

Research Article

Büşra Aris*

Dilation estimates for Orlicz amalgam spaces on the affine group

<https://doi.org/10.1515/gmj-2025-2056>

Received December 24, 2024; revised April 5, 2025; accepted April 10, 2024

Abstract: Let $\mathbb{A}$ be the affine group and let Φ_1, Φ_2 be Young functions. We study the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ defined on $\mathbb{A}$, where the local and global component spaces are the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$, respectively. In this paper, we obtain norm estimates of the dilation operator on the amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ using constructions related to the affine group $\mathbb{A}$.

Keywords: Wiener amalgam spaces, Orlicz spaces, dilation operator, affine group

MSC 2020: Primary 43A15, 43A70; secondary 42B35, 46E30

1 Introduction

An amalgam space consists of functions whose norm distinguishes between local and global properties. The first appearance of amalgam spaces was due to Wiener in his studies of generalized harmonic analysis [23–25]. Amalgam spaces of Lebesgue spaces were investigated by many authors [4, 5, 7, 19]. The most general definition of Wiener amalgam spaces was introduced by Feichtinger in 1980s [10–14].

Amalgam spaces have proven to be very fruitful within pure and applied mathematics, for example, in sampling theory [16] and in time-frequency analysis [15]. It turned out that continuity properties of certain operators can be conveniently described in the context of Wiener amalgam spaces [8, 9] and are mostly considered for the Lebesgue spaces on the real line. On the other hand, for $1 \leq p < \infty$, Heil and Kutyniok studied amalgam spaces $W(L^\infty(\mathbb{A}), L^p(\mathbb{A}))$ on the affine group $\mathbb{A}$ [17, 18], which is not abelian unlike the real line. They proved a useful convolution relation for the amalgam space $W(L^\infty(\mathbb{A}), L^1(\mathbb{A}))$.

It is well known that the affine group plays a prominent role in harmonic analysis, especially in wavelet theory. It is not an IN group, i.e., a locally compact group with a compact and invariant neighborhood of identity, and it includes all abelian groups as well as some non-abelian groups such as the reduced Heisenberg group which is important for time-frequency analysis. A key feature of the affine group is that the left Haar measure and the right Haar measure are not equal.

An Orlicz space is a type of function space which significantly generalizes the Lebesgue spaces L^p . Besides the L^p spaces, a variety of function spaces naturally arise in analysis such as $L \log^+ L$, which is a Banach space related to Hardy–Littlewood maximal functions. Orlicz spaces contain certain Sobolev spaces as subspaces.

In [2], Aris and Öztöpe considered Wiener amalgam spaces with respect to Orlicz spaces $W(L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ and $W(L^\infty(\mathbb{A}), L^\Phi(\mathbb{A}))$ on the affine group $\mathbb{A}$. They obtained some properties of Wiener amalgam spaces of Orlicz type and proved convolution relations for $W(L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ and $W(L^\infty(\mathbb{A}), L^\Phi(\mathbb{A}))$. In [1], the results of [2] were extended to the more general Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ by using the equivalent discrete norm. Meanwhile, in [3], the amalgam spaces $W(L^{\Phi_1}(\mathbb{R}^d), L^{\Phi_2}(\mathbb{R}^d))$ were considered and dilation estimates were studied for these spaces.

The aim of this paper is to give dilation properties of the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ on the affine group $\mathbb{A}$. In order to do this, we are motivated to study the equivalent discrete norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$

*Corresponding author: Büşra Aris, Department of Mathematics, Faculty of Science, İstanbul University, 34134 Vezneciler, İstanbul, Türkiye, e-mail: busra.aris@istanbul.edu.tr. <https://orcid.org/0000-0002-4699-4122>

by using a specific partition of unity of the affine group. We also improve the estimates of dilation operator by using interpolation arguments of Orlicz spaces. To the best of our knowledge, dilation properties of Orlicz amalgam spaces on the affine group have not been studied before.

The paper is organized as follows. In Section 2, we review the necessary definitions and some basic results of the Orlicz spaces $L^\Phi(\mathbb{A})$ on the affine group. In Section 3, we present the basic structure of Orlicz amalgam spaces on the affine group $\mathbb{A}$ which we denote by $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. Finally, in Section 4, we study the dilation properties of Orlicz amalgam spaces (Theorem 4.2 and Theorem 4.5). These results are also new for the Lebesgue spaces and the standard Orlicz spaces.

2 Preliminaries

Throughout the paper, we consider the affine group $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ with the multiplication

$$(a, b)(x, y) = \left(ax, \frac{b}{x} + y\right),$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers. The identity element and inverses in $\mathbb{A}$ are given by

$$e = (1, 0), \quad (a, b)^{-1} = \left(\frac{1}{a}, -ab\right)$$

for $(a, b) \in \mathbb{A}$, respectively. It is easy to see that $\mathbb{A}$ is a non-abelian group under its multiplication.

One can see that the left Haar measure on $\mathbb{A}$ is $d\mu = \frac{dx}{x} dy$. The affine group $\mathbb{A}$ is not unimodular.

We consider Orlicz spaces on the affine group $\mathbb{A}$. An Orlicz space is determined by a Young function. A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex, $\Phi(0) = 0$, and $\lim_{x \rightarrow \infty} \Phi(x) = \infty$. For a Young function Φ , Φ^{-1} is defined by

$$\Phi^{-1}(y) = \inf\{x > 0 : \Phi(x) > y\}, \quad y \geq 0,$$

where $\inf \emptyset = \infty$, and we have

$$\Phi(\Phi^{-1}(x)) \leq x \leq \Phi^{-1}(\Phi(x)), \quad x \geq 0. \quad (2.1)$$

For a Young function Φ , the complementary function Ψ of Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x > 0\}, \quad y \geq 0,$$

and Ψ is also a Young function. So (Φ, Ψ) is called a complementary Young pair. We have the Young inequality

$$xy \leq \Phi(x) + \Psi(y), \quad x, y \geq 0,$$

for complementary functions Φ and Ψ .

By our definition, a Young function can take the value ∞ at a certain point, and hence be discontinuous at such a point. However, we always consider the pair of complementary Young functions (Φ, Ψ) with Φ being real-valued and continuous on $[0, \infty)$ and increasing on $(0, \infty)$. Note that even though Φ is continuous, it may happen that Ψ is not continuous.

Let (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be complementary Young pairs. If $\Phi_1(x) \leq \Phi_2(x)$ for all $x \geq x_0 \geq 0$, then we have $\Psi_2(y) \leq \Psi_1(y)$ for all $y \geq y_0 = \Phi_1(x_0) \geq 0$.

Let Φ_1, Φ_2 be two Young functions. If there exist $c > 0$ and $x_0 \geq 0$ (depending on c) such that $\Phi_1(x) \leq \Phi_2(cx)$ for all $x \geq x_0$, then we say that Φ_2 is stronger than Φ_1 and denote this by $\Phi_1 < \Phi_2$. If, in addition, $c = 1$ and $x_0 = 0$, we say that Φ_2 is strictly stronger than Φ_1 . If $\Phi_1 < \Phi_2$ and $\Phi_2 < \Phi_1$, then we write $\Phi_1 \asymp \Phi_2$. Also, $\Phi_1 < \Phi_2$ if and only if $\Phi_2^{-1}(y) \leq c\Phi_1^{-1}(y)$ for all $y \geq y_0 = \Phi_1(x_0)$.

A Young function Φ satisfies the Δ_2 condition if there exist a constant $K > 0$ and an $x_0 \geq 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq x_0$. In this case, we write $\Phi \in \Delta_2$.

Let $\mathbb{A}$ be equipped with the left Haar measure $d\mu = \frac{dx}{x} dy$. Given a Young function Φ , the Orlicz space on $\mathbb{A}$ is defined by

$$L^\Phi(\mathbb{A}) = \left\{ f : \mathbb{A} \rightarrow \mathbb{C} \text{ measurable} : \int_{\mathbb{A}} \Phi(a|f(x, y)|) \frac{dx}{x} dy < \infty \text{ for some } a > 0 \right\}.$$

Then the Orlicz space is a Banach space under the Orlicz norm $\|\cdot\|_{L^\Phi(\mathbb{A})}$ defined for $f \in L^\Phi(\mathbb{A})$ by

$$\|f\|_{L^\Phi(\mathbb{A})} = \sup \left\{ \int_{\mathbb{A}} |f(x, y)g(x, y)| \frac{dx}{x} dy : \int_{\mathbb{A}} \Psi(|g(x, y)|) \frac{dx}{x} dy \leq 1 \right\},$$

where Ψ is the complementary Young function of Φ .

One can also define the Luxemburg norm $\|\cdot\|_{L^\Phi(\mathbb{A})}^o$ on $L^\Phi(\mathbb{A})$ by

$$\|f\|_{L^\Phi(\mathbb{A})}^o = \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi\left(\frac{|f(x, y)|}{k}\right) \frac{dx}{x} dy \leq 1 \right\}.$$

It is known that these norms are equivalent, that is,

$$\|\cdot\|_{L^\Phi(\mathbb{A})}^o \leq \|\cdot\|_{L^\Phi(\mathbb{A})} \leq 2\|\cdot\|_{L^\Phi(\mathbb{A})}^o$$

and

$$\|f\|_{L^\Phi(\mathbb{A})}^o \leq 1 \quad \text{if and only if} \quad \int_{\mathbb{A}} \Phi(|f(x, y)|) \frac{dx}{x} dy \leq 1.$$

If (Φ, Ψ) is a complementary Young pair and $\Phi \in \Delta_2$, then the dual space $L^\Phi(\mathbb{A})^*$ is $L^\Psi(\mathbb{A})$. If, in addition, $\Psi \in \Delta_2$, then the Orlicz space $L^\Phi(\mathbb{A})$ is a reflexive Banach space [21].

We denote the norm equivalence of Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ by $\|\cdot\|_X \asymp \|\cdot\|_Y$ or $X \asymp Y$. Although the same notation is used for the equivalence of Young functions, the meaning will be clear from the context.

We also have Hölder's inequality, which states that if $f \in L^\Phi(\mathbb{A})$ and $g \in L^\Psi(\mathbb{A})$, then $fg \in L^1(\mathbb{A})$ and

$$\|fg\|_{L^1(\mathbb{A})} \leq 2\|f\|_{L^\Phi(\mathbb{A})}^o \|g\|_{L^\Psi(\mathbb{A})}^o.$$

A normed space $(Y, \|\cdot\|_Y)$ consisting of measurable of complex-valued functions on a measurable space X is called solid if for each measurable function $f: X \rightarrow \mathbb{C}$ satisfying $|f| \leq |g|$ almost everywhere for some $g \in Y$, we have $f \in Y$ and $\|f\|_Y \leq \|g\|_Y$. Since the Young function Φ is increasing, the Orlicz space $L^\Phi(\mathbb{A})$ is a solid space (see [22]).

For $1 \leq p < \infty$ and the Young function $\Phi(x) = x^p$, the space $L^\Phi(\mathbb{A})$ is the classical Lebesgue space $L^p(\mathbb{A})$, and the norm $\|\cdot\|_{L^\Phi(\mathbb{A})}$ is equivalent to the norm of the usual Lebesgue spaces $\|\cdot\|_{L^p(\mathbb{A})}$.

If $p = 1$, then we obtain the space $L^1(\mathbb{A})$. In this case, the complementary Young function of $\Phi(x) = x$ is

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ \infty, & x > 1, \end{cases} \quad (2.2)$$

and $\|f\|_{L^\Phi(\mathbb{A})} = \|f\|_{L^1(\mathbb{A})}$ for all $f \in L^1(\mathbb{A})$. For the Young function Ψ given in (2.2), the space $L^\Psi(\mathbb{A})$ coincides with the space $L^\infty(\mathbb{A})$, and we have $\|f\|_{L^\Psi(\mathbb{A})} = \|f\|_{L^\infty(\mathbb{A})}$ for all $f \in L^\infty(\mathbb{A})$.

In addition, for the Young function

$$\Phi_s(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ \infty, & x > 1, \end{cases} \quad (2.3)$$

the space $L^{\Phi_s}(\mathbb{A})$ becomes $L^1(\mathbb{A}) \cap L^\infty(\mathbb{A})$ with the norm

$$\|f\|_{L^{\Phi_s}(\mathbb{A})} = \max\{\|f\|_{L^1(\mathbb{A})}, \|f\|_{L^\infty(\mathbb{A})}\}$$

for all $f \in L^{\Phi_s}(\mathbb{A})$. Then we have

$$\Phi_s(x)^{-1} = \begin{cases} x, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

For the Young function

$$\Phi_b(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ x - 1, & x > 1, \end{cases}$$

we obtain the space $L^1(\mathbb{A}) + L^\infty(\mathbb{A})$ with the norm

$$\|f\|_{L^{\Phi_b}(\mathbb{A})} = \inf_{f=f_1+f_2} (\|f_1\|_{L^1(\mathbb{A})} + \|f_2\|_{L^\infty(\mathbb{A})}).$$

We have $\Phi_b(x)^{-1} = x + 1$, $x \geq 0$. In addition, Φ_s and Φ_b are complementary Young functions [20, p. 52] which satisfy the Δ_2 condition.

Since

$$L^1(\mathbb{A}) \cap L^\infty(\mathbb{A}) \subset L^\Phi(\mathbb{A}) \subset L^1(\mathbb{A}) + L^\infty(\mathbb{A})$$

for any Young function Φ , we may consider $L^1(\mathbb{A}) \cap L^\infty(\mathbb{A})$ as the smallest Orlicz space, and $L^1(\mathbb{A}) + L^\infty(\mathbb{A})$ as the largest one [20, p. 100]. We denote the spaces $L^1(\mathbb{A}) \cap L^\infty(\mathbb{A})$ and $L^1(\mathbb{A}) + L^\infty(\mathbb{A})$ by $L^{\Phi_s}(\mathbb{A})$ and $L^{\Phi_b}(\mathbb{A})$, respectively. Also, we have $L^{\Phi_s}(K) \simeq L^\infty(K)$ for any compact subset $K \subset \mathbb{A}$.

We denote the translation operator by $L_{(a,b)}f(x, y) = f((a, b)^{-1}(x, y))$ and the dilation operator by $D_\lambda f(x, y) = f(\lambda x, \lambda y)$ for $(x, y), (a, b) \in \mathbb{A}$ and $\lambda > 0$. Both operators are well-defined, linear and bounded operators on Orlicz spaces [22].

Properties of the dilation operator D_λ when acting on Orlicz spaces were recently studied by Blasco and Osançlıoğlu in [6]. Let us modify their results for our aim.

Given $\lambda > 0$, another norm on the Orlicz space $L^\Phi(\mathbb{A})$ is defined by

$$\|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda} = \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi\left(\frac{|f(x, y)|}{k}\right) \frac{dx}{x} dy \leq \lambda \right\}.$$

When $\lambda = 1$, we obtain $\|f\|_{L^\Phi(\mathbb{A})}^{\circ, 1} = \|f\|_{L^\Phi(\mathbb{A})}^\circ$, and for $\lambda > 0$ we have

$$\|D_\lambda f\|_{L^\Phi(\mathbb{A})}^\circ = \|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda} \quad \text{for all } f \in L^\Phi(\mathbb{A}). \quad (2.4)$$

By convexity, it can be easily seen that

$$\lambda \|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda} \leq \|f\|_{L^\Phi(\mathbb{A})}^\circ \leq \|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda}, \quad 0 < \lambda \leq 1,$$

and

$$\|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda} \leq \|f\|_{L^\Phi(\mathbb{A})}^\circ \leq \lambda \|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda}, \quad \lambda \geq 1.$$

3 Orlicz amalgam spaces on the affine group

Let $\mathbb{A}$ be the affine group and let Φ_1, Φ_2 be Young functions. In this section, we give the definition and basic properties of the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ consisting of functions that are locally in $L^{\Phi_1}(\mathbb{A})$ and globally in $L^{\Phi_2}(\mathbb{A})$. In our theorems, the translation invariance and solidity of the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$ play important roles.

Next, we summarize some technical results from [1] that will be used in the next section.

Definition 3.1. Let Q be a fixed compact subset of $\mathbb{A}$ with nonempty interior and let $(x, y)Q$ be the set Q left translated by $(x, y) \in \mathbb{A}$. The Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ consists of all measurable functions $f: \mathbb{A} \rightarrow \mathbb{C}$ such that $f\chi_{(x, y)Q} \in L^{\Phi_1}(\mathbb{A})$ for each $(x, y) \in \mathbb{A}$, and the control function

$$F_f(x, y) = F_f^Q(x, y) = \|f\chi_{(x, y)Q}\|_{L^{\Phi_1}(\mathbb{A})}$$

belongs to $L^{\Phi_2}(\mathbb{A})$. The Orlicz amalgam norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is defined by

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))} := \|F_f\|_{L^{\Phi_2}(\mathbb{A})} = \|\|f\chi_{(x, y)Q}\|_{L^{\Phi_1}(\mathbb{A})}\|_{L^{\Phi_2}(\mathbb{A})}.$$

Like in Orlicz spaces, we define the Luxemburg norm $\|\cdot\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^o$ on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ by

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^o = \|\|f\chi_{(x, y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^o\|_{L^{\Phi_2}(\mathbb{A})}.$$

By the equivalence of the Orlicz norm and the Luxemburg norm in Orlicz spaces [22], we have

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^o \leq \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))} \leq 4\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^o.$$

Throughout the paper, we consider the Luxemburg norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

Note that the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is a Banach space and its definition is independent of the choice of the compact subset $Q \subset \mathbb{A}$, in the sense that different compact subsets yield equivalent Orlicz amalgam space norms. Moreover, since the Orlicz spaces $L^{\Phi_1}(\mathbb{A})$ and $L^{\Phi_2}(\mathbb{A})$ are solid spaces, $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is also a solid space.

Let us now recall Hölder's inequality for Orlicz amalgam spaces. If we have $f \in W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ and $g \in W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$, then $fg \in L^1(\mathbb{A})$ and

$$\|fg\|_{L^1(\mathbb{A})} \leq \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))} \|g\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^o.$$

Wiener amalgam spaces can be considered as spaces with a discrete norm. In fact, the norm which is given in Definition 3.1 and discrete norms are equivalent (see [1]).

Let $\{Q_h\}_{h>0}$ denote a fixed family of increasing, exhaustive neighborhoods of identity in $\mathbb{A}$. In particular, we take $Q_h = [e^{-h}, e^h] \times [-h, h]$. The Haar measure of the translated set $(x, y)Q_h$ is

$$\mu((x, y)Q_h) = \mu(Q_h) = \int_{-h}^h \int_{e^{-h}}^{e^h} \frac{dx}{x} dy = 4h^2.$$

Given $h > 0$, for $k, j \in \mathbb{Z}$, we define particular translates of Q_h and Q_{2h} as

$$\begin{aligned} B_{jk} &= (e^{2jh}, 2khe^{-h})Q_h, \\ B'_{jk} &= (e^{2jh}, 2khe^{-h})Q_{2h}. \end{aligned}$$

Note that $B_{jk} \subseteq B'_{jk}$.

To obtain an equivalent discrete norm on these spaces, the following lemma is a key observation.

Lemma 3.2 ([17]). *If $h > 0$, then:*

- (a) $\bigcup_{j,k \in \mathbb{Z}} B_{jk} = \mathbb{A}$,
- (b) *given $m, n \in \mathbb{Z}$, the box B'_{mn} can intersect at most $N = 5(2e^{3h} + 1)$ boxes B'_{jk} for $j, k \in \mathbb{Z}$.*

Hence the set $X = \{(e^{2jh}, 2khe^{-h}) : j, k \in \mathbb{Z}\}$ for $h > 0$ becomes a well-spread family [13, 14].

By Urysohn's lemma, there exist continuous functions $\phi_{jk} : \mathbb{A} \rightarrow \mathbb{R}$ such that $0 \leq \phi_{jk}(x, y) \leq 1$, $\text{supp}(\phi_{jk}) \subseteq B'_{jk}$ and $\phi_{jk}(x, y) = 1$ for $(x, y) \in B_{jk}$. Define

$$\psi_{jk} = \frac{\phi_{jk}}{\sum_{m,n \in \mathbb{Z}} \phi_{mn}}.$$

Thus $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$ is a bounded uniform partition of unity (BUPU). Then, by [11, Theorem 2], we have the following equivalence in $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$:

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))} \asymp \left\| \sum_{j,k \in \mathbb{Z}} \|f\psi_{jk}\|_{L^{\Phi_1}(\mathbb{A})} \chi_{B'_{jk}} \right\|_{L^{\Phi_2}(\mathbb{A})}.$$

To illustrate the usefulness of the discrete norm, we present a duality theorem and some inclusion relations for $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ given in [1].

Proposition 3.3. *Let $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)$ be complementary Young pairs with $\Phi_1, \Phi_2 \in \Delta_2$. Then the dual space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))^*$ is $W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))$.*

Proposition 3.4. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 < \Phi_2$, then $W(L^{\Phi_2}(\mathbb{A}), L^{\Phi}(\mathbb{A})) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi}(\mathbb{A}))$.*

Proposition 3.5. *Let Φ, Φ_1, Φ_2 be Young functions. If $\Phi_1 < \Phi_2$, then $W(L^{\Phi}(\mathbb{A}), L^{\Phi_1}(\mathbb{A})) \subseteq W(L^{\Phi}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.*

4 Dilation properties

In this section, we study dilation properties of Wiener amalgams of the Orlicz spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. By using the constructions related to the affine group, we obtain norm estimates of the dilation operator on the Orlicz amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$, differently from those in [3]. In particular, using the dilation properties of Orlicz spaces given in [6] and the interpolation of Orlicz spaces in [20], we improve the norm estimates.

Let $f_\lambda(x, y) = f(\lambda x, \lambda y)$, $\lambda > 0$, $(x, y) \in \mathbb{A}$. The following lemma provides an estimate for the norm of the dilation operator D_λ in the Orlicz space $L^\Phi(\mathbb{A})$. It will be used when proving subsequent theorems related to dilation properties of the Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$.

Note that the following result for $L^\Phi(\mathbb{R})$ is given by [6, Proposition 2.2]. If we adapt it to the space $L^\Phi(\mathbb{A})$, we obtain the following lemma.

Lemma 4.1. *Let Φ be a Young function, and define $C_\Phi(\lambda) := \|D_\lambda\|_{L^\Phi(\mathbb{A}) \rightarrow L^\Phi(\mathbb{A})}$. Then we have the following:*

- (i) $C_\Phi(\lambda) \geq \sup_{\mu > 0} \frac{\Phi^{-1}(\mu)}{\Phi^{-1}(\lambda\mu)}$.
- (ii) If $\Phi(st) \leq \Phi(s)\Phi(t)$ for all $s, t \geq 0$, then $C_\Phi(\lambda) \leq \frac{1}{\Phi^{-1}(\lambda)}$.

Furthermore, $C_\Phi(\lambda)$ is non-increasing, submultiplicative and $C_\Phi(1) = 1$.

Proof. (i) Let $A = [e^{-h}, e^h] \times [-h, h]$, $h > 0$ be a subset of $\mathbb{A}$. Since $0 < \mu(A) < \infty$, we can take the function $f = \chi_A$ as in [6, Lemma 2.1]. Then we have

$$D_\lambda(\chi_{[e^{-h}, e^h] \times [-h, h]}) = \chi_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]}.$$

Now, we calculate the norms $\|\chi_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]}^\circ\|_{L^\Phi(\mathbb{A})}$ and $\|\chi_{[e^{-h}, e^h] \times [-h, h]}^\circ\|_{L^\Phi(\mathbb{A})}$. By the definition of the Luxemburg norm, we obtain

$$\begin{aligned} \|\chi_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]}^\circ\|_{L^\Phi(\mathbb{A})} &= \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi \left(\frac{\chi_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]}(x, y)}{k} \right) \frac{dx}{x} dy \leq 1 \right\} \\ &= \inf \left\{ k > 0 : \int_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]} \Phi \left(\frac{1}{k} \right) \frac{dx}{x} dy \leq 1 \right\} \\ &= \inf \left\{ k > 0 : \Phi \left(\frac{1}{k} \right) \leq \frac{\lambda}{4h^2} \right\} = \frac{1}{\Phi^{-1}(\frac{\lambda}{4h^2})}. \end{aligned}$$

In a similar way, we have

$$\|\chi_{[e^{-h}, e^h] \times [-h, h]}^\circ\|_{L^\Phi(\mathbb{A})} = \frac{1}{\Phi^{-1}(\frac{1}{4h^2})}.$$

Therefore, we obtain

$$C_\Phi(\lambda) \geq \sup_{h > 0} \frac{\|\chi_{[\frac{e^{-h}}{\lambda}, \frac{e^h}{\lambda}] \times [-\frac{h}{\lambda}, \frac{h}{\lambda}]}^\circ\|_{L^\Phi(\mathbb{A})}}{\|\chi_{[e^{-h}, e^h] \times [-h, h]}^\circ\|_{L^\Phi(\mathbb{A})}} = \sup_{\mu > 0} \frac{\Phi^{-1}(\mu)}{\Phi^{-1}(\lambda\mu)}.$$

(ii) For $s > 0$ and $t = |D_\lambda f(x, y)|$, by changing the variables in the integral and using the Luxemburg norm, we obtain

$$\begin{aligned} \int_{\mathbb{A}} \Phi \left(\frac{s|D_\lambda f(x, y)|}{k} \right) \frac{dx}{x} dy &= \int_{\mathbb{A}} \Phi \left(\frac{s|f(\lambda x, \lambda y)|}{k} \right) \frac{dx}{x} dy \\ &\leq \frac{1}{\lambda} \int_{\mathbb{A}} \Phi(s) \Phi \left(\frac{|f(x, y)|}{k} \right) \frac{dx}{x} dy \\ &= \frac{\Phi(s)}{\lambda} \int_{\mathbb{A}} \Phi \left(\frac{|f(x, y)|}{k} \right) \frac{dx}{x} dy. \end{aligned}$$

Let us choose $s = \Phi^{-1}(\lambda)$. Since $\Phi(s) \leq \lambda$ by inequality (2.1), we have

$$\int_{\mathbb{A}} \Phi \left(\frac{\Phi^{-1}(\lambda)|D_\lambda f(x, y)|}{k} \right) \frac{dx}{x} dy \leq \int_{\mathbb{A}} \Phi \left(\frac{|f(x, y)|}{k} \right) \frac{dx}{x} dy.$$

By the definition of the Luxemburg norm, for $k = \|f\|_{L^\Phi(\mathbb{A})}^\circ$, we obtain

$$\|D_\lambda f\|_{L^\Phi(\mathbb{A})}^\circ \leq \frac{\|f\|_{L^\Phi(\mathbb{A})}^\circ}{\Phi^{-1}(\lambda)},$$

which implies that $C_\Phi(\lambda) = \|D_\lambda\|_{L^\Phi(\mathbb{A}) \rightarrow L^\Phi(\mathbb{A})} \leq \frac{1}{\Phi^{-1}(\lambda)}$. $\square$

Note that, by (2.4) and Lemma 4.1, we have

$$\|f\|_{L^\Phi(\mathbb{A})}^{\circ, \lambda} = \|f\|_{L^\Phi(\mathbb{A})}^\circ \leq \frac{1}{\Phi^{-1}(\lambda)} \|f\|_{L^\Phi(\mathbb{A})}^\circ, \quad f \in L^\Phi(\mathbb{A}), \quad \lambda > 0. \quad (4.1)$$

The first estimates for dilations on Orlicz amalgam spaces $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ are given in the following theorem.

Theorem 4.2. *Let Φ_1, Φ_2 be two Young functions. Then*

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \quad 0 < \lambda \leq 1, \quad (4.2)$$

and

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \lambda \frac{1}{\Phi_1^{-1}(\lambda)\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \quad \lambda \geq 1.$$

Proof. To compute the amalgam norm, we choose the compact subset $Q = [e^{-1}, e] \times [-1, 1] \subset \mathbb{A}$. Then we have

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ = \|F_{f_\lambda}^Q\|_{L^{\Phi_2}(\mathbb{A})}^\circ = \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^\circ \|f\|_{L^{\Phi_2}(\mathbb{A})}^\circ.$$

We first estimate the norm of the local component $\|\cdot\|_{L^{\Phi_1}(\mathbb{A})}^\circ$. We have

$$\begin{aligned} \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^\circ &= \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi_1 \left(\frac{|f(\lambda s, \lambda t)\chi_Q((x, y)^{-1}(s, t))|}{k} \right) \frac{ds}{s} dt \leq 1 \right\} \\ &= \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi_1 \left(\frac{|f(s, t)\chi_Q((x, y)^{-1}(\frac{s}{\lambda}, \frac{t}{\lambda}))|}{k} \right) \frac{ds}{s} \frac{dt}{\lambda} \leq 1 \right\} \\ &= \inf \left\{ k > 0 : \int_{\mathbb{A}} \Phi_1 \left(\frac{|f(s, t)\chi_Q((\lambda x, \lambda y)^{-1}(s, t))|}{k} \right) \frac{ds}{s} dt \leq \lambda \right\} \\ &= \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^{\circ, \lambda}. \end{aligned}$$

Let $0 < \lambda \leq 1$. By (4.1), we obtain

$$\begin{aligned} F_{f_\lambda}^Q(x, y) &= \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^\circ = \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^{\circ, \lambda} \\ &\leq \frac{1}{\Phi_1^{-1}(\lambda)} \|f\chi_{(\lambda x, \lambda y)Q}\|_{L^{\Phi_1}(\mathbb{A})}^\circ \\ &= \frac{1}{\Phi_1^{-1}(\lambda)} F_f^Q(\lambda x, \lambda y) = \frac{1}{\Phi_1^{-1}(\lambda)} D_\lambda F_f^Q(x, y), \end{aligned}$$

that is,

$$F_{f_\lambda}^Q(x, y) \leq \frac{1}{\Phi_1^{-1}(\lambda)} D_\lambda F_f^Q(x, y). \quad (4.3)$$

By the solidity of $L^{\Phi_2}(\mathbb{A})$ and (4.1), we have

$$\|F_{f_\lambda}^Q\|_{L^{\Phi_2}(\mathbb{A})}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)} \|D_\lambda F_f^Q\|_{L^{\Phi_2}(\mathbb{A})}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)} \frac{1}{\Phi_2^{-1}(\lambda)} \|F_f^Q\|_{L^{\Phi_2}(\mathbb{A})}^\circ,$$

which gives

$$\|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ$$

for $0 < \lambda \leq 1$.

Let $\lambda \geq 1$. We proceed by estimating the global component. By (4.1), (4.3) and the solidity of $L^{\Phi_2}(\mathbb{A})$, we have

$$\|F_{f_\lambda}^Q\|_{L^{\Phi_2}(\mathbb{A})}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)} \|D_\lambda F_f^{\lambda Q}\|_{L^{\Phi_2}(\mathbb{A})}^\circ \leq \frac{1}{\Phi_1^{-1}(\lambda)} \frac{1}{\Phi_2^{-1}(\lambda)} \|F_f^{\lambda Q}\|_{L^{\Phi_2}(\mathbb{A})}^\circ. \quad (4.4)$$

Now, we need to calculate the norm $\|f(L_{(\lambda x, \lambda y)} \chi_{\lambda Q})\|_{L^{\Phi_1}(\mathbb{A})}^\circ$ for $\lambda \geq 1$. To this end, we use the discrete norm on $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$. Since the sets B_{jk} is partition of $\mathbb{A}$, we can take $\chi_{B_{jk}}$ instead of ψ_{jk} . Then we have $\mu(\{(j, k) \in \mathbb{Z} \times \mathbb{Z} : B_{jk} \cap \lambda Q \neq \emptyset\}) < \infty$. Using [1, Proposition 3.2], we obtain

$$\begin{aligned} \|f\chi_{(x,y)\lambda Q}\|_{L^{\Phi_1}(\mathbb{A})}^\circ &= \left\| \sum_{j,k \in \mathbb{Z}} f\chi_{B_{jk}} \chi_{(x,y)\lambda Q} \right\|_{L^{\Phi_1}(\mathbb{A})}^\circ \\ &= \left\| \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z} \cap \lambda Q} f\chi_{B_{jk}} \right\|_{L^{\Phi_1}(\mathbb{A})}^\circ \\ &\leq \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z} \cap \lambda Q} \|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})}^\circ \chi_{B'_{jk}} \\ &\leq \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} \|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})}^\circ \chi_{B'_{jk}}, \end{aligned}$$

which implies

$$F_{f_\lambda}^Q(x, y) \leq \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} \|f\chi_{B_{jk}}\|_{L^{\Phi_1}(\mathbb{A})}^\circ \chi_{B'_{jk}}.$$

By the solidity of $L^{\Phi_2}(\mathbb{A})$ and (4.4), we have

$$\begin{aligned} \|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ &\leq \frac{1}{\Phi_1^{-1}(\lambda)} \frac{1}{\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \\ &\leq \frac{\lambda}{\Phi_1^{-1}(\lambda) \Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ. \end{aligned} \quad \square$$

The estimates given in Theorem 4.2 can be improved by using interpolation arguments in a manner similar to [8]. However, instead of the classical interpolation between Lebesgue type spaces, we must combine the interpolation of Orlicz spaces with the properties of Wiener amalgam spaces.

To achieve this, we first recall the interpolation result for Orlicz spaces which is given in [20, Lemma 14.2]. Let $[X, Y]_\rho$ denote the interpolation between certain Orlicz spaces X and Y .

Lemma 4.3. *Let Φ_0, Φ_1 be any Young functions. Then the function Φ defined by*

$$\Phi^{-1}(x) = \Phi_0^{-1}(u) \rho\left(\frac{\Phi_1^{-1}(u)}{\Phi_0^{-1}(u)}\right) \quad (4.5)$$

is a Young function and $L^\Phi(\mathbb{A}) = [L^{\Phi_0}(\mathbb{A}), L^{\Phi_1}(\mathbb{A})]_\rho$, where $\rho : [0, \infty) \rightarrow [0, \infty)$ is concave, continuous, positive on $(0, \infty)$, and such that

$$\rho(s) \leq \max\left\{\frac{s}{t}, 1\right\} \rho(t), \quad s, t > 0.$$

In particular, when $\rho(t) = t^\theta$, $0 < \theta < 1$, we have $\Phi^{-1} = (\Phi_0^{-1})^{1-\theta} (\Phi_1^{-1})^\theta$.

The interpolation argument for Wiener amalgam spaces is given in [10]. Here we focus on the interpolation result for Orlicz amalgam spaces. Then we have the following result.

Lemma 4.4. *Let B_0, B_1 be local components of Orlicz amalgam spaces and let Φ, Φ_0, Φ_1 be Young functions satisfying the Δ_2 condition, along with (4.5). Then we have*

$$[W(B_0, L^{\Phi_0}), W(B_1, L^{\Phi_1})]_\rho = W([B_0, B_1]_\rho, [L^{\Phi_0}, L^{\Phi_1}]_\rho) = W([B_0, B_1]_\rho, L^\Phi)$$

Note that the Orlicz space $L^\Phi(\mathbb{A})$ has an absolutely continuous norm for any Young function Φ which satisfies the Δ_2 condition. Hence Lemma 4.4 is a special case of [10, Theorem 2.2].

Let $\mathcal{O}$ be the set of Young functions such that the following conditions hold:

- (i) We have $\Phi_s, \Phi_b \in \mathcal{O}$.
- (ii) If $\Phi_1, \Phi_2 \in \mathcal{O}$, then either Φ_1 is strictly stronger than Φ_2 , or Φ_2 is strictly stronger than Φ_1 . In other words, either $\Phi_1(x) \leq \Phi_2(x)$, $x \geq 0$, or $\Phi_2(x) \leq \Phi_1(x)$, $x \geq 0$.

Now, we are ready to prove the main result of the paper. By using the techniques similar to those in [3], we obtain the following theorem.

Theorem 4.5. *Let $\Phi_1, \Phi_2 \in \mathcal{O}$ satisfy the Δ_2 condition. Then we have*

$$\|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\max\{\Phi_1^{-1}(\lambda), \Phi_2^{-1}(\lambda)\}} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \quad 0 < \lambda \leq 1, \quad (4.6)$$

and

$$\|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\min\{\Phi_1^{-1}(\lambda), \Phi_2^{-1}(\lambda)\}} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \quad \lambda \geq 1. \quad (4.7)$$

Here, $\|\cdot\|_X \leq \|\cdot\|_Y$ means that there exists a constant $c > 0$ such that $\|\cdot\|_X \leq c\|\cdot\|_Y$.

Proof. Let Φ_1, Φ_2 satisfy the assumptions of Proposition 4.5. We note that Φ_1 and Φ_2 give rise to intermediate spaces between the spaces generated by Φ_s and Φ_b , so the Young functions Φ_s and Φ_b play an essential role in the proof.

First, assume that $\Phi_2 \leq \Phi_1$. By inequalities (2.2) and (2.3), we have $L^1(\mathbb{A}) \cap L^\infty(\mathbb{A}) = L^\infty(\mathbb{A})$ for the local component. Therefore, we take the Young function $\Phi_1 = \Phi_s$, which implies that $\Phi_2 \leq \Phi_s$ in (2.2). Since $\Phi_s^{-1}(\lambda) = 1$ for all $\lambda > 0$, by (4.2) we obtain

$$\begin{aligned} \|f_\lambda\|_{W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ &\leq \frac{1}{\Phi_s^{-1}(\lambda)\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \\ &= \frac{1}{\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \\ &= \frac{1}{\max\{\Phi_s^{-1}(\lambda), \Phi_2^{-1}(\lambda)\}} \|f\|_{W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \end{aligned}$$

which is (4.6).

Now, we prove (4.6) for $\Phi_1 \neq \Phi_s$. In this case, we have $\Phi_2 \leq \Phi_1 < \Phi_s$ with $\Phi_b \leq \Phi_1 < \Phi_s$. By Proposition 3.4 and Proposition 3.5, we obtain

$$W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})) \subseteq W(L^{\Phi_b}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})).$$

Hence the amalgam space $W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))$ is the intermediate space, that is,

$$\begin{aligned} [W(L^{\Phi_s}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})), W(L^{\Phi_b}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))]_\rho &= W([L^{\Phi_s}, L^{\Phi_b}]_\rho, [L^{\Phi_2}, L^{\Phi_2}]_\rho) \\ &= W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A})). \end{aligned}$$

This, together with $\Phi_1^{-1} \leq \Phi_2^{-1}$ for $\Phi_2 \leq \Phi_1$, gives

$$\|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\max\{\Phi_1^{-1}(\lambda), \Phi_2^{-1}(\lambda)\}} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ.$$

Hence, we obtain (4.6) for the case $\Phi_1 \neq \Phi_s$.

Next, suppose that $\Phi_1 \leq \Phi_2$. Then, by [22, Theorem 2.2.2], we have $\Psi_2 \leq \Psi_1$, so that $\Psi_1^{-1} \leq \Psi_2^{-1}$ for the complementary Young functions. To prove (4.7), we apply relation (4.6) to the pair (Ψ_1, Ψ_2) . Since $\frac{1}{\lambda} \leq 1$, we have

$$\|g_{1/\lambda}\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^\circ \leq \frac{1}{\max\{\Psi_1^{-1}(\frac{1}{\lambda}), \Psi_2^{-1}(\frac{1}{\lambda})\}} \|g\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^\circ. \quad (4.8)$$

On the other hand, using (4.8), we obtain

$$\begin{aligned} |\langle f_\lambda, g \rangle| &= \lambda^{-1} |\langle f, g_{1/\lambda} \rangle| \\ &\leq 4\lambda^{-1} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \|g_{1/\lambda}\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^\circ \\ &\leq \lambda^{-1} \frac{1}{\max\{\Psi_1^{-1}(\frac{1}{\lambda}), \Psi_2^{-1}(\frac{1}{\lambda})\}} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \|g\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^\circ. \end{aligned}$$

Taking supremum over $\|g\|_{W(L^{\Psi_1}(\mathbb{A}), L^{\Psi_2}(\mathbb{A}))}^\circ = 1$, we get

$$\|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \leq \lambda^{-1} \frac{1}{\Psi_2^{-1}(\frac{1}{\lambda})} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ. \quad (4.9)$$

By the concavity of Ψ_2^{-1} , we have $\frac{1}{\Psi_2^{-1}(\frac{1}{\lambda})} \leq \Psi_2^{-1}(\lambda)$ for $\lambda \geq 1$. From (4.9), we obtain

$$\|f_\lambda\|_{W(L^{\Phi_1}, L^{\Phi_2})}^\circ \lesssim \lambda^{-1} \Psi_2^{-1}(\lambda) \|f\|_{W(L^{\Phi_1}, L^{\Phi_2})}^\circ.$$

By [22, Proposition 2.1.1], it follows that $u \leq \Phi^{-1}(u)\Psi^{-1}(u) < 2u$ for any $u > 0$ and any complementary Young pair (Φ, Ψ) . Hence we have

$$\begin{aligned} \|f_\lambda\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ &\lesssim 2\lambda^{-1}\lambda \frac{1}{\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \\ &\lesssim \frac{1}{\Phi_2^{-1}(\lambda)} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ \\ &= \frac{1}{\min\{\Phi_1^{-1}(\lambda), \Phi_2^{-1}(\lambda)\}} \|f\|_{W(L^{\Phi_1}(\mathbb{A}), L^{\Phi_2}(\mathbb{A}))}^\circ, \end{aligned}$$

which is (4.7).

The other cases can be handled in a similar way (see [3]). □

Acknowledgment: The author would like to thank Professor S. Öztop for a critical reading of the manuscript and helpful suggestions on the subject. The author is grateful to an anonymous referee for careful reading of the manuscript and for providing helpful comments.

Funding: This study was funded by Scientific Research Projects Coordination Unit of İstanbul University, project number FBA-2023-398-40.

References

- [1] B. Aris, Orlicz amalgam spaces on the affine group, *Hacet. J. Math. Stat.* **54** (2025), no. 2, 529–541.
- [2] B. Aris and S. Öztop, Wiener amalgam spaces with respect to Orlicz spaces on the affine group, *J. Pseudo-Differ. Oper. Appl.* **14** (2023), no. 2, Paper No. 23.
- [3] B. Aris, N. Teofanov and S. Öztop, Dilation estimates for Wiener amalgam spaces of Orlicz type (English summary), *Sampl. Theory Signal Process. Data Anal.* **22** (2024), no. 1, Paper No. 12.
- [4] A. Benedek and R. Panzone, The space L^p , with mixed norm, *Duke Math. J.* **28** (1961), 301–324.
- [5] J.-P. Bertrandias, C. Detry and C. Dupuis, Unions et intersections d'espaces L^p invariants par translation ou convolution, *Ann. Inst. Fourier (Grenoble)* **28** (1978), no. 2, 53–84.
- [6] O. Blasco and A. Osançlıoğlu, Notes on bilinear multipliers on Orlicz spaces, *Math. Nachr.* **292** (2019), no. 12, 2522–2536.
- [7] R. C. Busby and H. A. Smith, Product-convolution operators and mixed-norm spaces, *Trans. Amer. Math. Soc.* **263** (1981), no. 2, 309–341.
- [8] E. Cordero and F. Nicola, Sharpness of some properties of Wiener amalgam and modulation spaces, *Bull. Aust. Math. Soc.* **80** (2009), no. 1, 105–116.
- [9] E. Cordero and L. Rodino, *Time-Frequency Analysis of Operators*, De Gruyter Stud. Math. 75, De Gruyter, Berlin, 2020.
- [10] H. G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation, in: *Functional Analysis and Approximation* (Oberwolfach 1980), Internat. Ser. Numer. Math. 60, Birkhäuser, Basel (1981), 153–165.
- [11] H. G. Feichtinger, Banach convolution algebras of Wiener type, in: *Functions, Series, Operators, Vol. I, II* (Budapest 1980), Colloq. Math. Soc. János Bolyai 35, North-Holland, Amsterdam (1983), 509–524.
- [12] H. G. Feichtinger, Banach spaces of distributions defined by decomposition methods. II, *Math. Nachr.* **132** (1987), 207–237.
- [13] H. G. Feichtinger and P. Gröbner, Banach spaces of distributions defined by decomposition methods. I, *Math. Nachr.* **123** (1985), 97–120.
- [14] H. G. Feichtinger and K. H. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions. I, *J. Funct. Anal.* **86** (1989), no. 2, 307–340.
- [15] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 2001.
- [16] C. Heil, *A Basis Theory Primer*, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2011.
- [17] C. Heil and G. Kutyniok, The homogeneous approximation property for wavelet frames, *J. Approx. Theory* **147** (2007), no. 1, 28–46.
- [18] C. Heil and G. Kutyniok, Convolution and Wiener amalgam spaces on the affine group, in: *Recent Advances in Computational Sciences*, World Scientific, Hackensack (2008), 209–217.
- [19] F. Holland, Harmonic analysis on amalgams of L^p and 1^q , *J. Lond. Math. Soc. (2)* **10** (1975), 295–305.

- [20] L. Maligranda, *Orlicz Spaces and Interpolation*, Sem. Math. 5, Universidade Estadual de Campinas, Campinas, 1989.
- [21] A. Osançlıoğlu and S. Öztop, Weighted Orlicz algebras on locally compact groups, *J. Aust. Math. Soc.* **99** (2015), no. 3, 399–414.
- [22] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textb. Pure Appl. Math. 146, Marcel Dekker, New York, 1991.
- [23] N. Wiener, On the representation of functions by trigonometrical integrals, *Math. Z.* **24** (1926), no. 1, 575–616.
- [24] N. Wiener, Tauberian theorems, *Ann. of Math. (2)* **33** (1932), no. 1, 1–100.
- [25] N. Wiener, *The Fourier Integral and Certain of its Applications*, Cambridge Math. Libr., Cambridge University, Cambridge, 1988.