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Choquet integrals, weighted Hausdorff content and maximal operators

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Abstract. The boundedness of maximal operators on the weighted Choquet space and the Choquet–Morrey space is established. These results are used to study Carleson embeddings for weighted Sobolev spaces.

Keywords. Choquet integral, maximal operator, Carleson embedding.

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1 Introduction

The purpose of this note is to establish the boundedness of maximal operators on the weighted Choquet space and the Choquet–Morrey space. Let us first introduce some notation.

The Choquet integral of $\phi \geq 0$ with respect to a set function $\mathscr{L}$ is defined by

$$\int \phi d\mathscr{L} = \int_0^\infty \mathscr{L}\{x \in \mathbb{R}^n : \phi > t\} dt.$$

Let $0 \leq \beta < n$; the maximal function of f of order β is defined by

$$M_\beta f(x) = \sup_{x \in Q} |Q|^{\frac{\beta}{n}-1} \int_Q |f(y)| dy, \tag{1}$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes and $|Q|$ denotes the n-dimensional volume of Q. For convenience, M_0 is replaced by M, that is, M is the Hardy–Littlewood maximal operator.

If $E \subset \mathbb{R}^n$, $\omega \in A_1(\mathbb{R}^n)$ and $0 < \alpha \leq n$, then the α-dimensional weighted Hausdorff content of E (see [7]) is defined by

$$H_\omega^\alpha(E) = \inf \sum_{j=1}^\infty \frac{\omega(Q_j)}{|Q_j|} l(Q_j)^\alpha, \tag{2}$$

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where the infimum is taken over all coverings of E by countable families of cubes Q_j with sides parallel to the coordinate axes. Here $l(Q)$ denotes the side length of the cube Q and $\omega(Q) = \int_Q w(x)dx$. The Muckenhoupt class $A_1(\mathbb{R}^n)$ is defined by

$$M(\omega)(x) \leq C\omega(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Remark 1. If we take the infimum in (2) only over coverings of E by dyadic cubes, we obtain an equivalent quantity $H^\alpha_{w,d}$ called the dyadic α-dimensional weighted Hausdorff content; see Proposition 3.4.2 in [7].

Now we can state our main results.

Theorem 1. *Let $\omega \in A_1(\mathbb{R}^n)$, $0 \leq \beta < n$, $0 < \alpha < n$ and $0 < \alpha + \beta < n$. If $\alpha/(n-\beta) < p < (n-\alpha)/\beta$, then*

$$\int_{\mathbb{R}^n} (M_\beta f)^p dH^\alpha_\omega \leq C \int_{\mathbb{R}^n} |f|^p dH^{\alpha+p\beta}_\omega,$$

where C depends only on α, β, n, p and the constant $A_1(\mathbb{R}^n)$.

Let $0 \leq \lambda$, $0 < p$ and $0 < \alpha \leq n$. We define the Choquet–Morrey space $H^{\alpha,\lambda}_p$ by

$$\|f\|_{H^{\alpha,\lambda}_p} := \sup_{x_0 \in \mathbb{R}^n, r>0} r^{-\lambda} \int_0^\infty H^\alpha(\{y \in B(x_0, r) : |f(y)|^p > t\})dt < \infty.$$

Theorem 2. *Let $0 \leq \beta < n$, $0 < \alpha < n$ and $0 < \alpha + \beta < n$. If $\alpha/(n-\beta) < p < (n-\alpha)/\beta$ and $0 \leq \lambda < \alpha$, then*

$$\|M_\beta f\|_{H^{\alpha,\lambda}_p} \leq C\|f\|_{H^{\alpha+p\beta,\lambda}_p},$$

where C depends only on α, β, n, p and λ. Let $p = \alpha/(n-\beta)$. Then

$$H^\alpha(\{y \in B(x,r) : |M_\beta f(y)| > t\}) \leq Ct^{-\alpha/(n-\beta)} r^\lambda \|f\|_{H^{\frac{n\alpha}{n-\beta},\lambda}_{\frac{\alpha}{n-\beta}}},$$

where C depends only on α, β, n and λ.

Obviously, our results extend some well-known results for maximal operators; see [2] and [6].

2 Proof of the main theorems

Proof of Theorem 1. We follow the same scheme as in [6]. Without loss of generality, we can assume that $f \geq 0$.

Obviously, to prove Theorem 1, we only need to consider the dyadic case by Remark 1. By the definition of $H^{\alpha}_{\omega,d}$, for each integer k, we can take a family of non-overlapping dyadic cubes $Q_j^{(k)}$ such that

$$\{x : 2^k < f(x) \leq 2^{k+1}\} \subset \bigcup_j Q_j^{(k)}$$

and

$$\sum_j \frac{\omega(Q_j^{(k)})}{|Q_j^{(k)}|} l(Q_j^{(k)})^{\alpha+p\beta} \leq 2H^{\alpha+p\beta}_{\omega,d}(\{x : 2^k < f \leq 2^{k+1}\}).$$

Set $g = \sum_k 2^{(k+1)p} \chi_{A_k}$, where $A_k = \bigcup_j Q_j^{(k)}$ and χ_E is the characteristic function of the set E. Thus $f^p \leq g$.

Let $0 \leq \gamma < n$ and $\alpha/(n-\gamma) < q$. We first claim that for any cube Q,

$$\int M_\gamma(\chi_Q)^q dH^{\alpha}_{\omega} \leq C \frac{\omega(Q)}{|Q|} l(Q)^{\alpha+q\gamma}. \tag{3}$$

Indeed, let x_Q be the center of Q. Then

$$M_\gamma(\chi_Q)(x) \leq C_0 \frac{l(Q)^n}{(l(Q)+|x-x_Q|)^{n-\gamma}}, \quad x \in \mathbb{R}^n.$$

By the property of $A_1(\mathbb{R}^n)$ and the definition of H^{α}_{ω}, we then get

$$\begin{aligned}
&\int_0^\infty H^{\alpha}_{\omega}(\{y : |M_\gamma \chi_Q(y)|^q > t\})dt \\
&\leq \int_0^\infty H^{\alpha}_{\omega}\left(\left\{y : C_0 \frac{l(Q)^n}{(l(Q)+|y-x_Q|)^{n-\gamma}} > t^{1/q}\right\}\right)dt \\
&\leq \int_0^{(C_0 l(Q)^\gamma)^q} H^{\alpha}_{\omega}\left(\left\{y : C_0 \frac{l(Q)^n}{|y-x_Q|^{n-\gamma}} > t^{1/q}\right\}\right)dt \\
&= \int_0^{(C_0 l(Q)^\gamma)^q} H^{\alpha}_{\omega}\left(\left\{y : |y-x_Q| < \left(\frac{C_0 l(Q)^n}{t^{1/q}}\right)^{\frac{1}{n-\gamma}}\right\}\right)dt \\
&\leq \int_0^{(C_0 l(Q)^\gamma)^q} \frac{\omega\big(B\big(x_Q, \big(\frac{C_0 l(Q)^n}{t^{1/q}}\big)^{\frac{1}{n-\gamma}}\big)\big)}{\big|B\big(x_Q, \big(\frac{C_0 l(Q)^n}{t^{1/q}}\big)^{\frac{1}{n-\gamma}}\big)\big|} \left(\frac{C_0 l(Q)^n}{t^{1/q}}\right)^{\frac{\alpha}{n-\gamma}} dt
\end{aligned}$$

$$\leq C\frac{\omega(Q)}{|Q|}\int_0^{(C_0 l(Q)^\gamma)^q}\Big(\frac{C_0 l(Q)^n}{t^{1/q}}\Big)^{\frac{\alpha}{n-\gamma}}dt$$
$$\leq C\frac{\omega(Q)}{|Q|}l(Q)^{\alpha+q\gamma}.$$

The last inequality holds since $\alpha/(n-\gamma)<q$. Thus (3) is proved.

Let us proceed to the proof. Assume first that $1\leq p<(n-\alpha)/\beta$. Then

$$(M_\beta f)^p\leq M_{p\beta}(f^p)\leq M_{p\beta}(g)\leq\sum_k 2^{(k+1)p}\sum_j M_{p\beta}(\chi_{Q_j^{(k)}}).$$

Taking $q=1$ and $\gamma=p\beta$ in (3) and using the subadditivity of weighted Choquet integrals, we have

$$\begin{aligned}\int(M_\beta f)^p dH^\alpha_{\omega,d}&\leq C\sum_k 2^{(k+1)p}\sum_j\int M_{p\beta}(\chi_{Q_j^{(k)}})dH^\alpha_{\omega,d}\\&\leq C\sum_k 2^{(k+1)p}\frac{\omega(Q_j^{(k)})}{|Q_j^{(k)}|}l(Q_j^{(k)})^{\alpha+p\beta}\\&\leq C\sum_k 2^{(k+1)p}H^{\alpha+p\beta}_{\omega,d}(\{x:2^k<f(x)\leq 2^{k+1}\})\\&\leq C\sum_k\frac{2^{2p}}{2^p-1}\int_{2^{(k-1)p}}^{2^{kp}}H^{\alpha+p\beta}_{\omega,d}(\{x:f(x)^p>t\})dt\\&\leq C\int f^p dH^{\alpha+p\beta}_{\omega,d}.\end{aligned}$$

Assume now that $\alpha/(n-\beta)<p<1$. Note that $f\leq\sum_k 2^{k+1}\chi_{A_k}$,

$$M_\beta f\leq\sum_k 2^{k+1}\sum_j M_\beta(\chi_{Q_j^{(k)}}).$$

Taking $q=p$ and $\gamma=\beta$ in (3), we obtain

$$\int(M_\beta)^p dH^\alpha_{\omega,d}\leq C\sum_k 2^{(k+1)p}\frac{\omega(Q_j^{(k)})}{|Q_j^{(k)}|}l(Q_j^{(k)})^{\alpha+p\beta}\leq C\int f^p dH^{\alpha+p\beta}_{\omega,d}.$$

The proof is complete. □

Let us now turn to the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality, we can assume $\lambda > 0$. Choose any $x_0 \in \mathbb{R}^n$ and $r > 0$, and write

$$f(x) = f_0(x) + \tilde{f_0}(x),$$

where $f_0 = \chi_{B(x_0,2r)}$.

For the term f_0, from Theorem 1, we have

$$\int_0^\infty H^\alpha(\{y \in B(x_0, r) : |M_\beta f_0(y)|^p > t\})dt \leq C \int_0^\infty H^{\alpha+p\beta}(\{y \in B(x_0, 2r) : |f(y)|^p > t\})dt.$$

Hence

$$\|M_\beta f_0\|_{H_p^{\alpha,\lambda}} \leq C\|f\|_{H_p^{\alpha+p\beta,\lambda}}.$$

Now consider the term $\tilde{f_0}$. Let $q = pn/(\alpha + p\beta) > 1$. Then for any $y \in B(x_0, r)$, we have

$$\begin{aligned} M_\beta(\tilde{f_0})(y) &\leq C \sum_{k=1}^\infty (2^k r)^{\beta-n} \int_{B(x_0,2^{k+1}r)} |f(x)|dx \\ &\leq C \sum_{k=1}^\infty (2^k r)^{\beta-n/q} \left(\int_{B(x_0,2^{k+1}r)} |f(x)|^q dx \right)^{1/q} \\ &\leq C \sum_{k=1}^\infty (2^k r)^{\beta-n/q} \left(\int_{B(x_0,2^{k+1}r)} |f(x)|^p dH^{\alpha+p\beta} \right)^{1/p} \\ &\leq C \sum_{k=1}^\infty (2^k r)^{(\lambda-\alpha)/p} \|f\|^{1/p}_{H_p^{\alpha+p\beta,\lambda}} \\ &\leq C r^{(\lambda-\alpha)/p} \|f\|^{1/p}_{H_p^{\alpha+p\beta,\lambda}}, \end{aligned}$$

since $\lambda < \alpha$ and in the third inequality we use the inequality (see [6])

$$\int f(x)dx \leq \frac{n}{\alpha} \left(\int f^{\alpha/n} dH^\alpha \right)^{n/\alpha}$$

for $0 < \alpha \leq n$.

Thus

$$\int_0^\infty H^\alpha(\{y \in B(x_0, r) : |M_\beta(\tilde{f_0})(y)|^p > t\})dt \leq \int_0^\infty H^\alpha(\{y \in B(x_0, r) : C r^{(\lambda-\alpha)} \|f\|_{H_p^{\alpha+p\beta,\lambda}} > t\})dt$$

$$\leq \int_0^{Cr^{(\lambda-\alpha)}\|f\|_{H_p^{\alpha+p\beta,\lambda}}} H^\alpha(B(x_0,r))dt$$
$$\leq Cr^\lambda \|f\|_{H_p^{\alpha+p\beta,\lambda}}.$$

Therefore

$$\|M_\beta \tilde{f}_0\|_{H_p^{\alpha,\lambda}} \leq C\|f\|_{H_p^{\alpha+p\beta,\lambda}}.$$

From these inequalities we obtain

$$\|M_\beta f\|_{H_p^{\alpha,\lambda}} \leq C(\|M_\beta f_0\|_{H_p^{\alpha,\lambda}} + \|M_\beta \tilde{f}_0\|_{H_p^{\alpha,\lambda}}) \leq C\|f\|_{H_p^{\alpha+p\beta,\lambda}}.$$

As for the case $p = \alpha/(n-\beta)$, the proof is the same replacing Theorem 1 by the corresponding weak estimate, that is, for any $t > 0$

$$H^\alpha(\{y : |M_\beta f(y)| > t\}) \leq Ct^{-\alpha/(n-\beta)} \int f^{\frac{\alpha}{n-\beta}} dH^{\frac{n\alpha}{n-\beta}}. \qquad (4)$$

The proof of (4) can be found in [6]. □

3 An application

In this section, we study the Carleson embeddings for weighted Sobolev spaces.

To state our result, let us introduce some notation.

Let ω be a weight. If $E \subset \mathbb{R}^n$ for $n \geq 2$, we define the weighted, p-variational capacity $\mathrm{cap}_\omega^p(E)$ by (see [7, p. 117])

$$\mathrm{cap}_\omega^p(E) = \inf\left\{\int_{\mathbb{R}^n} |\nabla\varphi|^p \omega dx : \varphi \in C_0^\infty(\mathbb{R}^n),\ 0 \leq \varphi \leq 1,\right.$$
$$\left. E \subset \mathrm{Int}(\{x \in \mathbb{R}^n : \varphi(x) = 1\})\right\},$$

where $\mathrm{Int}(E)$ stands for the interior of a set $E \subset \mathbb{R}^n$; $c_\omega^p(\mu;t)$ denotes the p-variational capacity minimizing function of $t \in (0,\infty)$ associated with a non-negative measure μ on $\mathbb{R}_+^{n+1}$:

$$c_\omega^p(\mu;t) = \inf\{\mathrm{cap}_\omega^p : \text{bounded open } O \subset \mathbb{R}^n,\ \mu(T(O)) > t\},$$

where

$$T(O) = \{(t,x) \in \mathbb{R}_+^{n+1} : B(x,t) \subset O\},$$
$$\mathbb{R}_+^{n+1} = \mathbb{R}^{n+1} \bigcap \{x_{n+1} > 0\}.$$

Next, we give the Carleson embeddings for weighted Sobolev spaces.

Theorem 3. *Let* $1 \le q < \infty, 1 \le p < n$, $\omega \in A_1(\mathbb{R}^n)$ *and* μ *be a non-negative measure on* $\mathbb{R}^{n+1}_+$. *Then the following four conditions are equivalent:*

(a) $\left(\int_{\mathbb{R}^{n+1}_+} |e^{t^2\Delta} f(x)|^q d\mu(x,t)\right)^{1/q} \le C\|\nabla f\|_{L^p(\omega)}$, $\forall f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n)$;

(b) $\sup_{\lambda>0} \lambda\big(\mu(\{(t,x) \in \mathbb{R}^{n+1}_+ : |e^{t^2\Delta} f(x)| > \lambda\})\big)^{1/q} \le C\|\nabla f\|_{L^p(\omega)}$, $\forall f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n)$;

(c) $\sup_{t>0} \dfrac{t^{p/q}}{c^p_\omega(\mu;t)} < \infty$;

(d) $\sup\left\{\dfrac{(\mu(T(O)))^{p/q}}{\mathrm{cap}^p_\omega(O)} : \text{bounded open } O \subset \mathbb{R}^n\right\} < \infty.$

Here the weighted homogeneous Sobolev space $\dot{W}^{1,p}_\omega(\mathbb{R}^n)$ *(where* $p \ge 1$*) is the completion of* f*'s in* $C^\infty_0(\mathbb{R}^n)$ *(all infinitely differentiable functions with compact support in* $\mathbb{R}^n$*) with respect to the norm* $\|f\|_{\dot{W}^{1,p}_\omega(\mathbb{R}^n)} = \|\nabla f\|_{L^p_\omega(\mathbb{R}^n)} < \infty$, *and the heat kernel on* $\mathbb{R}^n \times \mathbb{R}^n$ *is defined by*

$$e^{t\Delta}(x,y) = (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big), \quad t \in (0,\infty),\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We remark that Theorem 3 extends Theorem 1.2 in [8].

To prove Theorem 3, we need the following results.

Lemma 1. *Let* $\omega \in A_1(\mathbb{R}^n)$ *and* $1 \le p < \infty$. *Then*

$$\int_0^\infty \mathrm{cap}^p_\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) d\lambda^p \le C\|\nabla f\|_{L^p(\omega)},\ \forall f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n).$$

Proof. We first prove the following inequality:

$$\int_0^\infty \mathrm{cap}^p_\omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) d\lambda^p \le C\|\nabla f\|_{L^p(\omega)},\ \forall f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n). \tag{5}$$

Indeed, by the monotonicity of capacity, the integral on the left-hand side in (5) does not exceed

$$I := \sum_{j=-\infty}^{\infty} 2^{jp}\,\mathrm{cap}^p_\omega(\{x : |f(x)| > 2^j\}).$$

Let $\lambda_\epsilon \in C^\infty(\mathbb{R})$, $0 \le \lambda_\epsilon \le 1$, $\lambda_\epsilon(t) = 1$ for $t \ge 1$, $\lambda_\epsilon(t) = 0$ for $t \le 0$, $0 \le \lambda'_\epsilon(t) \le 1+\epsilon$, $\epsilon > 0$ and let

$$f_j(x) = \lambda_\epsilon(2^{1-j}|f(x)| - 1).$$

Without loss of generality, we can assume that $f \in C_0^\infty(\mathbb{R}^n)$. Obviously, one has $f_j \in C_0^\infty(\mathbb{R}^n)$, $0 \le f_j \le 1$ and

$$E_j := \{x : |f(x)| > 2^j\} \subset \mathrm{Int}(\{x \in \mathbb{R}^n : f_j(x) = 1\}),$$

by the definition of $\mathrm{cap}_\omega^p(E_j)$, we then have

$$\begin{aligned} I &\le C \sum_{j=-\infty}^{\infty} 2^{jp} \int_{\mathbb{R}^n} |\nabla f_j(x)|^p \omega(x) dx \\ &= C \sum_{j=-\infty}^{\infty} 2^{jp} \int_{\{x \in \mathbb{R}^n : 2^{j-1} \le |f(x)| < 2^j\}} |\nabla f_j(x)|^p \omega(x) dx \\ &\le C \sum_{j=-\infty}^{\infty} \int_{\{x \in \mathbb{R}^n : 2^{j-1} \le |f(x)| < 2^j\}} [\lambda_\epsilon'(2^{1-j}|f(x)| - 1)]^p |\nabla f(x)|^p \omega(x) dx \\ &\le C(1+\epsilon)^p \int |\nabla f(x)|^p \omega(x) dx. \end{aligned}$$

Letting ϵ tend to zero, we obtain (5).

Let us proceed to the proof. We consider two cases for p.

Case 1, $p = 1$. From (3.5.4) in [7, p. 117] we know that $\mathrm{cap}_\omega^1(E) \simeq H_\omega^{n-1}(E)$. This relation, together with (5) and Theorem 1 with respect to the $(n-1)$-dimensional Hausdorff capacity yields

$$\int_0^\infty \mathrm{cap}_\omega^1(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) d\lambda \le C \|\nabla f\|_{L^1(\omega)}, \ \forall f \in \dot{W}_\omega^{1,1}(\mathbb{R}^n).$$

Case 2, $1 < p < \infty$. By the boundedness of M on $L^p(\omega)$ and applying the same argument as in the proof of Theorem 1.4 in [4], we know that for $f \in C_0^\infty(\mathbb{R}^n)$

$$|\nabla(Mf)(x)| \le M(|\nabla f|)(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Hence

$$\||\nabla(Mf)|\|_{L^p(\omega)} \le C\|M(|\nabla f|)\|_{L^p(\omega)} \le C\|\nabla f\|_{L^p(\omega)}.$$

From this and (5), we can obtain the desired result. □

Lemma 2. *Let $\omega \in A_1(\mathbb{R}^n)$ and $1 \le p < \infty$. Given $f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n)$, $\lambda > 0$ and a non-negative measure μ on $\mathbb{R}^{n+1}_+$, let*

$$E_\lambda(f) = \left\{(t, y) \in \mathbb{R}^{n+1}_+ : |e^{t^2\Delta} f(y)| > \lambda\right\}$$

and

$$O_\lambda(f) = \left\{x \in \mathbb{R}^n : \sup_{|x-y|<t} |e^{t^2\Delta} f(y)| > \lambda\right\}.$$

Then the following four statements are true:

(a) *For any natural number k,*

$$\mu\left(E_\lambda(f) \cap T(B(0,k))\right) \le \mu\left(T(O_\lambda(f) \cap B(0,k))\right).$$

(b) *For any natural number k,*

$$\mathrm{cap}^p_\omega\left(O_\lambda(f) \cap B(0,k)\right) \ge c^p_\omega\left(\mu;\ \mu\left(T(O_\lambda(f) \cap B(0,k))\right)\right).$$

(c) *There exists a dimensional constant $\theta_1 > 0$ such that*

$$\sup_{|x-y|<t} |e^{t^2\Delta} f(y)| \le \theta_1 Mf(x), \quad x \in \mathbb{R}^n.$$

(d) *There exists a dimensional constant $\theta_2 > 0$ such that*

$$(t, x) \in T(O) \Rightarrow e^{t^2\Delta}|f|(x) \ge \theta_2,$$

when O is a bounded open set contained in $\mathrm{Int}(\{x \in \mathbb{R}^n : f(x) = 1\})$.

The proof of Lemma 2 can be found in [8, p. 285].

Proof of Theorem 3. Adapting the same arguments as in [8, pp. 290–291], and using Lemmas 1 and 2, we can obtain the desired result. □

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