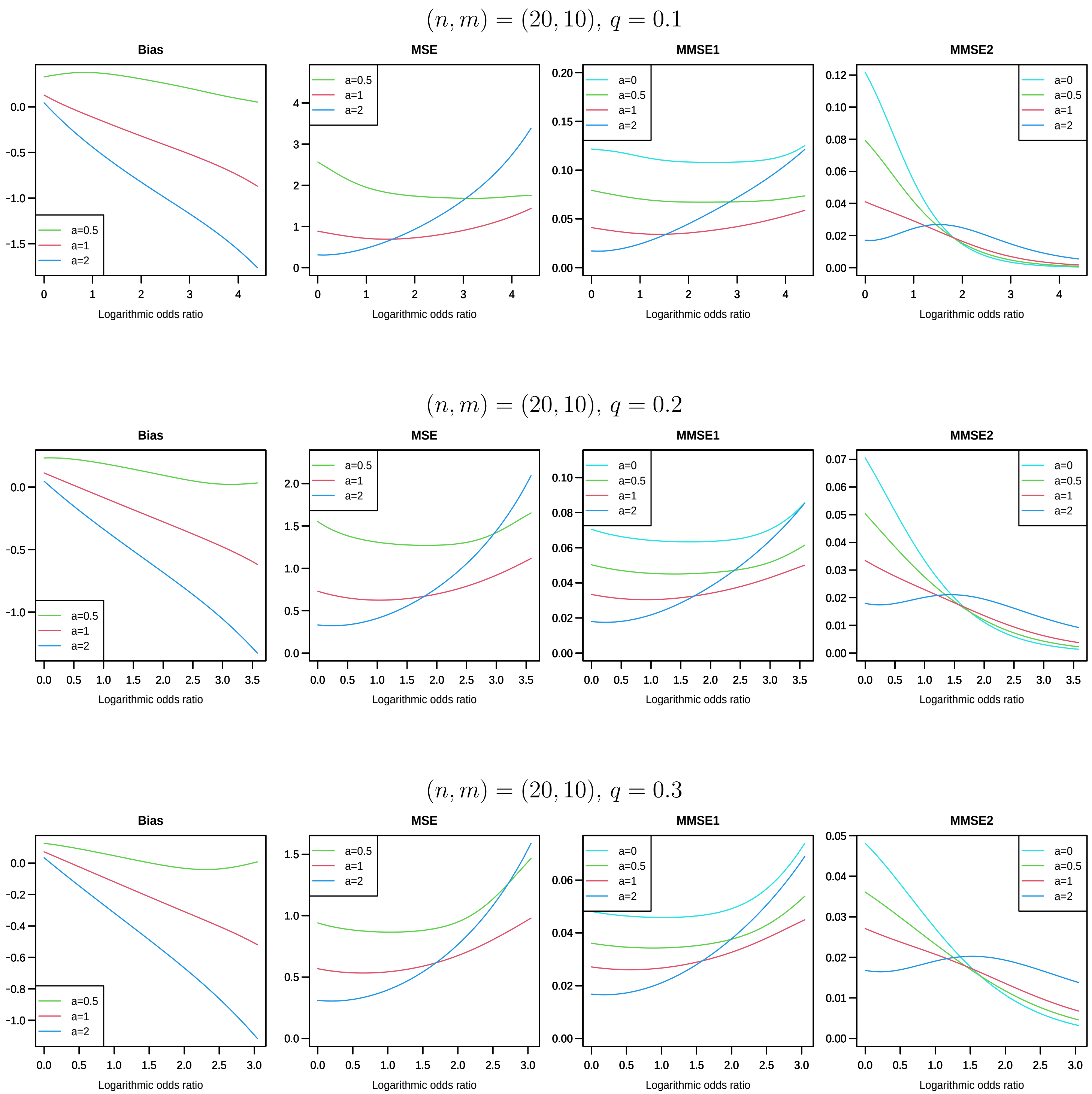
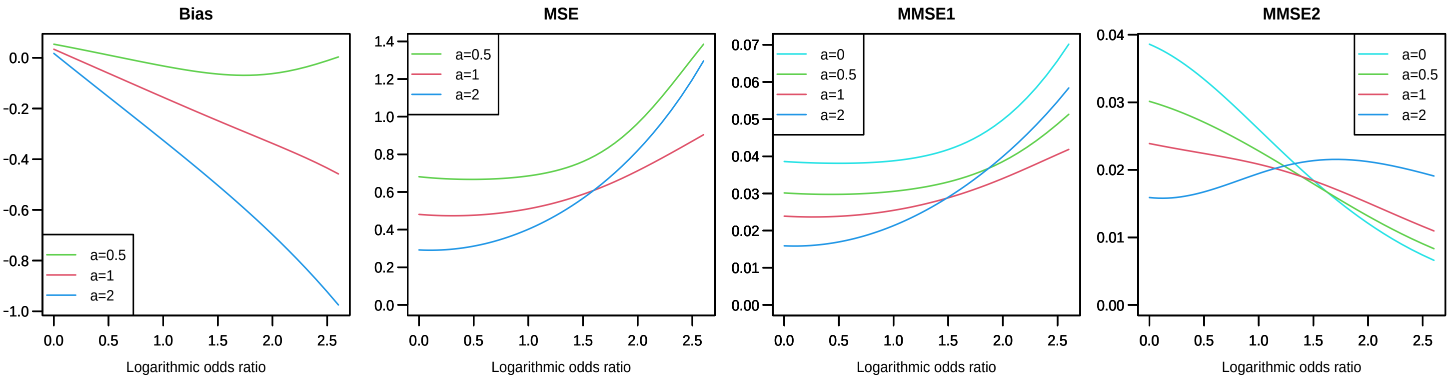
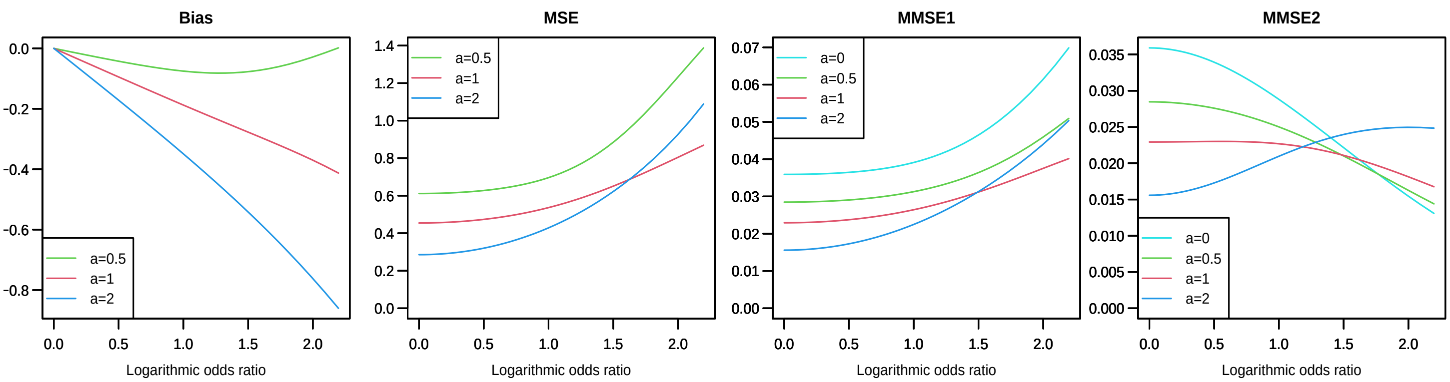


Figure S4 The choice of prior by graphical comparison of three risks and bias numerically calculated; the case of $(n, m) = (20, 10)$.

$(n, m) = (20, 10),\ q = 0.4$



$(n, m) = (20, 10),\ q = 0.5$



$(n, m) = (20, 10),\ q = 0.6$

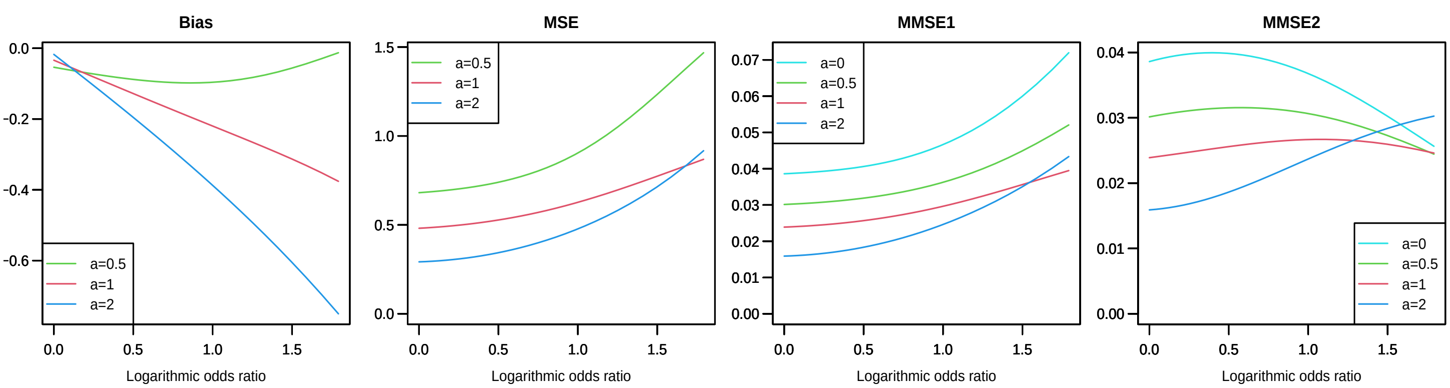
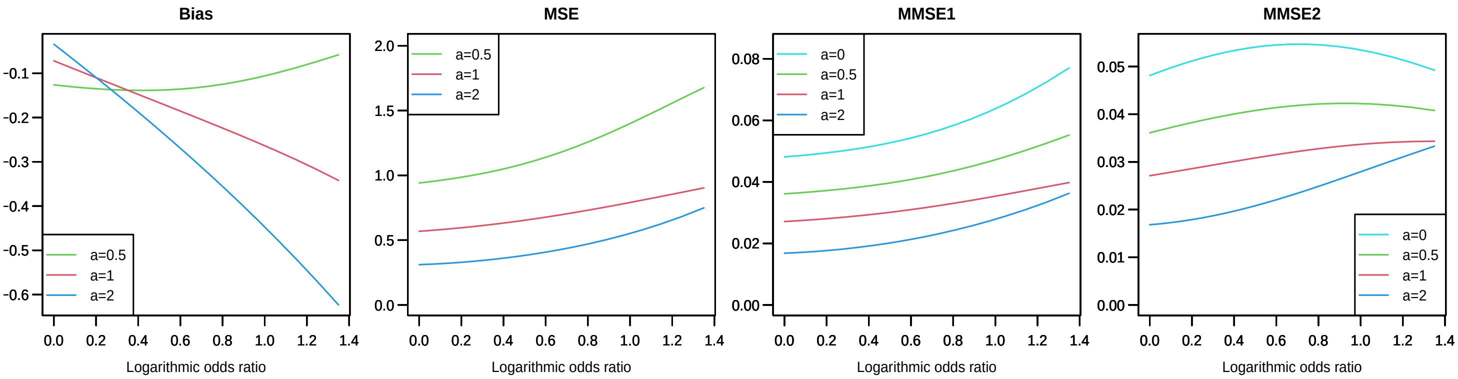
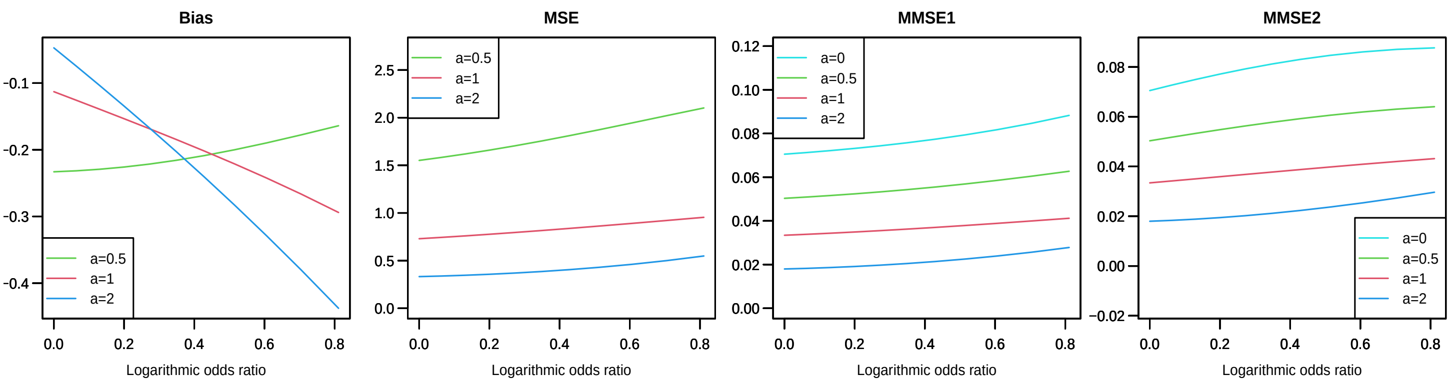


Figure S4 (continued).

$(n, m) = (20, 10),\ q = 0.7$



$(n, m) = (20, 10),\ q = 0.8$



$(n, m) = (20, 10),\ q = 0.9$

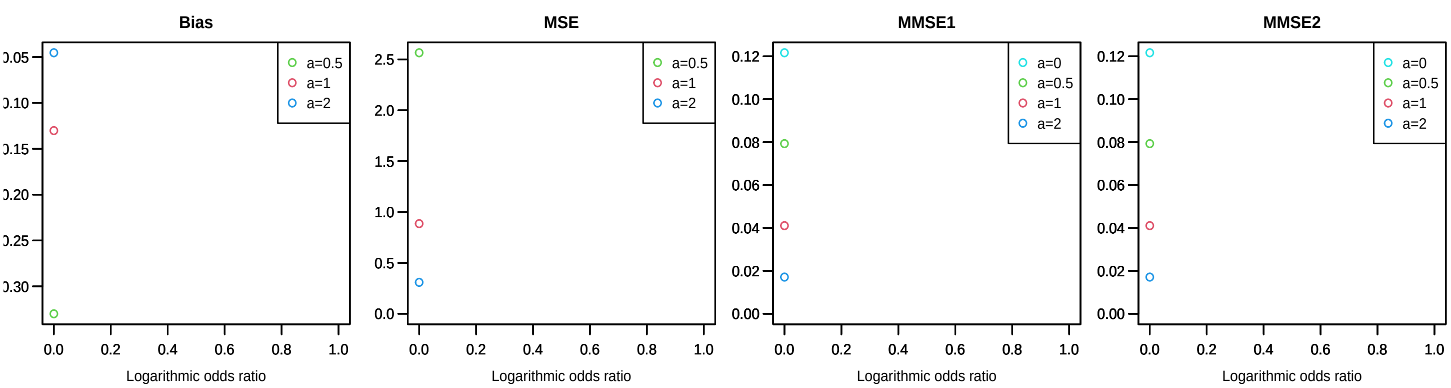
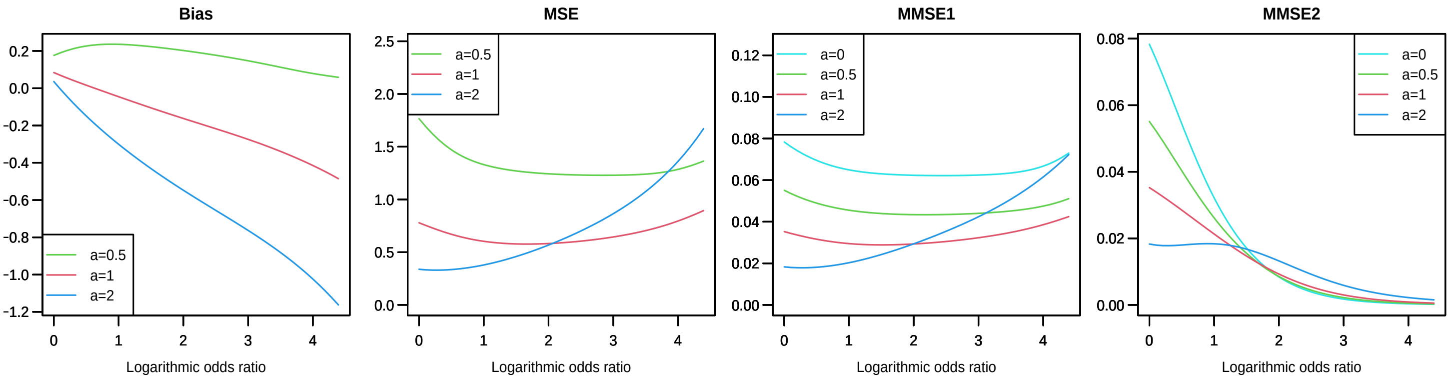
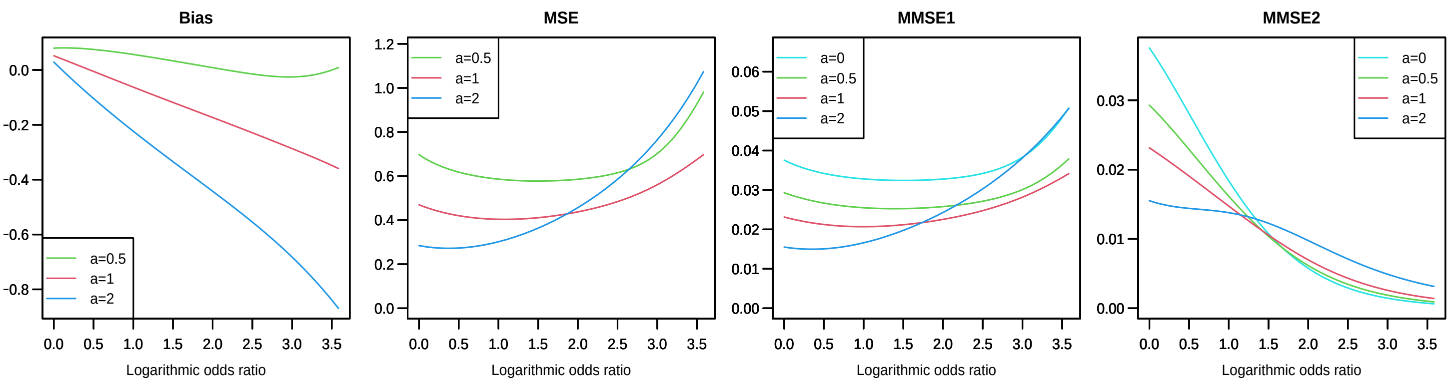


Figure S4 (continued).

$(n, m) = (30, 20),\ q = 0.1$



$(n, m) = (30, 20),\ q = 0.2$



$(n, m) = (30, 20),\ q = 0.3$

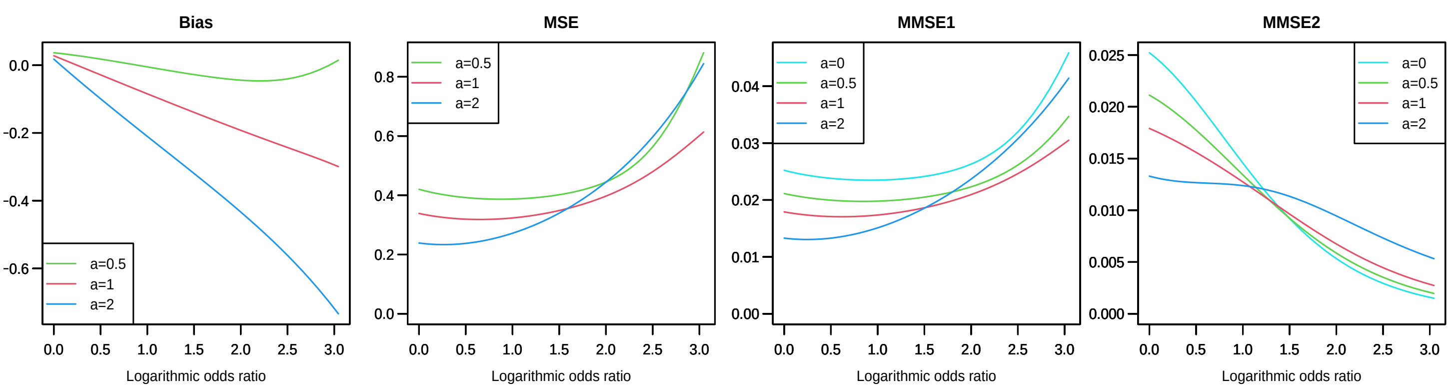
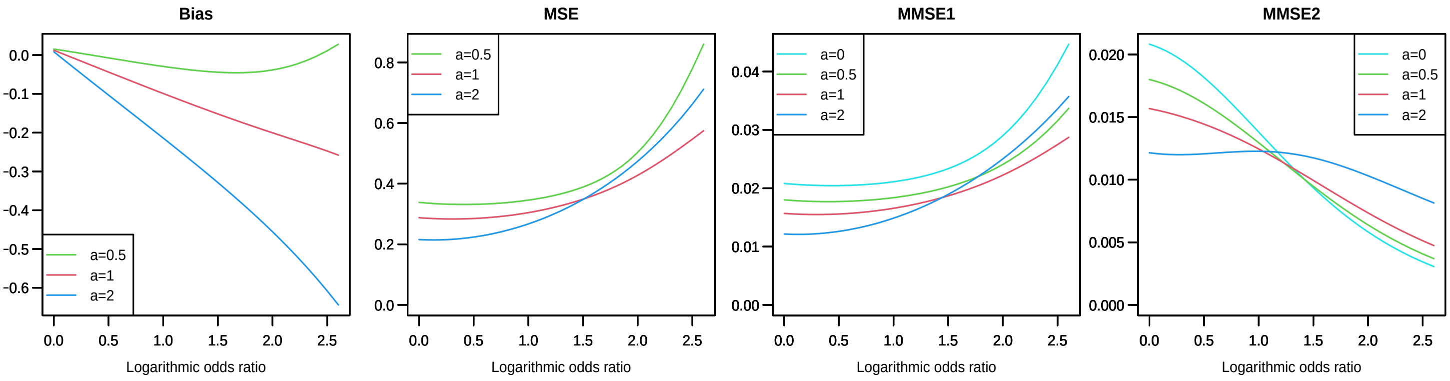
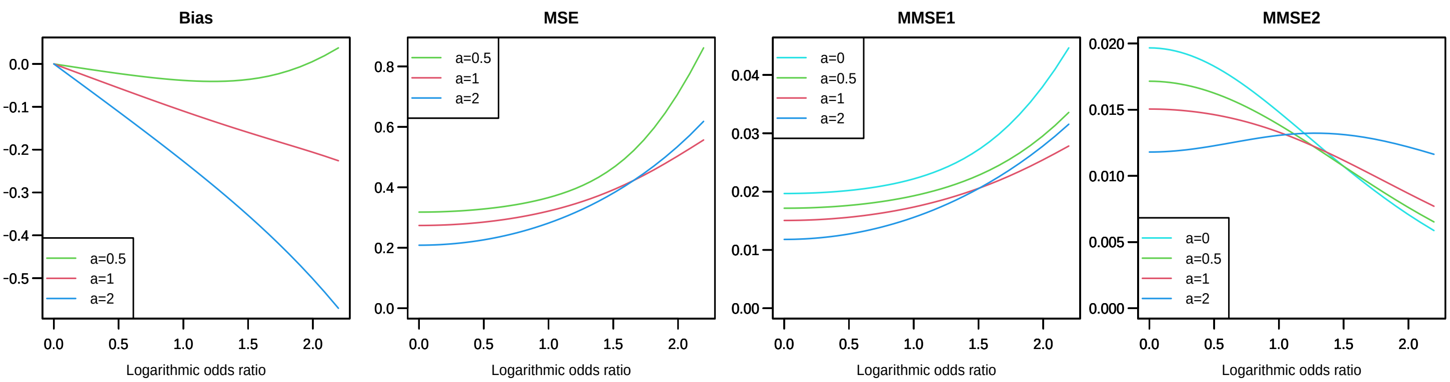


Figure S5 The choice of prior by graphical comparison of three risks and bias numerically calculated; the case of $(n, m) = (30, 20)$.

$(n, m) = (30, 20),\ q = 0.4$



$(n, m) = (30, 20),\ q = 0.5$



$(n, m) = (30, 20),\ q = 0.6$

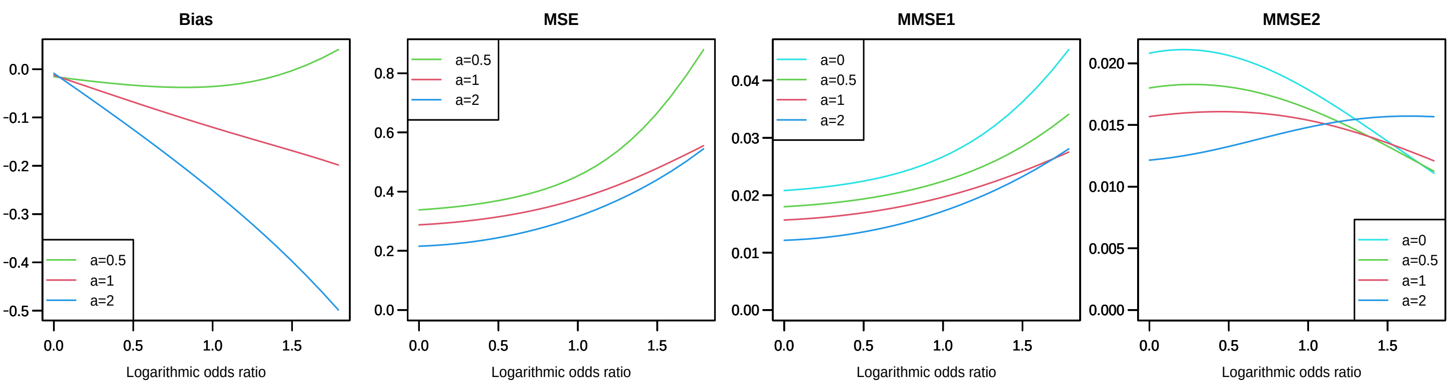
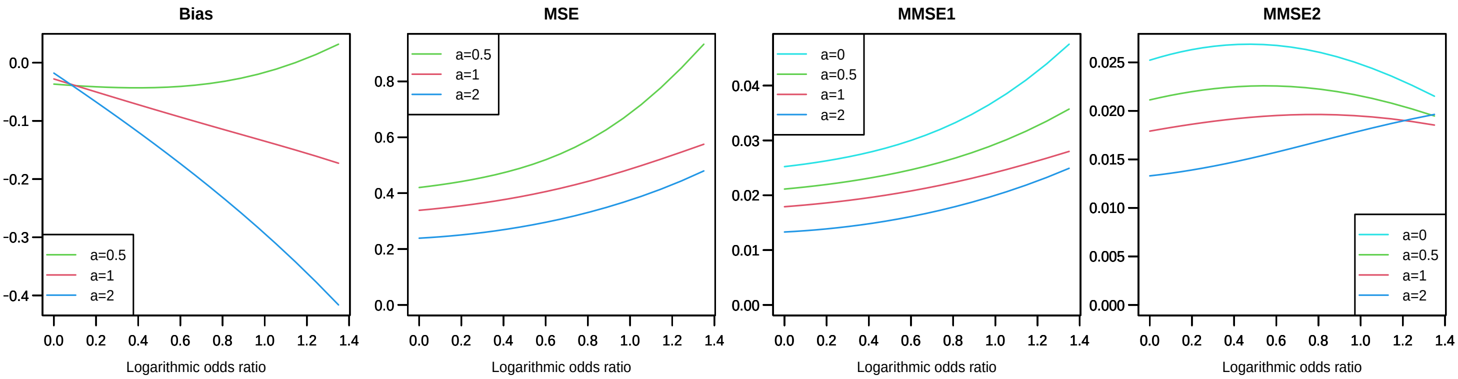
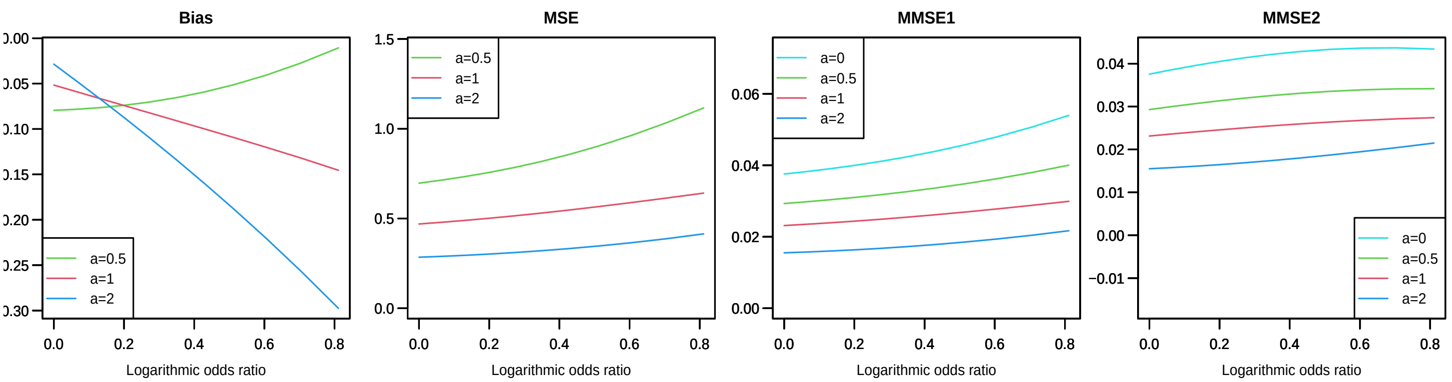


Figure S5 (continued).

$(n, m) = (30, 20),\ q = 0.7$



$(n, m) = (30, 20),\ q = 0.8$



$(n, m) = (30, 20),\ q = 0.9$

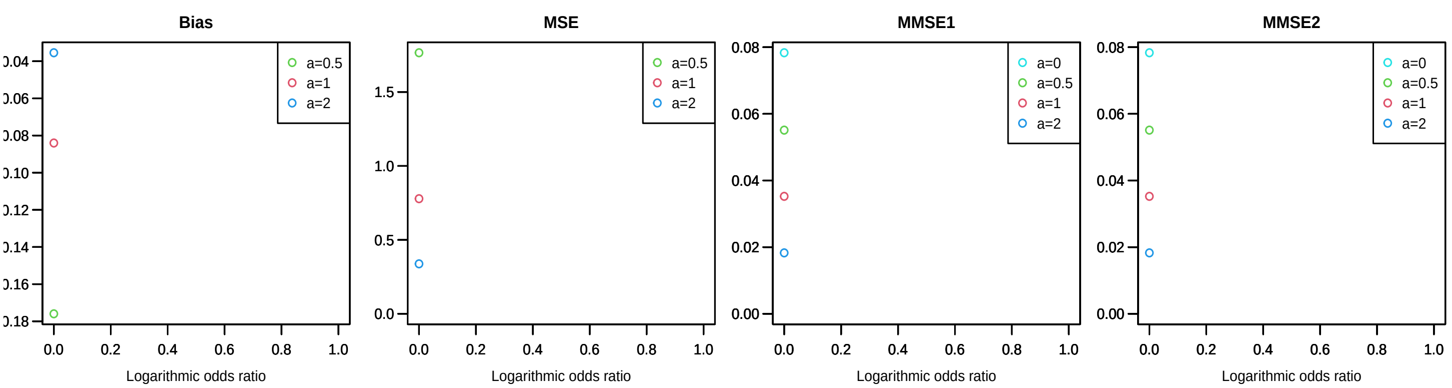


Figure S5 (continued).

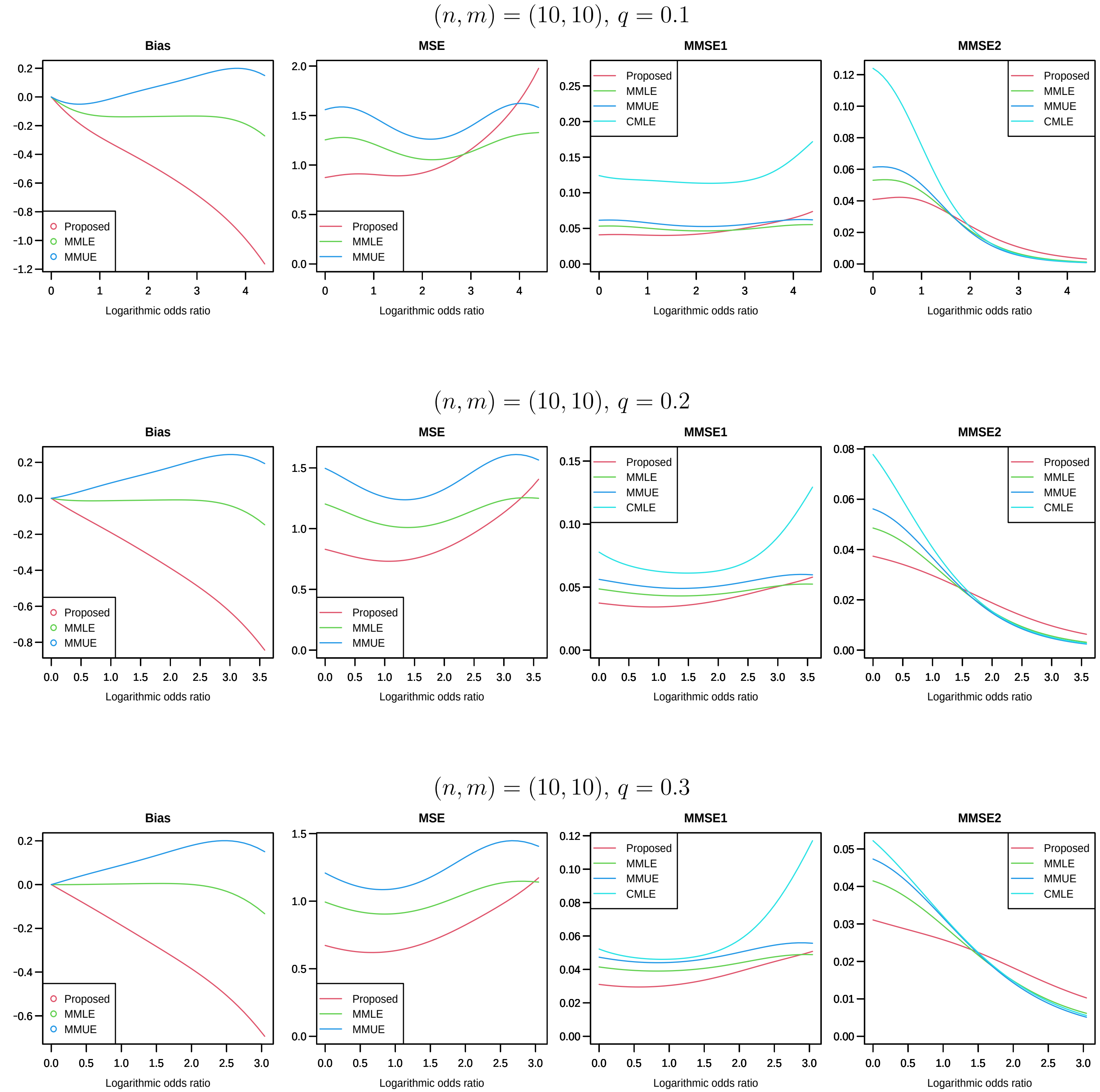
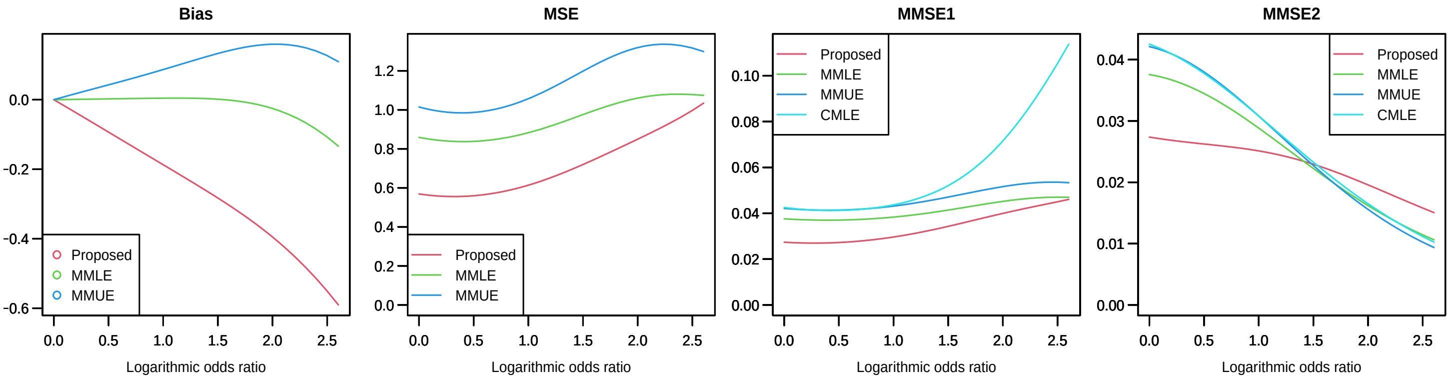
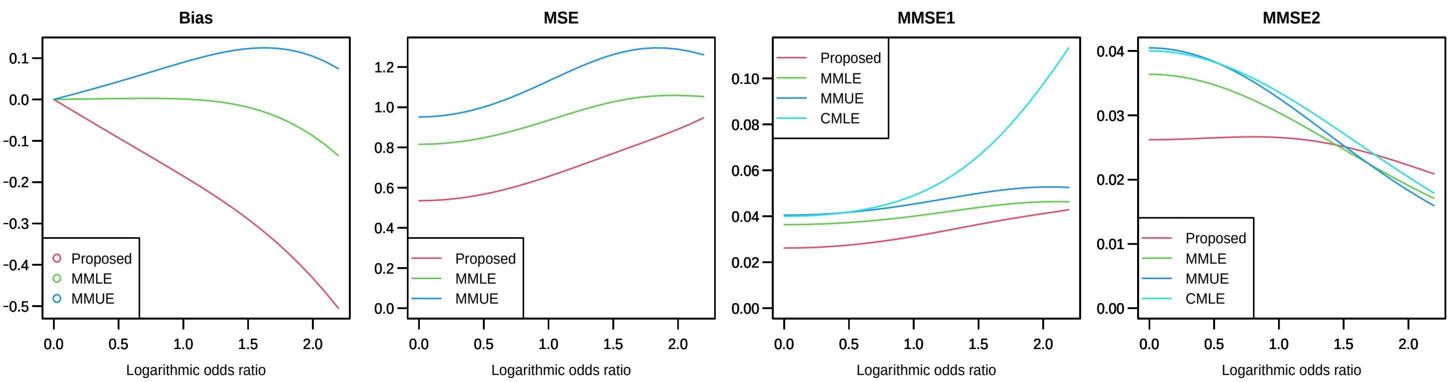


Figure S6 Graphical comparison of the three risks and bias numerically calculated from the four estimators; the case of $(n, m) = (10, 10)$.

$(n, m) = (10, 10),\ q = 0.4$



$(n, m) = (10, 10),\ q = 0.5$



$(n, m) = (10, 10),\ q = 0.6$

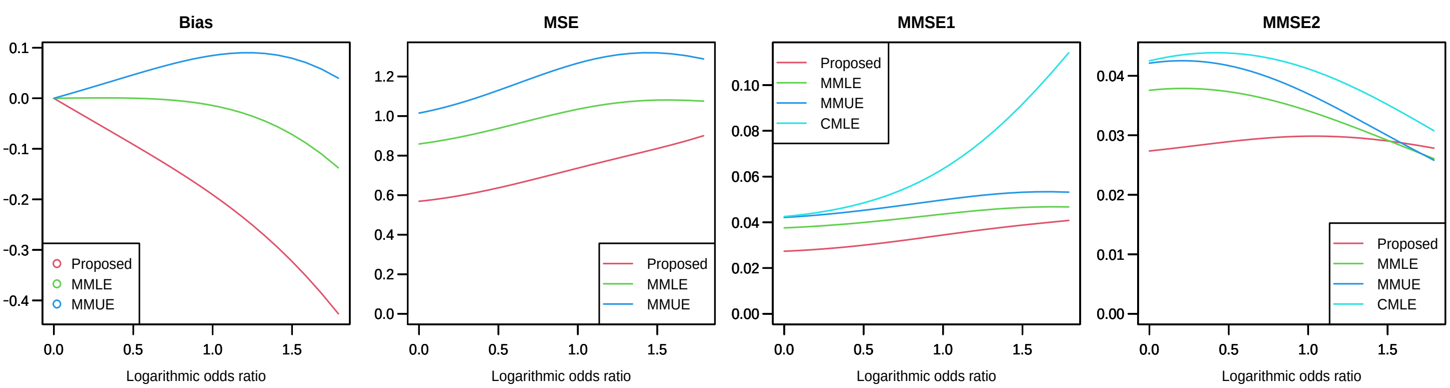
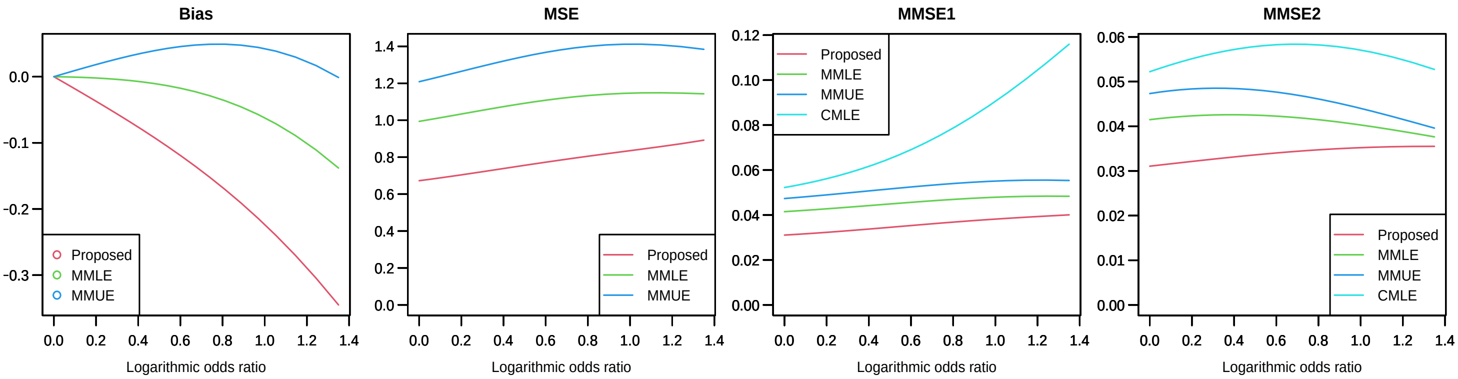
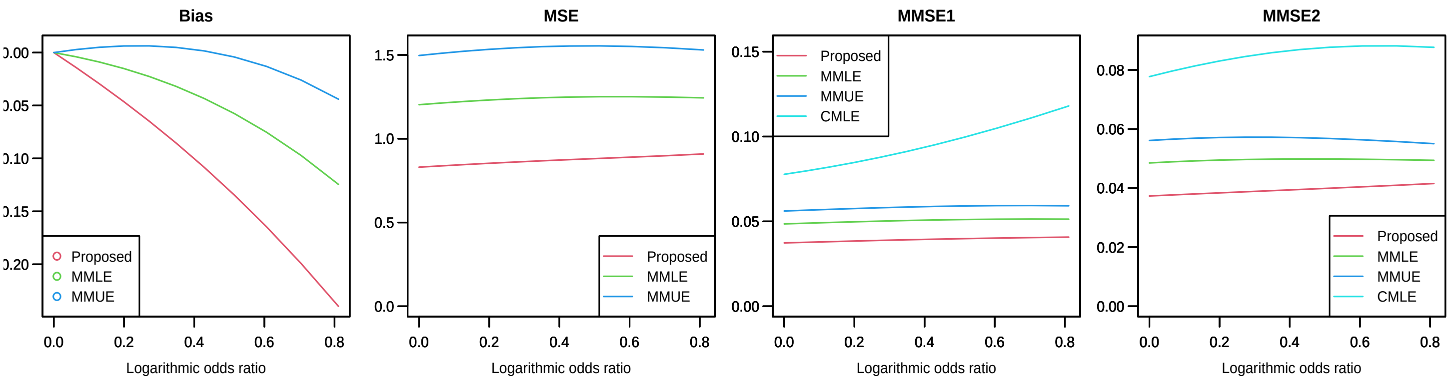


Figure S6 (continued).

$$(n, m) = (10, 10),\ q = 0.7$$



$$(n, m) = (10, 10),\ q = 0.8$$



$$(n, m) = (10, 10),\ q = 0.9$$

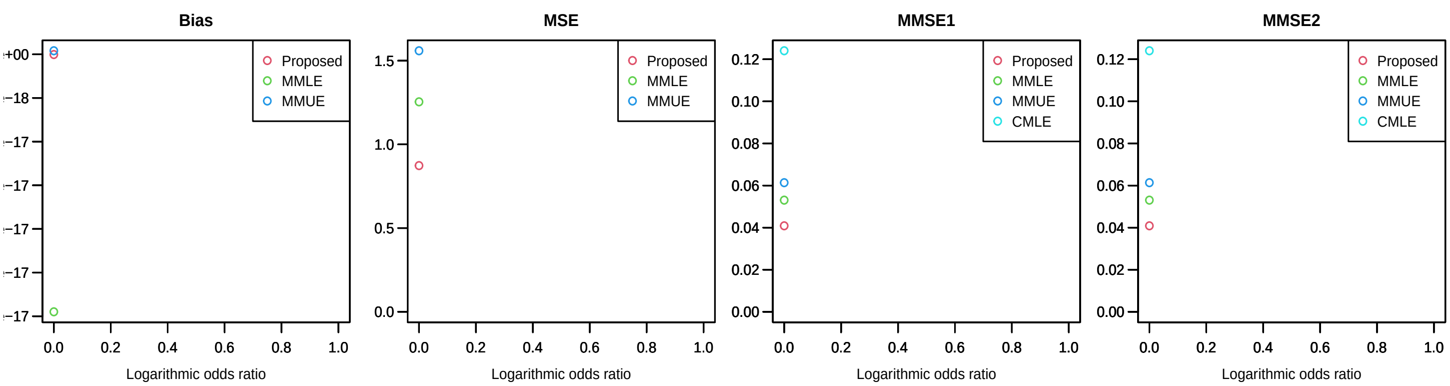


Figure S6 (continued).

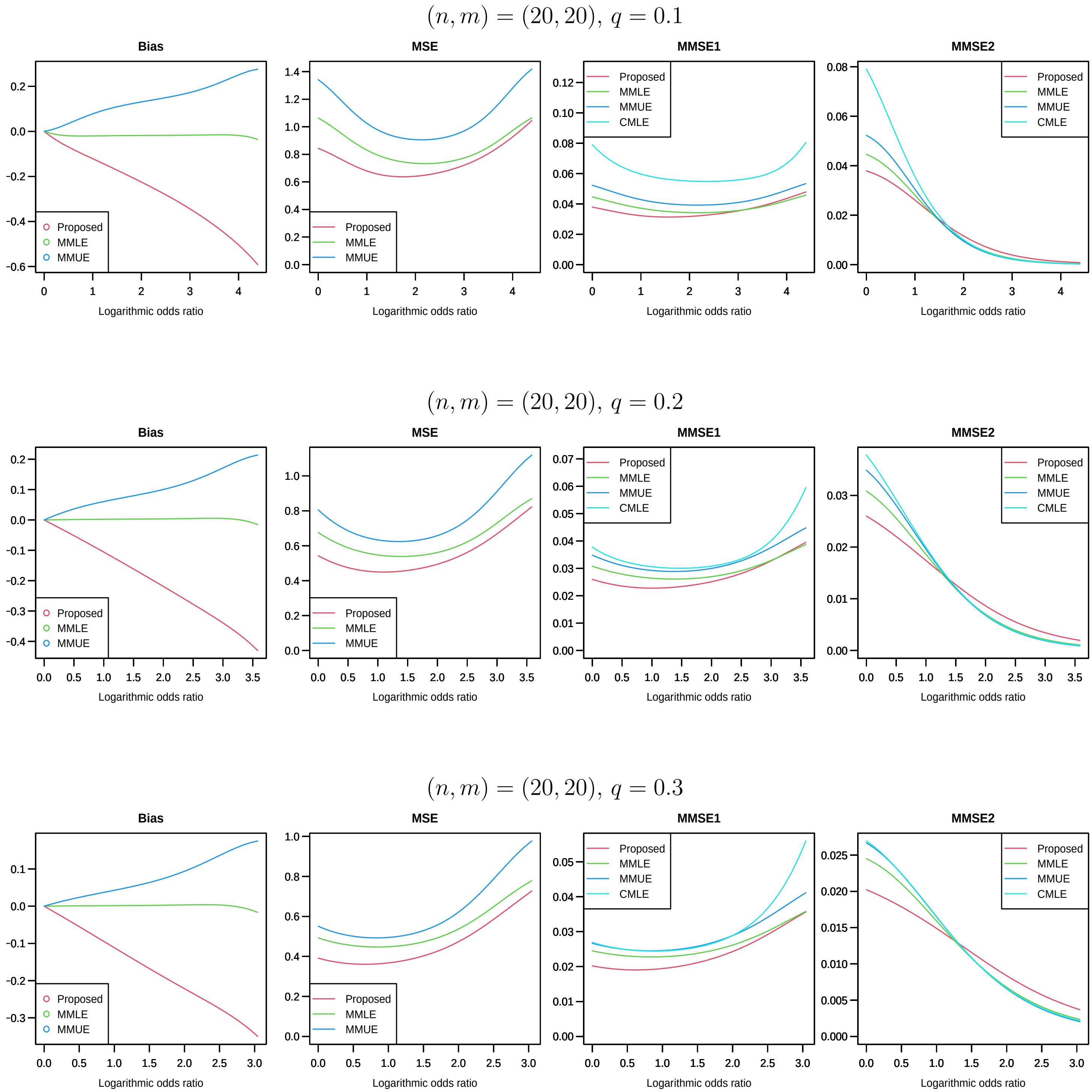


Figure S7 Graphical comparison of the three risks and bias numerically calculated from the four estimators; the case of $(n, m) = (20, 20)$.

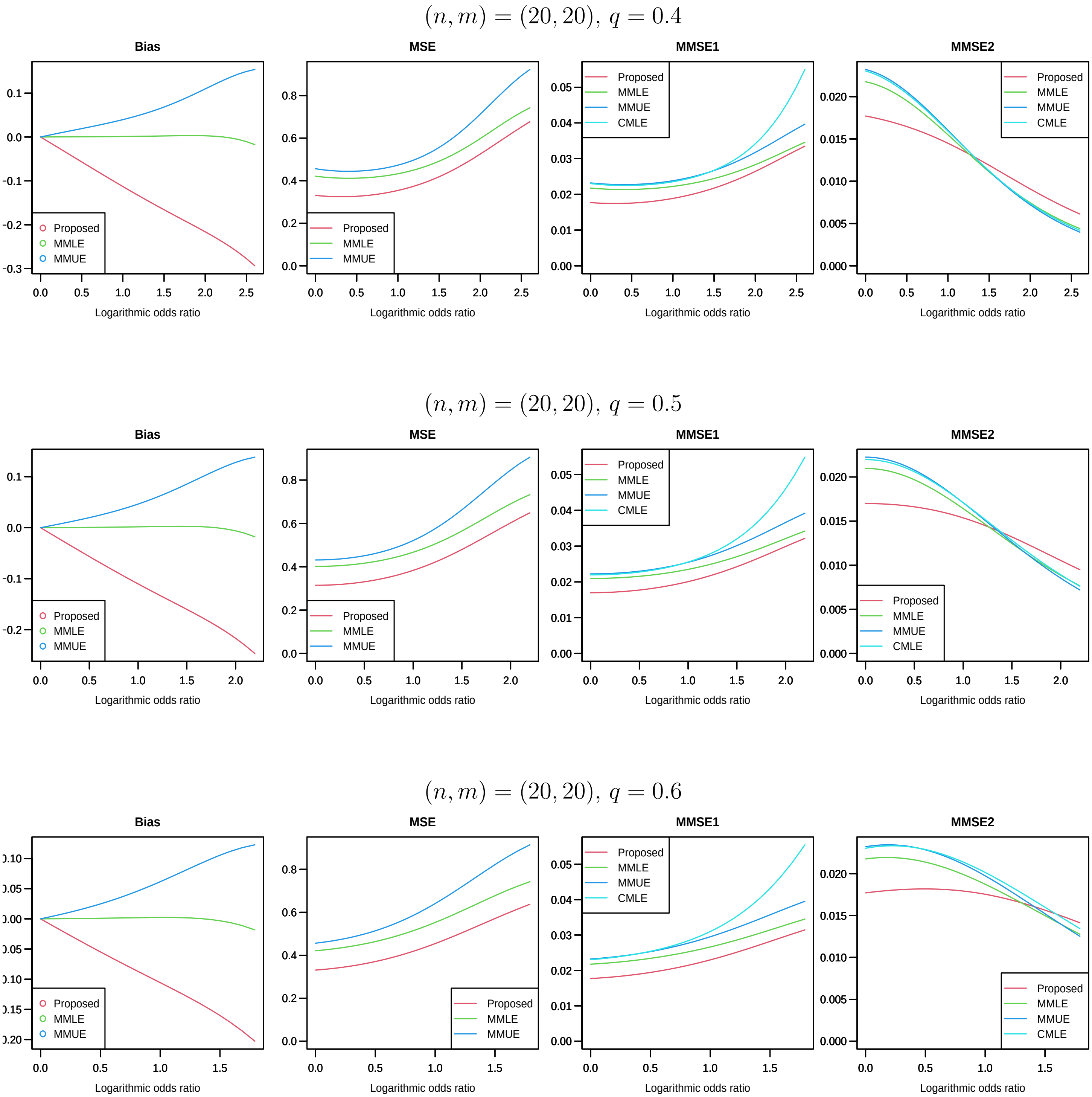


Figure S7 (continued).

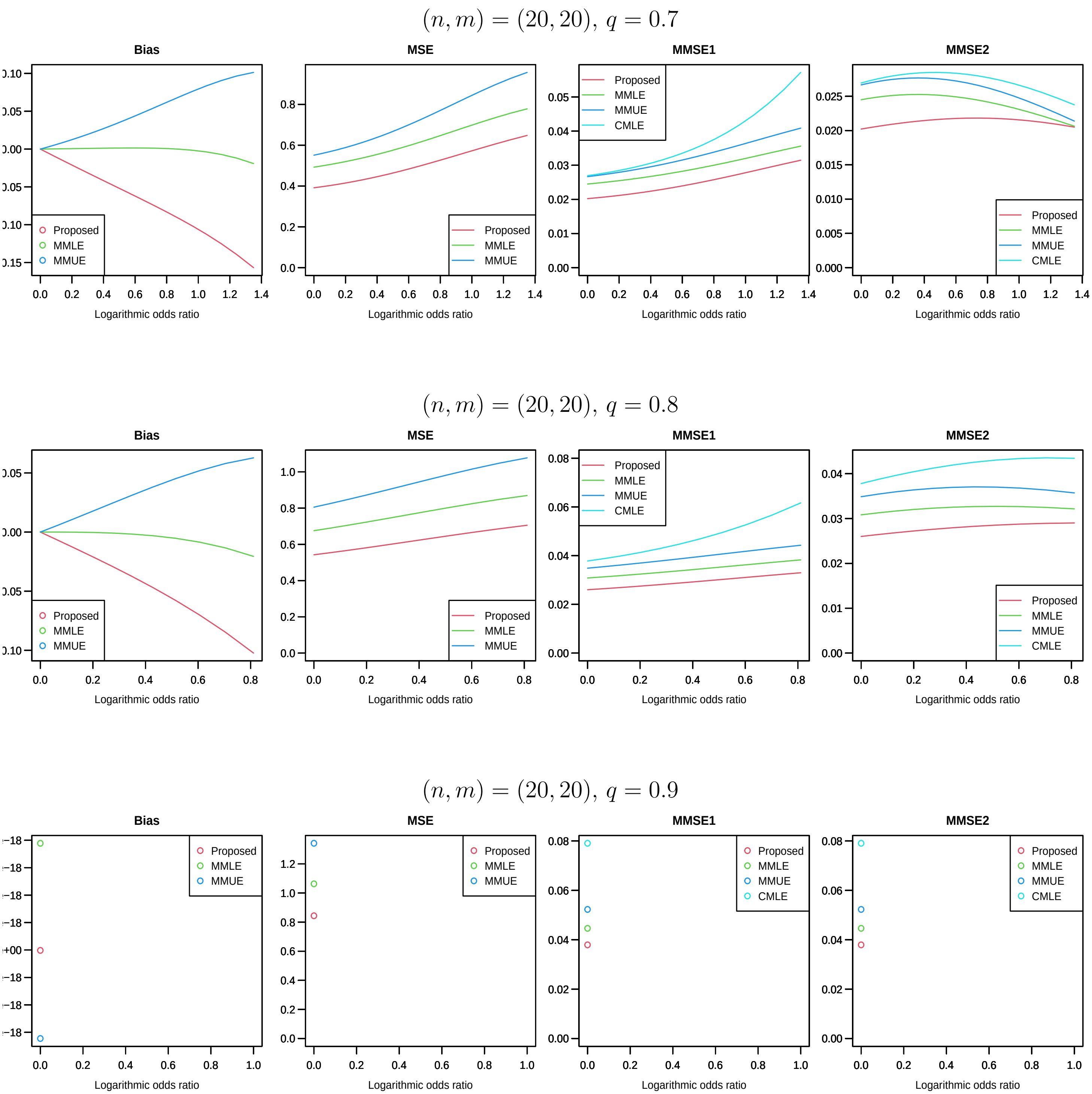


Figure S7 (continued).

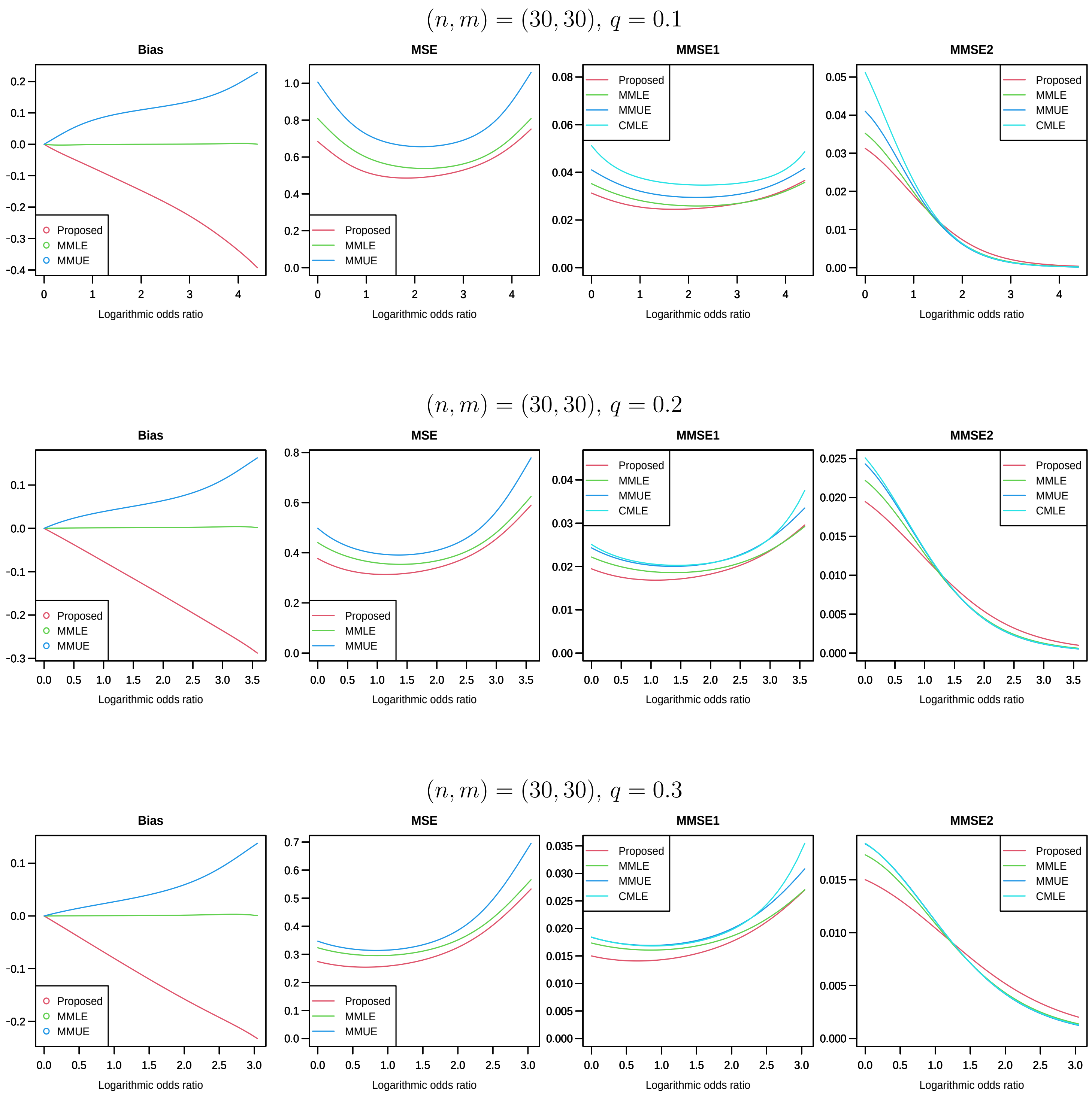
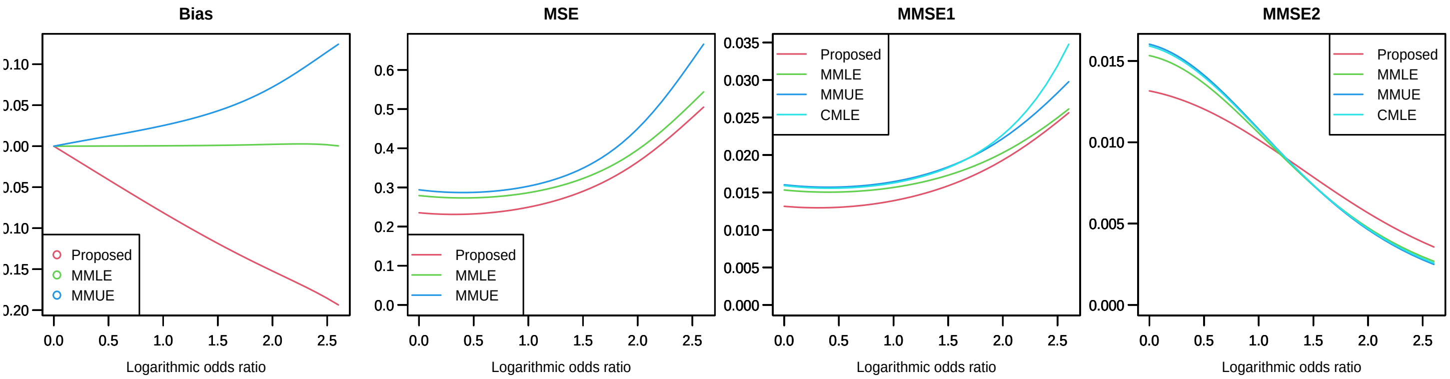
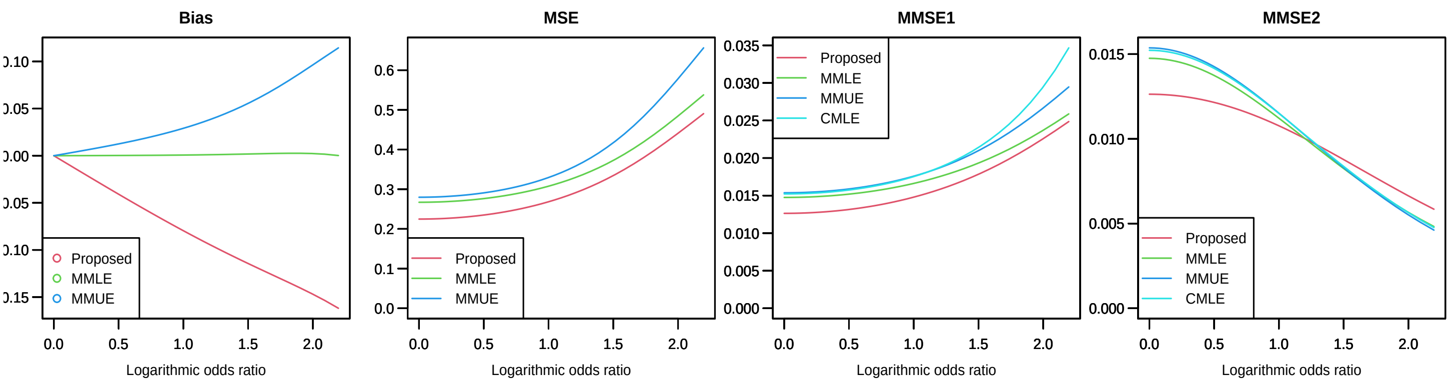


Figure S8 Graphical comparison of the three risks and bias numerically calculated from the four estimators; the case of $(n, m) = (30, 30)$.

$(n, m) = (30, 30),\ q = 0.4$



$(n, m) = (30, 30),\ q = 0.5$



$(n, m) = (30, 30),\ q = 0.6$

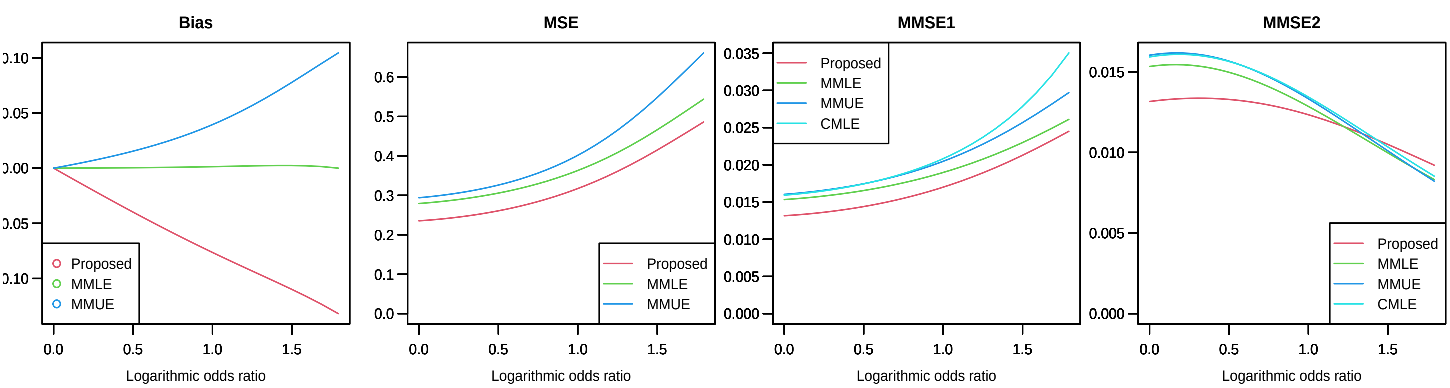
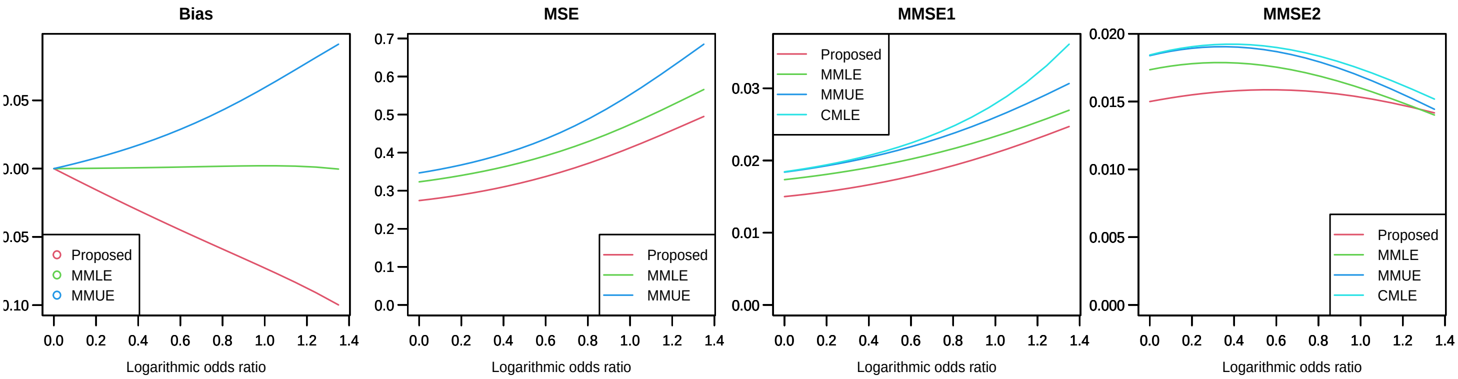
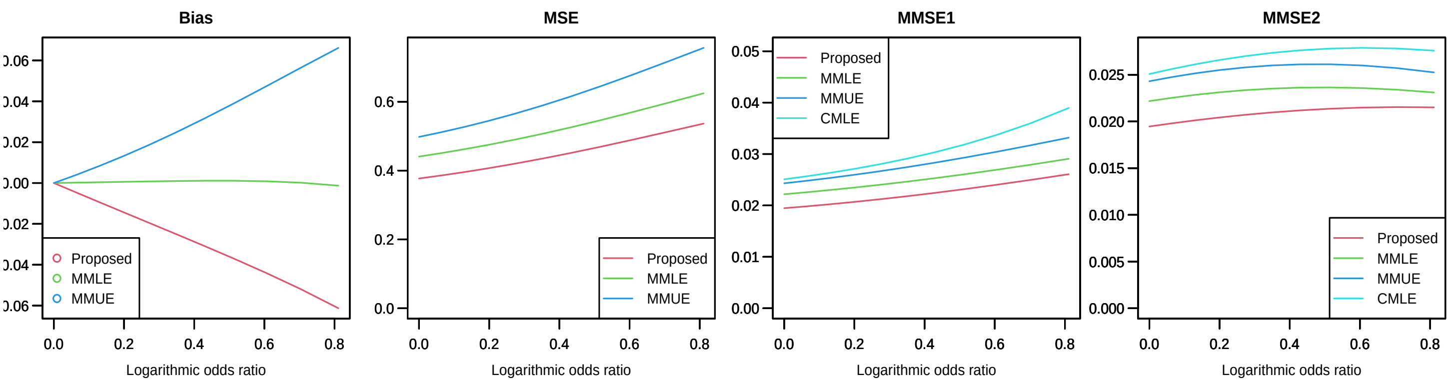


Figure S8 (continued).

$(n, m) = (30, 30),\ q = 0.7$



$(n, m) = (30, 30),\ q = 0.8$



$(n, m) = (30, 30),\ q = 0.9$

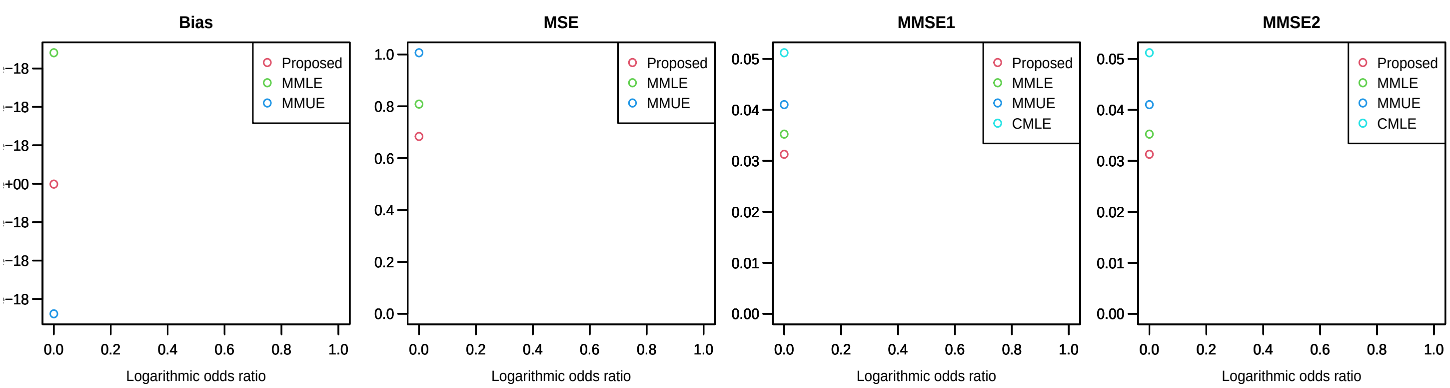


Figure S8 (continued).

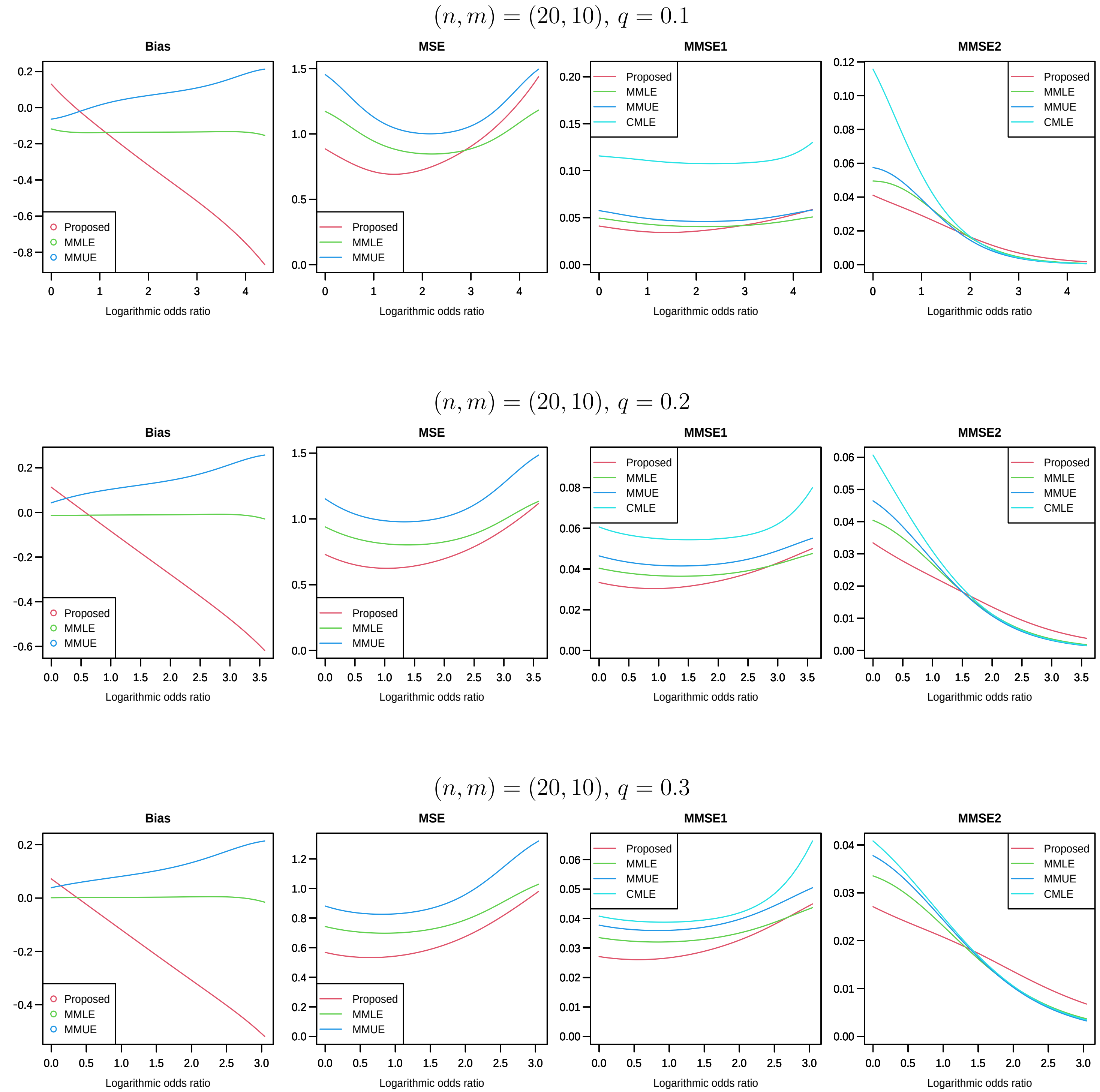
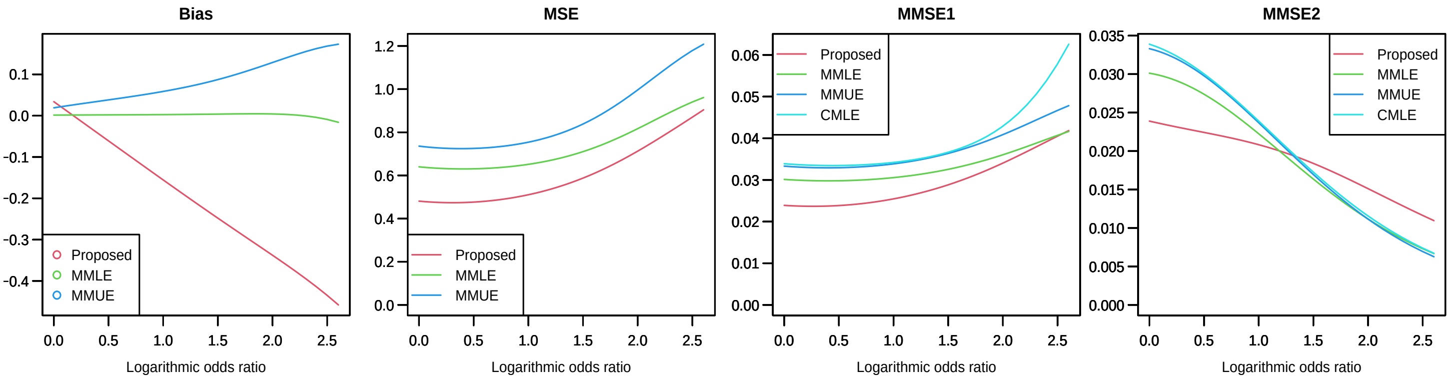
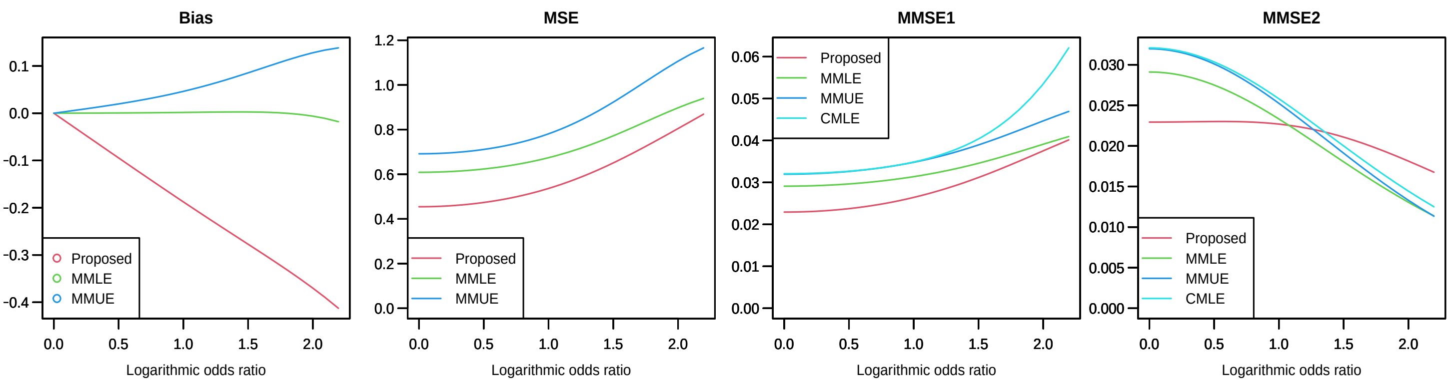


Figure S9 Graphical comparison of the three risks and bias numerically calculated from the four estimators; the case of $(n, m) = (20, 10)$.

$(n, m) = (20, 10),\ q = 0.4$



$(n, m) = (20, 10),\ q = 0.5$



$(n, m) = (20, 10),\ q = 0.6$

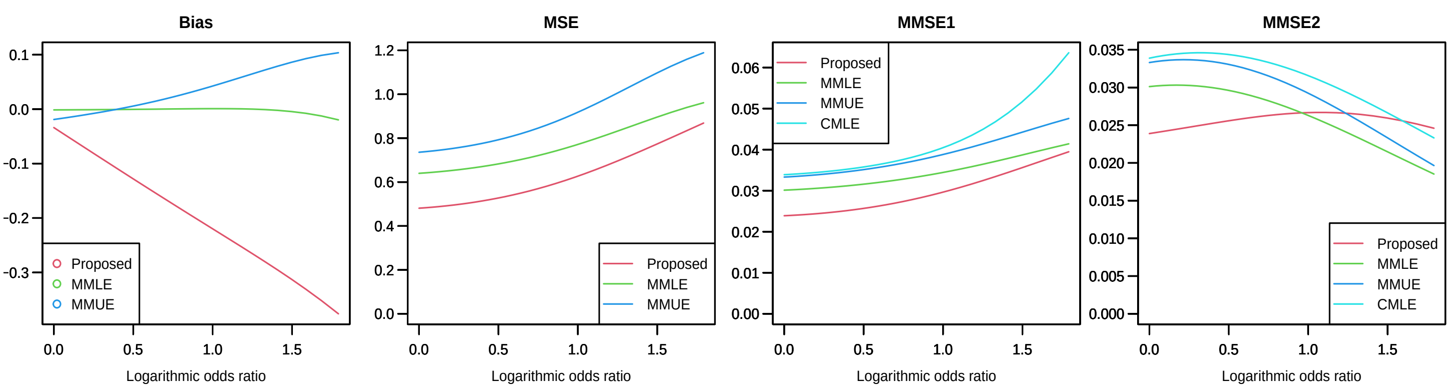
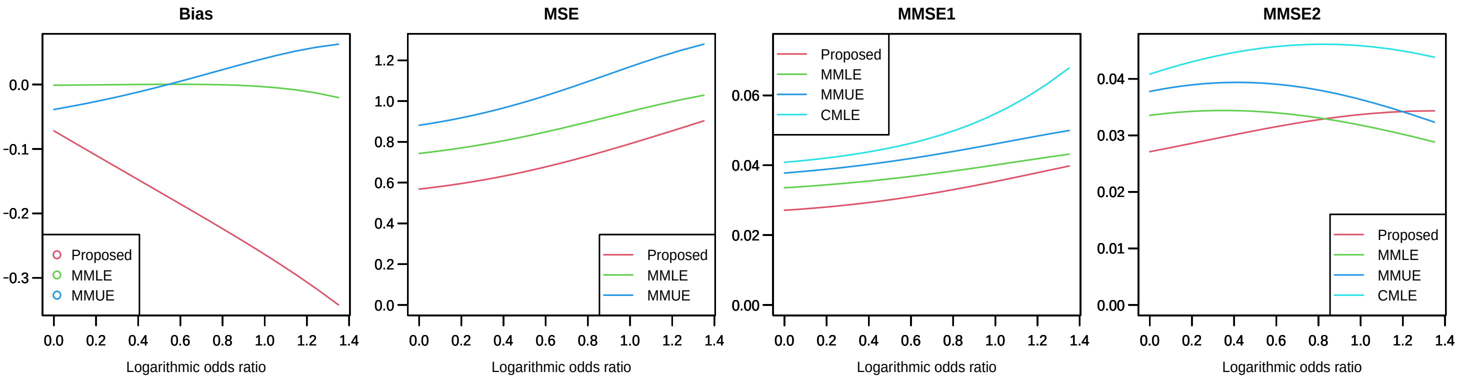
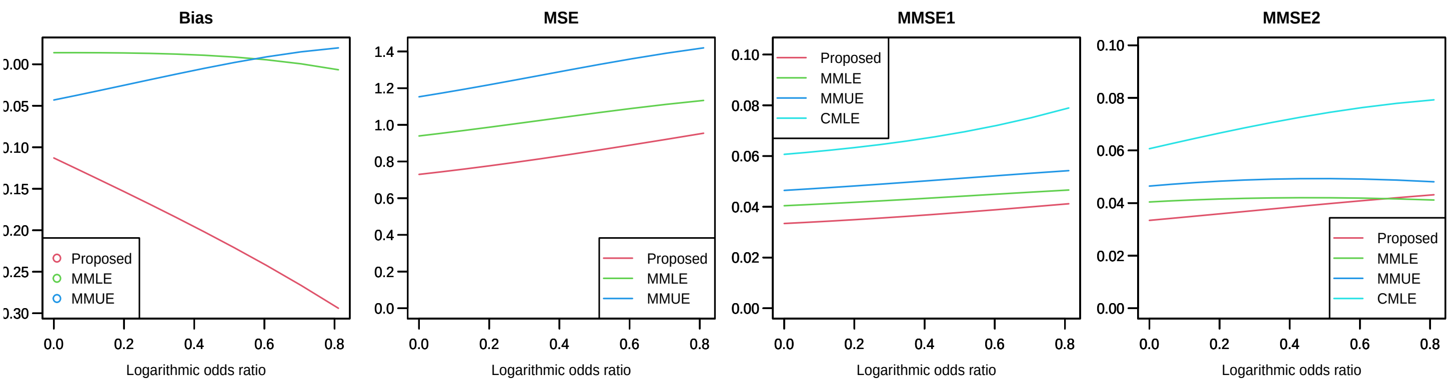


Figure S9 (continued).

$(n, m) = (20, 10),\ q = 0.7$



$(n, m) = (20, 10),\ q = 0.8$



$(n, m) = (20, 10),\ q = 0.9$

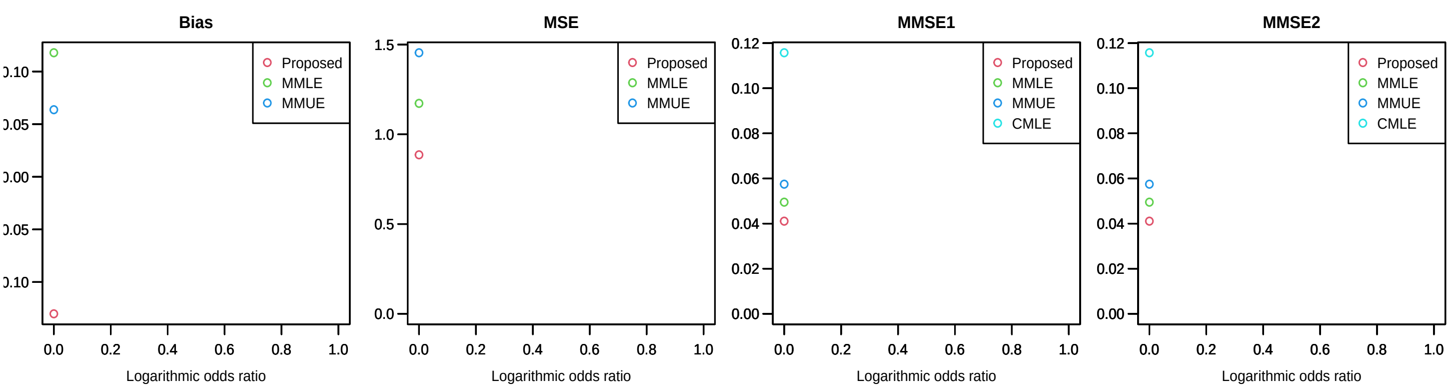


Figure S9 (continued).

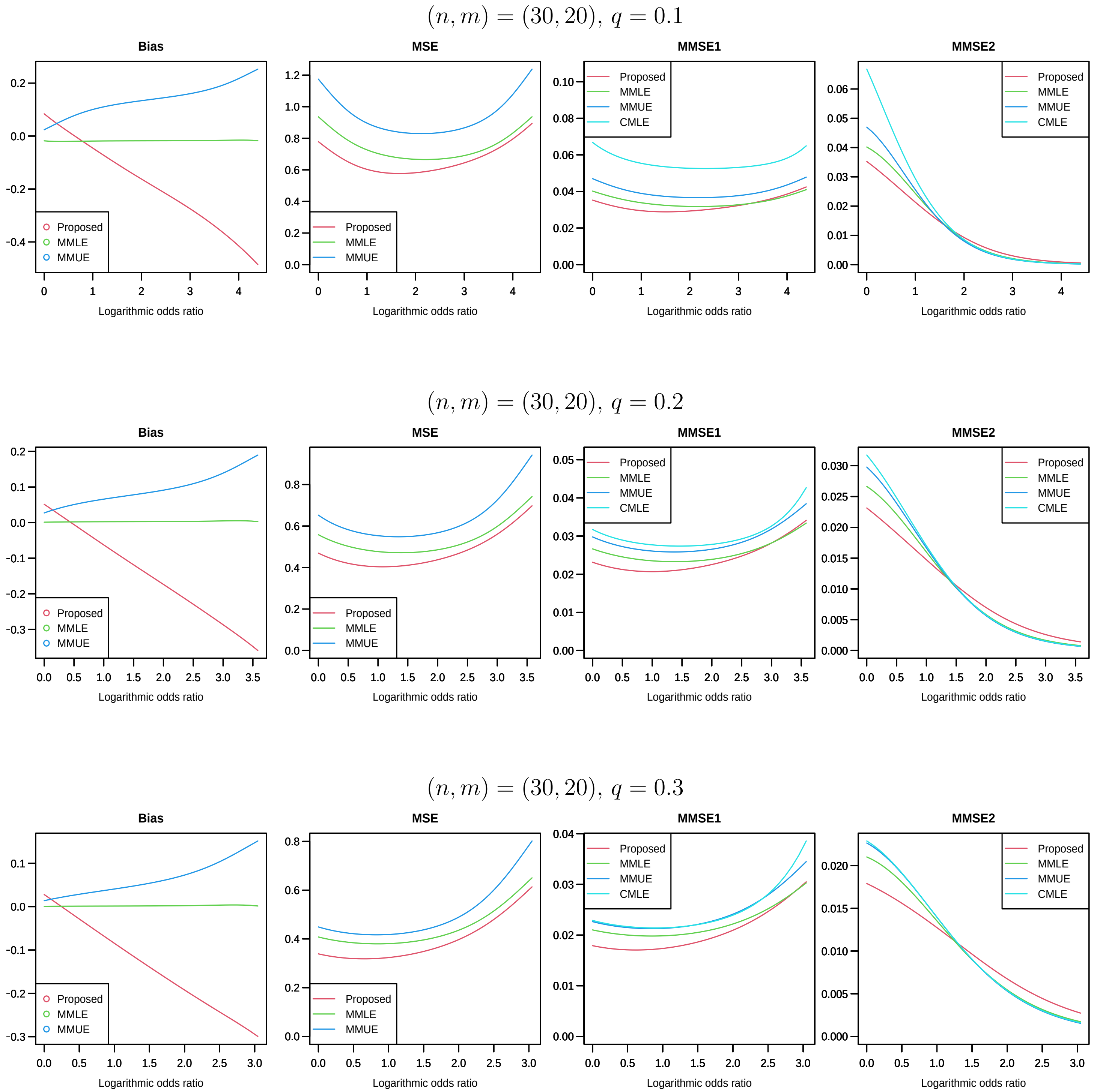


Figure S10 Graphical comparison of the three risks and bias numerically calculated from the four estimators; the case of $(n, m) = (30, 20)$.

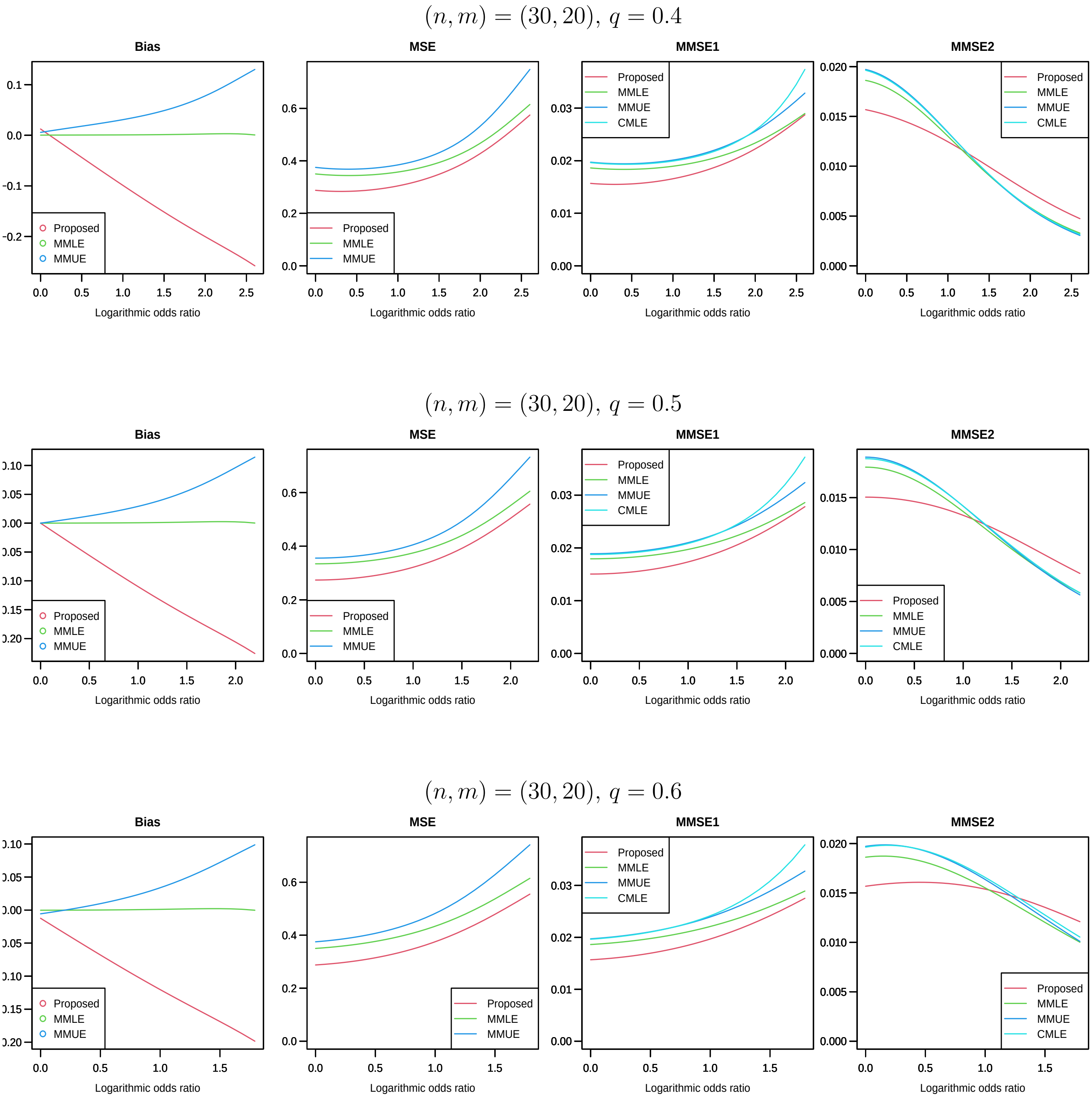
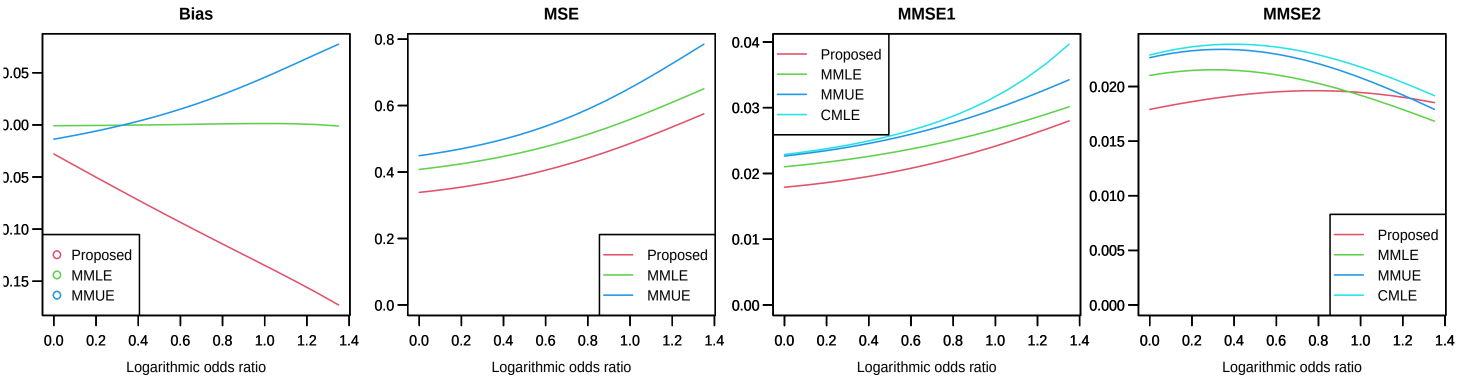
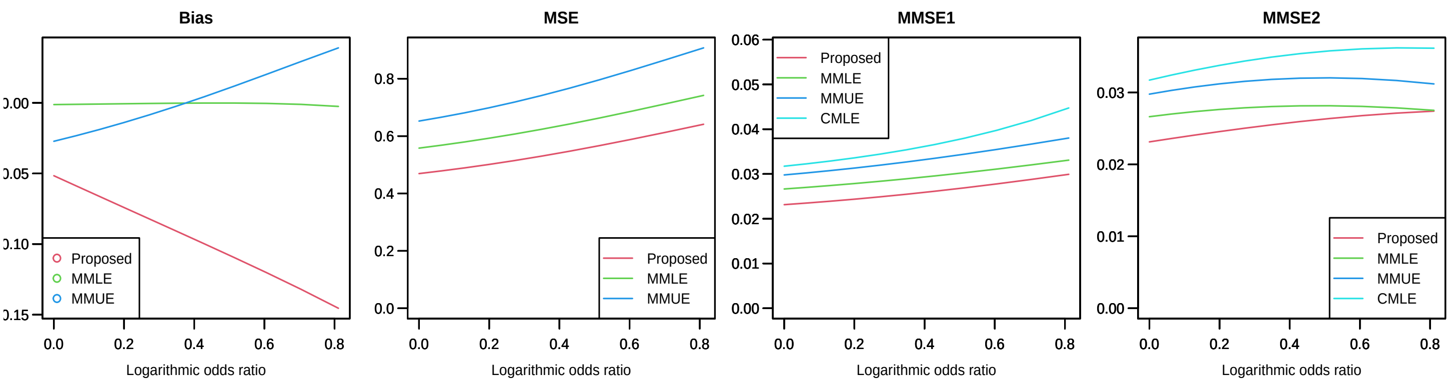


Figure S10 (continued).

$(n, m) = (30, 20),\ q = 0.7$



$(n, m) = (30, 20),\ q = 0.8$



$(n, m) = (30, 20),\ q = 0.9$

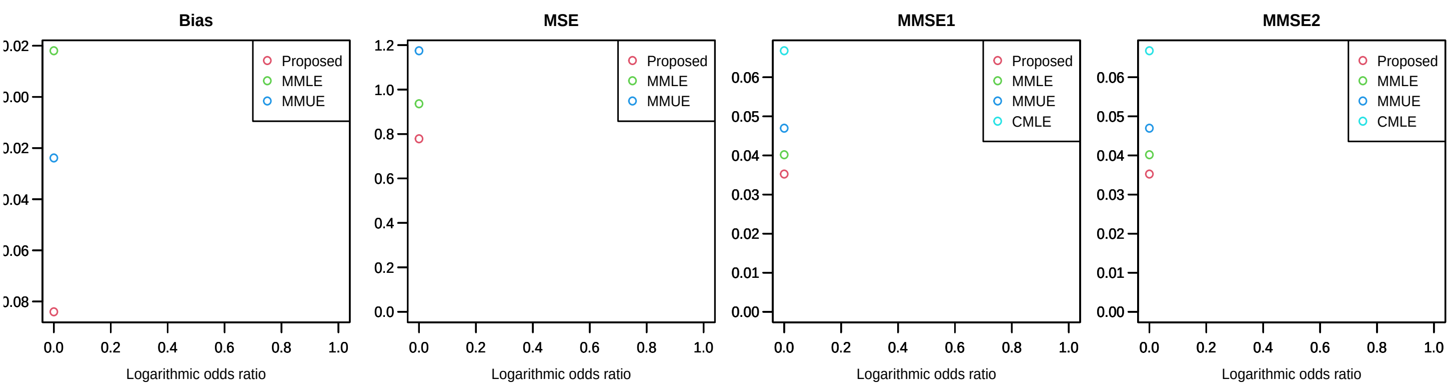


Figure S10 (continued).

We present a transformation of equation (5) to enhance its computational tractability in statistical analysis software. Additionally, we provide a sample code in software R version 4.4.0 for estimating a logarithmic odds ratio using this transformed equation. Execution of this code yields estimates for the four logarithmic odds ratios presented in Examples 1 and 2.

Transformation of (5)

Substituting equations (3) and (4) into equation (5), we obtaine:

$$\begin{aligned}
\hat{\beta}_a &= \int_{-\infty}^{\infty} \beta \pi_a(\beta|x,y) d\beta \\
&= \int_{-\infty}^{\infty} \binom{n}{x}\binom{m}{y} \frac{\beta \exp(x\beta)}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \binom{n}{z}\binom{m}{t-z} \exp(z\beta)} \frac{1}{\mathrm{Be}(a,a)} \frac{\exp(a\beta)}{[1+\exp(\beta)]^{2a}} \frac{1}{b} d\beta \\
&= \frac{\binom{n}{x}\binom{m}{y} \frac{1}{\mathrm{Be}(a,a)} \int_{-\infty}^{\infty} \frac{\beta \exp(x\beta)}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \binom{n}{z}\binom{m}{t-z} \exp(z\beta)} \frac{\exp(a\beta)}{[1+\exp(\beta)]^{2a}} d\beta}{\binom{n}{x}\binom{m}{y} \frac{1}{\mathrm{Be}(a,a)} \int_{-\infty}^{\infty} \frac{\exp(x\beta)}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \binom{n}{z}\binom{m}{t-z} \exp(z\beta)} \frac{\exp(a\beta)}{[1+\exp(\beta)]^{2a}} d\beta} \\
&= \frac{\int_{-\infty}^{\infty} \frac{\beta \exp(x\beta)}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \binom{n}{z}\binom{m}{t-z} \exp(z\beta)} \frac{\exp(a\beta)}{[1+\exp(\beta)]^{2a}} d\beta}{\int_{-\infty}^{\infty} \frac{\exp(x\beta)}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \binom{n}{z}\binom{m}{t-z} \exp(z\beta)} \frac{\exp(a\beta)}{[1+\exp(\beta)]^{2a}} d\beta}. \qquad \text{(S.1)}
\end{aligned}$$

Equation (S.1) is further transformed by separating it into a numerator and denominator.

Numerator:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \beta \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta \\
&= \int_{0}^{\infty} \beta \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta \\
&\quad + \int_{-\infty}^{0} \beta \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta \\
&= \int_{0}^{\infty} \beta \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta \\
&\quad + \int_{0}^{\infty} (-\beta) \frac{\exp\{(x+a)(-\beta) - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z(-\beta)]} d\beta \\
&= \int_{0}^{\infty} \beta \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta
\end{aligned}$$

$$- \int_0^\infty \beta \frac{\exp\{-(x+a)\beta - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} - z\beta]} d\beta$$

$$= \int_0^\infty \Bigg(\frac{\exp[(x-\min(n,t)+a]\beta + \log(\beta) - 2a\log[1+\exp(\beta)])}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [z-\min(n,t)]\beta\}}$$

$$- \frac{\exp\{[\max(0,t-m)-x-a]\beta + \log(\beta) - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [\max(0,t-m)-z]\beta\}} \Bigg) d\beta. \qquad \text{(S.2)}$$

Denominator:

$$\int_{-\infty}^\infty \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta$$

$$= \int_0^\infty \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta$$

$$+ \int_{-\infty}^0 \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta$$

$$= \int_0^\infty \frac{\exp\{(x+a)\beta - 2a\log[1+\exp(\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} + z\beta]} d\beta$$

$$+ \int_0^\infty \frac{\exp\{-(x+a)\beta - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp[\log\binom{n}{z} + \log\binom{m}{t-z} - z\beta]} d\beta$$

$$= \int_0^\infty \Bigg(\frac{\exp[(x-\min(n,t)+a]\beta - 2a\log[1+\exp(\beta)])}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [z-\min(n,t)]\beta\}}$$

$$+ \frac{\exp\{[\max(0,t-m)-x-a]\beta - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [\max(0,t-m)-z]\beta\}} \Bigg) d\beta. \qquad \text{(S.3)}$$

Substituting equations (S.3) and (S.2) into equation (S.1), we obtaine:

$$\hat{\beta}_a = \int_0^\infty \Bigg(\frac{\exp[(x-\min(n,t)+a]\beta + \log(\beta) - 2a\log[1+\exp(\beta)])}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [z-\min(n,t)]\beta\}}$$

$$- \frac{\exp\{[\max(0,t-m)-x-a]\beta + \log(\beta) - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [\max(0,t-m)-z]\beta\}} \Bigg) d\beta \Bigg/$$

$$\int_0^\infty \Bigg(\frac{\exp[(x-\min(n,t)+a]\beta - 2a\log[1+\exp(\beta)])}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [z-\min(n,t)]\beta\}}$$

$$+ \frac{\exp\{[\max(0,t-m)-x-a]\beta - 2a\log[1+\exp(-\beta)]\}}{\sum_{z=\max(0,t-m)}^{\min(n,t)} \exp\{\log\binom{n}{z} + \log\binom{m}{t-z} + [\max(0,t-m)-z]\beta\}} \Bigg) d\beta. \qquad \text{(S.4)}$$

The R code is provided to estimate the logarithmic odds ratios using equation (S.4).

R code

```
f<-function(n,m,x,y,a){t<-x+y
 f1<-function(b){
  s<-0; for(z in max(0,t-m):min(n,t)){
   s<-s+exp(lchoose(n,z)+lchoose(m,t-z)+(z-min(n,t))*b)}; return(s)}
 f2<-function(b){
  s<-0; for(z in max(0,t-m):min(n,t)){
   s<-s+exp(lchoose(n,z)+lchoose(m,t-z)+(max(0,t-m)-z)*b)}; return(s)}
 fN<-function(b){
  exp((x-min(n,t)+a)*b+log(b)-2*a*log(1+exp(b)))/f1(b)-
  exp((max(0,t-m)-x-a)*b+log(b)-2*a*log(1+exp(-b)))/f2(b)}
 fD<-function(b){
  exp((x-min(n,t)+a)*b-2*a*log(1+exp(b)))/f1(b)+
  exp((max(0,t-m)-x-a)*b-2*a*log(1+exp(-b)))/f2(b)}
 Prop<-integrate(fN,0,Inf)$value/integrate(fD,0,Inf)$value
 CMLE<-log(fisher.test(matrix(c(x,n-x,y,m-y),2,2))$estimate)
 MMLE<-log((x+0.5)*(m-y+0.5)/((y+0.5)*(n-x+0.5)))
 ff<-function(lo,up,l,r){sum(dbinom(lo:up,l,r))-0.5}
 g<-function(h,w,o){if(w==0)return(0); if(w==o)return(1)
  uniroot(h,c(0,1))$root}
 pL<-g(function(p)ff(x,n,n,p),x,n); pU<-g(function(p)ff(0,x,n,p),x,n)
 qL<-g(function(q)ff(y,m,m,q),y,m); qU<-g(function(q)ff(0,y,m,q),y,m)
 tp<-mean(c(pL,pU)); tq<-mean(c(qL,qU)); MMUE<-log(tp*(1-tq)/(tq*(1-tp)))
 result<-round(c(Prop,MMLE,MMUE,CMLE),3)
 names(result)<-c("Prop.","MMLE","MMUE","CMLE"); return(result)}

#Example 1 n<-11; m<-13; x<-8; y<-2; a<-1
f(11,13,8,2,1)

#Example 2 n<-13; m<-12; x<-8; y<-7; a<-1
f(13,12,8,7,1)
```