Research Article

Dong Hyun Cho*

A generalized conditional Wiener integral with a drift and its applications

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Abstract: Let $C[0,T]$ denote the space of real-valued continuous functions on $[0,T]$ and let

$$Z_{\vec{e},\infty}(x)=\biggl(x(0),\int_0^T e_1(t)\,dx(t),\int_0^T e_2(t)\,dx(t),\dots\biggr)\quad\text{for } x\in C[0,T],$$

where $\{e_j\}_{j=1}^{\infty}$ is a sequence of appropriate functions on $[0,T]$. In this paper, we derive a simple evaluation formula for calculating Radon–Nikodym derivatives similar to the conditional Wiener integrals of functions on $C[0,T]$ given $Z_{\vec{e},\infty}$ which has an initial weight and a kind of drift. As applications of the formula, we evaluate the derivatives of various functions containing the time integral which is of interest in quantum mechanics, especially in Feynman integration theory.

Keywords: Analogue of Wiener measure, cylinder function, conditional Wiener integral, simple formula for conditional Wiener integral, time integral

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1 Introduction

Let $C_0[0,T]$ denote the Wiener space, the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$. A time integral is simply the Riemann integral of a function of the continuous random variable $X(x,t)=x(t)$ with respect to the parameter t for $x\in C_0[0,T]$. The Feynman–Kac functional on the Wiener space $C_0[0,T]$ is given by $\exp\{-\int_0^T V(t,X(x,t))\,dt\}$ including the time integral, where V is a complex-valued potential. Calculations involving the conditional Wiener integrals of the Feynman–Kac functional are important in the study of the Feynman integral [9], and it can provide a solution of the integrals equation which is formally equivalent to the Schrödinger equation [6]. In particular, when $0=t_0<t_1<t_2<\cdots<T$ and $\xi_j\in\mathbb{R}$ for $j=0,1,2,\dots$, the conditional Wiener integrals of functions involving the time integral, in which the paths pass through ξ_j at each time t_j, are very useful in describing the Brownian motion. As for one of simple formulas used to calculate the conditional Wiener integrals stated above, Park and Skoug [10] derived a simple formula for conditional Wiener integrals containing the time integral with the conditioning function $(\int_0^T e_1(t)\,dx(t),\int_0^T e_2(t)\,dx(t),\dots)$ for $x\in C_0[0,T]$, where the e_j are in $L^2[0,T]$. In their simple formula, they expressed the conditional Wiener integrals directly in terms of ordinary Wiener integrals, which generalizes the evaluations of conditional Wiener integrals with a finite dimensional conditioning function. We note that the Wiener measure used in [10] has no drifts with the variance function $\beta(t)=t$ for $t\in[0,T]$.

On the other hand, let $C[0,T]$ denote the space of continuous real-valued functions on $[0,T]$. Ryu [12, 13] introduced a finite positive measure $w_{\alpha,\beta;\varphi}$ on $C[0,T]$, where $\alpha,\beta:[0,T]\to\mathbb{R}$ are appropriate functions and φ is a finite positive measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. We note that $w_{\alpha,\beta;\varphi}$ is equivalent to the Wiener measure on $C_0[0,T]$ if $\alpha(t)=0$, $\beta(t)=t$ and $\varphi=\delta_0$, which is the Dirac measure concentrated at 0. When $e_1,\dots,e_n$ are of bounded variations on $[0,T]$ and $Z_{\vec{e},n}(x)=(x(t_0),\int_0^T e_1(t)\,dx(t),\dots,\int_0^T e_n(t)\,dx(t))$ for $x\in C[0,T]$, the author

***Corresponding author: Dong Hyun Cho,** Department of Mathematics, Kyonggi University, Suwon 16227, Republic of Korea, e-mail: j94385@kyonggi.ac.kr. https://orcid.org/0000-0001-8989-6648

of [4] derived a simple evaluation formula for generalized conditional Wiener integrals of functions on $C[0,T]$ with the conditioning function $Z_{\vec{e},n}$. As applications of the formula, he calculated the conditional Wiener integrals of the time integral, the cylinder functions and the functions in a Banach algebra which generalizes the Cameron–Storvick's one [1]. We note that the value space of $Z_{\vec{e},n}$ is finite dimensional.

Let $Z_{\vec{e},\infty}(x) = (x(0), \int_0^T e_1(t)\,dx(t), \int_0^T e_2(t)\,dx(t), \dots)$ for $x \in C[0,T]$, where the e_j are orthonormal in the space of functions that are square integrable with respect to β. In this paper, we derive a simple evaluation formula for calculating Radon–Nikodym derivatives similar to the conditional Wiener integrals of functions on $C[0,T]$ with the conditioning function $Z_{\vec{e},\infty}$. Regarding the applications of the formula, we evaluate the derivatives of various functions involving the time integral, such as cylinder-type functions and especially the function $\exp\{\int_0^T x(t)\,d\beta(t)\}$ on $C[0,T]$, which is a specific type of the Feynman–Kac functional and plays a significant role in Feynman integration theory. We note that the value space of $Z_{\vec{e},\infty}$ is infinite dimensional and $Z_{\vec{e},\infty}$ has a kind of drift with the more generalized variance function β. Furthermore, while every path in $C_0[0,T]$ starts at the origin, the paths in our underlying space $C[0,T]$ may not. Our measure in this paper may not be a probability measure, so that the results of this work generalize those of [4, 10].

To summarize, with the conditioning function $Z_{\vec{e},\infty}$ on $C[0,T]$, this paper extends the simple evaluation formula in [4] stated above. Using the extended formula, we will evaluate the Radon–Nikodym derivatives of various functions involving time integrals, particularly focusing on a specific type of Feynman–Kac functional $\exp\{\int_0^T x(t)\,d\beta(t)\}$.

2 A generalized analogue of Wiener space

Let $C[0,T]$ denote the space of continuous real-valued functions on the interval $[0,T]$. Let α be absolutely continuous on $[0,T]$ and let β be continuous, strictly increasing on $[0,T]$. For $\vec{t}_k = (t_0, t_1, \dots, t_k)$ with $0 = t_0 < t_1 < \cdots < t_k \le T$, let $J_{\vec{t}_k} : C[0,T] \to \mathbb{R}^{k+1}$ be the function given by $J_{\vec{t}_k}(x) = (x(t_0), x(t_1), \dots, x(t_k))$. For any Borel set $B_j (j = 0, 1, \dots, k)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}_k}^{-1}(\prod_{j=0}^k B_j)$ of $C[0,T]$ is called an interval I and let $\mathcal{I}$ be the set of all such intervals I. Define a premeasure $m_{\alpha,\beta;\varphi}$ on $\mathcal{I}$ by

$$m_{\alpha,\beta;\varphi}(I) = \int_{B_0} \int_{\prod_{j=1}^k B_j} \mathcal{W}(\vec{t}_k, \vec{u}_k, u_0)\, dm_L^k(\vec{u}_k)\, d\varphi(u_0),$$

where m_L denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, and for $u_0 \in \mathbb{R}$, $\vec{u}_k = (u_1, \dots, u_k) \in \mathbb{R}^k$,

$$\mathcal{W}(\vec{t}_k, \vec{u}_k, u_0) = \left[\frac{1}{\prod_{j=1}^k 2\pi[\beta(t_j) - \beta(t_{j-1})]}\right]^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^k \frac{[u_j - \alpha(t_j) - u_{j-1} + \alpha(t_{j-1})]^2}{\beta(t_j) - \beta(t_{j-1})}\right\}.$$

The Borel σ-algebra $\mathcal{B}(C[0,T])$ of $C[0,T]$ with the supremum norm coincides with the smallest σ-algebra generated by $\mathcal{I}$, and there exists a unique positive finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for all $I \in \mathcal{I}$. This measure $w_{\alpha,\beta;\varphi}$ is called a generalized analogue of Wiener measure on $(C[0,T], \mathcal{B}(C[0,T]))$ according to φ (see [12, 13]).

Let $\nu_{\alpha,\beta}$ denote the Lebesgue–Stieltjes measure defined by $\nu_{\alpha,\beta}(E) = \int_E d(|\alpha| + \beta)(t)$ for each Lebesgue measurable subset E of $[0,T]$, where $|\alpha|$ denotes the total variation of α. Define $L^2_{\alpha,\beta}[0,T]$ to be the space of functions on $[0,T]$ that are square-integrable with respect to $\nu_{\alpha,\beta}$ (see [11]); that is,

$$L^2_{\alpha,\beta}[0,T] = \left\{f : [0,T] \to \mathbb{R} \;\middle|\; \int_0^T [f(t)]^2\, d\nu_{\alpha,\beta}(t) < \infty\right\}.$$

The space $L^2_{\alpha,\beta}[0,T]$ is a (real) Hilbert space and has the inner product

$$\langle f, g\rangle_{\alpha,\beta} = \int_0^T f(t)g(t)\, d\nu_{\alpha,\beta}(t).$$

We note that $L^2_{\alpha,\beta}[0,T] \subseteq L^2_{0,\beta}[0,T]$, where $L^2_{0,\beta}[0,T]$ denotes the space $L^2_{\alpha,\beta}[0,T]$ with $\alpha \equiv 0$. Throughout the remainder of this paper, we give additional conditions for α and β. Assume that $\beta' > 0$ and $\frac{|\alpha|'}{\beta'}$ is bounded. In this case, $L^2_{\alpha,\beta}[0,T] = L^2_{0,\beta}[0,T]$, and the equality means that they are equal as vector spaces and the two norms on them are equivalent so that they have the same topology, but that they need not be equal isometrically.

Let $S[0,T]$ be the collection of step functions on $[0,T]$ and let $\int_0^T \phi(t)\,dx(t)$ denote the Riemann–Stieltjes integral. For $f \in L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}$ be a sequence of step functions in $S[0,T]$ with $\lim_{n\to\infty}\|\phi_n - f\|_{\alpha,\beta} = 0$. Define $I_{\alpha,\beta}(f)$ by the $L^2(C[0,T])$-limit

$$I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty} \int_0^T \phi_n(t)\,dx(t)$$

for all $x \in C[0,T]$ for which this limit exists or $I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty}\int_0^T \phi_n(t)\,dx(t)$ pointwisely if exists. We note that for $f \in L^2_{\alpha,\beta}[0,T]$, $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Moreover, we have the following theorems [2].

Theorem 2.1. *If f is of bounded variation on $[0,T]$, then $I_{\alpha,\beta}(f)(x) = \int_0^T f(t)\,dx(t)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.*

Throughout this paper, for $x \in C[0,T]$, we redefine $I_{\alpha,\beta}(f)(x) = \int_0^T f(t)\,dx(t)$ if $\int_0^T f(t)\,dx(t)$ exists.

Theorem 2.2. *Let $f, g \in L^2_{\alpha,\beta}[0,T]$. Then we have the following:*
(1) $\int_{C0,T]} I_{\alpha,\beta}(f)(x)\,dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R}) I_{\alpha,\beta}(f)(\alpha)$,
(2) $\int_{C[0,T]} [I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]\,dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\langle f,g\rangle_{0,\beta} + [I_{\alpha,\beta}(f)(\alpha)][I_{\alpha,\beta}(g)(\alpha)]]$,
(3) *$I_{\alpha,\beta}(f)$ is Gaussian with the mean $I_{\alpha,\beta}(f)(\alpha)$ and the variance $\|f\|^2_{0,\beta}$ if $\varphi(\mathbb{R}) = 1$. In this case, the covariance of $I_{\alpha,\beta}(f)$ and $I_{\alpha,\beta}(g)$ is given by $\langle f,g\rangle_{0,\beta}$.*

Theorem 2.3. *Let $\{f_1,\dots,f_n\}$ be a set of functions in $L^2_{\alpha,\beta}[0,T]$, which are nonzero and orthogonal in $L^2_{0,\beta}[0,T]$. Then, for a Borel measurable function $f:\mathbb{R}^n \to \mathbb{C}$,*

$$\begin{aligned}&\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x),\dots,I_{\alpha,\beta}(f_n)(x))\,dw_{\alpha,\beta;\varphi}(x)\\ &\overset{*}{=} \varphi(\mathbb{R})\bigg[\prod_{j=1}^n \frac{1}{2\pi\|f_j\|^2_{0,\beta}}\bigg]^{\frac12} \int_{\mathbb{R}^n} f(\vec{u}) \exp\bigg\{-\frac12\sum_{j=1}^n \frac{[u_j - I_{\alpha,\beta}(f_j)(\alpha)]^2}{\|f_j\|^2_{0,\beta}}\bigg\}\,dm_L^n(\vec{u}),\end{aligned}$$

where $\vec{u} = (u_1,\dots,u_n)$, and $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal. Moreover, if $\varphi(\mathbb{R}) = 1$, then $I_{\alpha,\beta}(f_1),\dots,I_{\alpha,\beta}(f_n)$ are independent.

Let W be an $\mathbb{R}^{\aleph_0}$-valued Borel measurable function defined for $w_{\alpha,\beta;\varphi}$ a.e. on $C[0,T]$, and let $F: C[0,T] \to \mathbb{C}$ be integrable. Let m_W be the image measure on the Borel class $\mathcal{B}(\mathbb{R}^{\aleph_0})$ of $\mathbb{R}^{\aleph_0}$ induced by W. By the Radon–Nikodym theorem, there exists an m_W-integrable function ψ defined on $\mathbb{R}^{\aleph_0}$ which is unique up to m_W a.e. such that for every $B \in \mathcal{B}(\mathbb{R}^{\aleph_0})$,

$$\int_{W^{-1}(B)} F(x)\,dw_{\alpha,\beta;\varphi}(x) = \int_B \psi(\vec{\xi})\,dm_W(\vec{\xi}).$$

Define the function ψ as a generalized conditional Wiener integral of F for a given W and denote it by $GE[F|W]$. We note that $GE[F|W]$ is a Radon–Nikodym derivative rather than a conditional expectation (or a conditional Wiener integral) since m_W need not be a probability measure.

3 A simple formula for the generalized conditional Wiener integrals

In this section, we derive a simple evaluation formula for the generalized conditional Wiener integral as defined in the previous section.

Let $\{e_j : j = 1,2,\dots\}$ be an orthonormal (preferably not complete) subset of $L^2_{0,\beta}[0,T]$ such that it is orthogonal in $L^2_{\alpha,\beta}[0,T]$. Such a set always exists for the following reason.

Example 3.1. Let $\{t_n\}_{n=0}^{\infty}$ be a strictly increasing sequence in the interval $[0, T]$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = T$. For $j = 1, 2, \ldots$, let

$$g_j(s) = \frac{1}{\sqrt{\beta(t_j) - \beta(t_{j-1})}} \chi_{[t_{j-1}, t_j]}(s) \quad \text{for } s \in [0, T]. \tag{3.1}$$

It is clear that the set $\{g_1, g_2, \ldots\}$ is an orthonormal subset of $L^2_{0,\beta}[0, T]$ such that it is orthogonal in $L^2_{\alpha,\beta}[0, T]$. We also note that each g_j is of bounded variation on $[0, T]$.

Let V be the closure of the subspace of $L^2_{0,\beta}[0, T]$ generated by $\{e_1, e_2, \ldots\}$ and let $V^\perp$ be the orthogonal complement of V. Let $\mathcal{P}_{\vec{e},\infty,\beta} : L^2_{0,\beta}[0, T] \to V$ and $\mathcal{P}^\perp_{\vec{e},\infty,\beta} : L^2_{0,\beta}[0, T] \to V^\perp$ be the orthogonal projections. By [7, Problem 8, p. 167], $\mathcal{P}_{\vec{e},\infty,\beta}$ and $\mathcal{P}^\perp_{\vec{e},\infty,\beta}$ exist, and they can be expressed by

$$\mathcal{P}_{\vec{e},\infty,\beta} v = \sum_{j=1}^{\infty} \langle v, e_j \rangle_{0,\beta} e_j, \quad \mathcal{P}^\perp_{\vec{e},\infty,\beta} v = v - \mathcal{P}_{\vec{e},\infty,\beta} v \quad \text{for } v \in L^2_{0,\beta}[0, T].$$

Since $\|\cdot\|_{0,\beta}$ and $\|\cdot\|_{\alpha,\beta}$ are equivalent, V is also closed under the norm $\|\cdot\|_{\alpha,\beta}$. Moreover, we have the following lemma.

Lemma 3.2. *For $v \in L^2_{0,\beta}[0, T]$, we have*

$$I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(\alpha) = \sum_{j=1}^{\infty} \langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(\alpha) \tag{3.2}$$

and there exists $M > 0$ such that

$$|I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(\alpha)|^2 \le M \sum_{j=1}^{\infty} \langle v, e_j \rangle^2_{0,\beta} \le M \|v\|^2_{0,\beta}.$$

Proof. For any positive integer n, by the Hölder's inequality, we have

$$\left| I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(\alpha) - \sum_{j=1}^{n} \langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(\alpha) \right|^2 \le \nu_{\alpha,\beta}([0, T]) \left\| \mathcal{P}_{\vec{e},\infty,\beta} v - \sum_{j=1}^{n} \langle v, e_j \rangle_{0,\beta} e_j \right\|^2_{\alpha,\beta}.$$

Letting $n \to \infty$, we have (3.2) since $\|\cdot\|_{0,\beta}$ and $\|\cdot\|_{\alpha,\beta}$ are equivalent. Take $M_1 > 0$ with $\|f\|_{\alpha,\beta} \le M_1 \|f\|_{0,\beta}$ for all $f \in L^2_{0,\beta}[0, T]$. By (3.2), from the orthogonality of the e_j with respect to $\nu_{\alpha,\beta}$ we have

$$\begin{aligned} |I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(\alpha)|^2 &= \lim_{n\to\infty} \left| I_{\alpha,\beta}\left(\sum_{j=1}^{n} \langle v, e_j \rangle_{0,\beta} e_j \right)(\alpha) \right|^2 \\ &\le \nu_{\alpha,\beta}([0, T]) \lim_{n\to\infty} \sum_{j=1}^{n} \langle v, e_j \rangle^2_{0,\beta} \|e_j\|^2_{\alpha,\beta} \\ &\le M_1^2 \nu_{\alpha,\beta}([0, T]) \sum_{j=1}^{\infty} \langle v, e_j \rangle^2_{0,\beta}. \end{aligned}$$

Let $M = M_1^2 \nu_{\alpha,\beta}([0, T])$. Then the second inequality of this lemma follows from the Bessel inequality. □

For convenience, let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})} \varphi$ throughout this paper. We now provide the theorem which is needed in the sequel.

Theorem 3.3. *Let $v \in L^2_{0,\beta}[0, T]$. Then the series $\sum_{j=1}^{\infty} \langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(x)$ converges for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0, T]$ and*

$$I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(x) = \sum_{j=1}^{\infty} \langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(x).$$

Proof. Since φ_0 is a probability measure, so is $w_{\alpha,\beta;\varphi_0}$. For each positive integer j, let $X_j(x) = \langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0, T]$. By Theorems 2.1, 2.2 and 2.3, X_j is Gaussian with the mean $\langle v, e_j \rangle_{0,\beta} I_{\alpha,\beta}(e_j)(\alpha)$ and the variance $\langle v, e_j \rangle^2_{0,\beta}$, and $\{X_j\}_{j=1}^{\infty}$ is a sequence of independent random variables. Furthermore, we have $\sum_{j=1}^{\infty}\mathrm{Var}[X_j]<\infty$ by Lemma 3.2. By [8, Proposition 2.3.3], $\sum_{j=1}^{\infty}[X_j(x)-E[X_j]]$ converges pointwisely for $w_{\alpha,\beta;\varphi_0}$ a.e. $x\in C[0,T]$. Since $\sum_{j=1}^{\infty}E[X_j]$ converges by Lemma 3.2, it follows that $\sum_{j=1}^{\infty}X_j(x)$ exists for $w_{\alpha,\beta;\varphi_0}$ a.e. $x\in C[0,T]$. Since the null sets with respect to $w_{\alpha,\beta;\varphi}$ are equivalent to the null sets with respect to $w_{\alpha,\beta;\varphi_0}$, it follows that $\sum_{j=1}^{\infty}X_j(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$. Because $\sum_{j=1}^{n}\langle v,e_j\rangle_{0,\beta}e_j$ converges to $\mathcal{P}_{\vec{e},\infty,\beta}v$ in $L^2_{\alpha,\beta}[0,T]$, we have $I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}v)(x)=\sum_{j=1}^{\infty}X_j(x)$ in $L^2(C[0,T])$. We conclude that $I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}v)(x)=\sum_{j=1}^{\infty}X_j(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$ by [2, Corollary 3.11]. □

Let $z_0(x)=x(0)$ for $x\in C[0,T]$. For $j=1,2,\dots$, define z_j and $Z_{\vec{e},\infty}$ by $z_j(x)=I_{\alpha,\beta}(e_j)(x)$ and

$$Z_{\vec{e},\infty}(x)=(z_0(x),z_1(x),z_2(x),\dots)$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$. For $s\in[0,T]$, $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$ and $\vec{\xi}=(\xi_0,\xi_1,\xi_2,\dots)\in\mathbb{R}^{\aleph_0}$, let

$$c_j(s)=\langle e_j,\chi_{[0,s]}\rangle_{0,\beta},\quad x_{\vec{e},\infty,\beta}(s)=z_0(x)+I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,s]})(x)$$

and

$$\vec{\xi}_{\vec{e},\infty,\beta}(s)=\xi_0+\sum_{j=1}^{\infty}\xi_jc_j(s).$$

We have

$$x_{\vec{e},\infty,\beta}(s)=z_0(x)+\sum_{j=1}^{\infty}c_j(s)z_j(x)$$

by Theorem 3.3. Moreover, since $\vec{\xi}_{\vec{e},\infty,\beta}(s)$ is the evaluation of $x_{\vec{e},\infty,\beta}(s)$ for $z_j(x)=\xi_j (j=0,1,\dots)$, $\vec{\xi}_{\vec{e},\infty,\beta}(s)$ exists for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi}\in\mathbb{R}^{\aleph_0}$.

For the continuities of $x_{\vec{e},\infty,\beta}$ and $\vec{\xi}_{\vec{e},\infty,\beta}$, assume that β' is bounded on a subinterval (a,b) of $[0,T]$ throughout the remainder of this work.

Theorem 3.4. *For $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$ and $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi}\in\mathbb{R}^{\aleph_0}$, both $x_{\vec{e},\infty,\beta}$ and $\vec{\xi}_{\vec{e},\infty,\beta}$ belong to $C[0,T]$.*

Proof. Let $\mu_{s,t}=I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[s,t]})(a)$ and $\sigma_{s,t}=\|\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[s,t]}\|_{0,\beta}$ for $a<s\le t<b$. Then according to Theorem 2.3 we have

$$\begin{aligned}I&\equiv\int_{C[0,T]}[I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,s]})(x)-I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,t]})(x)]^4\,dw_{\alpha,\beta;\varphi_0}(x)\\&=\int_{C[0,T]}[I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[s,t]})(x)]^4\,dw_{\alpha,\beta;\varphi_0}(x)\\&=\Big(\frac{1}{2\pi\sigma_{s,t}^2}\Big)^{\frac12}\int_{\mathbb{R}}(u+\mu_{s,t})^4\exp\Big\{-\frac{u^2}{2\sigma_{s,t}^2}\Big\}\,dm_L(u)\\&=3\sigma_{s,t}^4+6\mu_{s,t}^2\sigma_{s,t}^2+\mu_{s,t}^4.\end{aligned}$$

By Lemma 3.2 and the mean value theorem, we get

$$I\le(3+6M+M^2)[\beta(t)-\beta(s)]^2\le M_1^2(3+6M+M^2)(t-s)^2,$$

where for some $M_1\ge0$, $|\beta'|\le M_1$ on (a,b). By [14, Theorem 6.3], we conclude that $I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)$ is continuous for $w_{\alpha,\beta;\varphi_0}$ a.e. $x\in C[0,T]$, and so is $x_{\vec{e},\infty,\beta}$. Since the null sets with respect to $w_{\alpha,\beta;\varphi}$ are equivalent to the null sets with respect to $w_{\alpha,\beta;\varphi_0}$, it follows that $x_{\vec{e},\infty,\beta}$ is continuous for $w_{\alpha,\beta,\varphi}$ a.e. $x\in C[0,T]$. Since $\vec{\xi}_{\vec{e},\infty,\beta}$ is the evaluation of $x_{\vec{e},\infty,\beta}$ for $z_j(x)=\xi_j(j=0,1,\dots)$, it follows that $\vec{\xi}_{\vec{e},\infty,\beta}$ is continuous for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi}\in\mathbb{R}^{\aleph_0}$, completing the proof. □

We note that for $w_{\alpha,\beta;\varphi}$ a.e. $x\in C[0,T]$ and $j=1,2,\dots,$

$$z_j(x)=\int_0^T e_j(u)\,dx(u)$$

and

$$x_{\vec{e},\infty,\beta}(s) = x(0) + \sum_{j=1}^{\infty} c_j(s) \int_0^T e_j(u)\,dx(u)$$

by Theorems 2.1 and 3.3 if each e_j is of bounded variation on $[0,T]$. Moreover, we have the following properties:

(P1) For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and $s \in [0,T]$, by Theorem 2.2, we have

$$x(s) - x_{\vec{e},\infty,\beta}(s) = x(s) - x(0) - I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,s]})(x) = I_{\alpha,\beta}(\chi_{[0,s]} - \mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,s]})(x) = I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s]})(x).$$

(P2) For $0 \le s_1 \le s_2 \le T$,

$$\langle \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = \beta(s_1) - \beta(0) - \sum_{j=1}^{\infty} c_j(s_1)c_j(s_2).$$

Theorem 3.5. *If $\varphi(\mathbb{R}) = 1$, then $\{I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v) : v \in L^2_{0,\beta}[0,T]\}$ and z_j are independent for $j = 0, 1, 2, \dots$. In particular, $\{I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s]}) : s \in [0,T]\}$ and z_j are stochastically independent.*

Proof. Since $\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v \in V^{\perp}$ and $e_j \in V$, we have $\langle \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v, e_j\rangle_{0,\beta} = 0$, so that the independence of $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)$ and z_j for $j = 1, 2, \dots$, follows from Theorem 2.2. To complete the proof, it suffices to prove that z_0 and $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)$ are independent. Since $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)$ is defined via $L^2(C[0,T])$-limit, we can take a sequence $\{\phi_n\}_{n=1}^{\infty}$ of step functions in $S[0,T]$ with $\lim_{n\to\infty}\int_0^T \phi_n(t)\,dx(t) = I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(x)$ pointwise for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. For each $n \in \mathbb{N}$, let $\phi_n(t) = \sum_{j=1}^{m_n} d_{n,j}\chi_{I_{n,j}}(t)$ for $t \in [0,T]$, where $d_{n,j} \in \mathbb{R}$ and the intervals $I_{n,j} \subseteq [0,T]$ with endpoints $t_{n,j-1}$ and $t_{n,j}$ are mutually disjoint. Let $\mathcal{F}$ denote the Fourier transform. Then, for $\xi_1, \xi_2 \in \mathbb{R}$, by Theorem 2.2, [5, Lemma 3] and the dominated convergence theorem, we get

$$\begin{aligned}
\mathcal{F}(z_0, I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v))(\xi_1,\xi_2) &= \int_{C[0,T]} \exp\{i[\xi_1 x(0) + \xi_2 I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(x)]\}\,dw_{\alpha,\beta;\varphi}(x)\\
&= \int_{C[0,T]} \exp\Big\{i\Big[\xi_1 x(0) + \xi_2 \lim_{n\to\infty}\int_0^T \phi_n(t)\,dx(t)\Big]\Big\}\,dw_{\alpha,\beta;\varphi}(x)\\
&= \lim_{n\to\infty}\int_{C[0,T]} \exp\Big\{i\Big[\xi_1 x(0) + \xi_2 \sum_{j=1}^{m_n} d_{n,j}[x(t_{n,j}) - x(t_{n,j-1})]\Big]\Big\}\,dw_{\alpha,\beta;\varphi}(x)\\
&= \mathcal{F}(z_0)(\xi_1)\int_{C[0,T]} \exp\Big\{i\xi_2 \lim_{n\to\infty}\int_0^T \phi_n(t)\,dx(t)\Big\}\,dw_{\alpha,\beta;\varphi}(x)\\
&= \mathcal{F}(z_0)(\xi_1)\int_{C[0,T]} \exp\{i\xi_2 I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(x)\}\,dw_{\alpha,\beta;\varphi}(x)\\
&= \mathcal{F}(z_0)(\xi_1)\mathcal{F}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\xi_2),
\end{aligned}$$

which completes the proof. □

By Theorem 3.5, we have the following corollary.

Corollary 3.6. *The stochastic processes $\{I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s]}) : 0 \le s \le T\}$ and $\{x_{\vec{e},\infty,\beta}(s) : 0 \le s \le T\}$ are stochastically independent if $\varphi(\mathbb{R}) = 1$.*

Using the same process used in the proof of [10, Theorem 2] and [5, Theorem 4] with aid of (P1), Theorem 3.5 and Corollary 3.6, we obtain the following theorem.

Theorem 3.7. *If $F : C[0,T] \to \mathbb{C}$ is integrable, then for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$ we have*

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \int_{C[0,T]} F(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x) + \vec{\xi}_{\vec{e},\infty,\beta})\,dw_{\alpha,\beta;\varphi_0}(x).$$

Remark 3.8. We note that if $V = L^2_{0,\beta}[0,T]$, that is, $\{e_1, e_2, \dots\}$ is completely orthonormal in $L^2_{0,\beta}[0,T]$, then $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x) = 0$ so that for $Z_{\vec{e},\infty}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, $GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = F(\vec{\xi}_{\vec{e},\infty,\beta})$ by Theorem 3.7.

Let $\{t_n\}_{n=0}^{\infty}$ be a strictly increasing sequence in $[0,T]$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = T$. Let $\mathcal{A}_T$ be the set of all convergent sequences $\vec{\xi} = (\xi_0, \xi_1, \xi_2, \ldots) \in \mathbb{R}^{\aleph_0}$ with $\lim_{n\to\infty} \xi_n \equiv \xi_T$. For $s, t \in [t_{j-1}, t_j]$, let $\gamma_j(t) = \frac{\beta(t)-\beta(t_{j-1})}{\beta(t_j)-\beta(t_{j-1})}$ and $\Phi_j(s,t) = [\beta(t_j) - \beta(s)]\gamma_j(t)$. For $s \in [0,T]$ and $\vec{\xi} = (\xi_0, \xi_1, \xi_2, \ldots) \in \mathbb{R}^{\aleph_0}$, let

$$\Xi(\infty, \vec{\xi})(s) = \xi_0 + \sum_{j=1}^{\infty} \chi_{(t_{j-1},t_j]}(s)\bigg[\sum_{l=1}^{j-1} \xi_l \sqrt{\beta(t_l) - \beta(t_{l-1})} + \frac{\beta(s) - \beta(t_{j-1})}{\sqrt{\beta(t_j) - \beta(t_{j-1})}}\xi_j\bigg] + \chi_{\{T\}}(s) \sum_{l=1}^{\infty} \xi_l \sqrt{\beta(t_l) - \beta(t_{l-1})} \quad \text{(if exists)}.$$

For $x \in C[0,T]$, define the polygonal function $P_{\infty,\beta}(x)$ of x by

$$P_{\infty,\beta}(x)(s) = \chi_{\{0\}}(s)x(t_0) + \chi_{\{T\}}(s)x(T) + \sum_{j=1}^{\infty} \chi_{(t_{j-1},t_j]}(s)[x(t_{j-1}) + \gamma_j(s)[x(t_j) - x(t_{j-1})]] \tag{3.3}$$

for $s \in [0,T]$. Similarly, for $\vec{\xi} = (\xi_0, \xi_1, \xi_2, \ldots) \in \mathcal{A}_T$, the polygonal function $P_{\infty,\beta}(\vec{\xi})$ of $\vec{\xi}$ on $[0,T]$ is defined by (3.3) with replacing $x(t_0)$, $x(t_j)$ and $x(T)$ by ξ_0, ξ_j and ξ_T, respectively. Throughout this paper, we will use the notation $\vec{g}$ in place of $\vec{e}$ when e_j is replaced by g_j which is given by (3.1). We note that $\Xi(\infty, \vec{\xi})$, $P_{\infty,\beta}(x)$ and $P_{\infty,\beta}(\vec{\xi})$ belong to $C[0,T]$ if they exist.

Theorem 3.9. *Let $F : C[0,T] \to \mathbb{C}$ be integrable. Define $X_\infty : C[0,T] \to \mathbb{R}^{\aleph_0}$ by $X_\infty(x) = (x(t_0), x(t_1), x(t_2), \ldots)$. Then:*

(1) *For $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have*

$$GE[F|Z_{\vec{g},\infty}](\vec{\xi}) = \int_{C[0,T]} F(x - P_{\infty,\beta}(x) + \Xi(\infty, \vec{\xi}))\, dw_{\alpha,\beta;\varphi_0}(x), \tag{3.4}$$

where $m_{Z_{\vec{g},\infty}}$ is the measure on $\mathcal{B}(\mathbb{R}^{\aleph_0})$ induced by $Z_{\vec{g},\infty}$.

(2) *For m_{X_∞} a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$,*

$$GE[F|X_\infty](\vec{\xi}) = \int_{C[0,T]} F(x - P_{\infty,\beta}(x) + P_{\infty,\beta}(\vec{\xi}))\, dw_{\alpha,\beta;\varphi_0}(x). \tag{3.5}$$

Proof. For $s \in [0,T)$, we have $x_{\vec{g},\infty,\beta}(s) = P_{\infty,\beta}(x)(s)$ and $\vec{\xi}_{\vec{g},\infty,\beta}(s) = \Xi(\infty, \vec{\xi})(s)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ by [4, Corollary 3.5]. If $s = T$, then

$$\begin{aligned}
x_{\vec{g},\infty,\beta}(T) &= z_0(x) + \sum_{l=1}^{\infty} c_l(T) I_{\alpha,\beta}(g_l)(x) \\
&= x(0) + \lim_{n\to\infty} \sum_{l=1}^{n} \frac{1}{\beta(t_l) - \beta(t_{l-1})} \int_0^T \chi_{[t_{l-1},t_l]}(u)\, d\beta(u) \int_0^T \chi_{[t_{l-1},t_l]}(u)\, dx(u) \\
&= x(0) + \lim_{n\to\infty} \sum_{l=1}^{n} [x(t_l) - x(t_{l-1})] \\
&= \lim_{n\to\infty} x(t_n) = x(T) = P_{\infty,\beta}(x)(T)
\end{aligned}$$

and for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} = (\xi_0, \xi_1, \ldots) \in \mathbb{R}^{\aleph_0}$, we obtain

$$\begin{aligned}
\vec{\xi}_{\vec{g},\infty,\beta}(T) &= \xi_0 + \sum_{l=1}^{\infty} \xi_l c_j(T) \\
&= \xi_0 + \sum_{l=1}^{\infty} \frac{\xi_l}{\sqrt{\beta(t_l) - \beta(t_{l-1})}} \int_0^T \chi_{[t_{l-1},t_l]}(u)\, d\beta(u) \\
&= \xi_0 + \sum_{l=1}^{\infty} \xi_l \sqrt{\beta(t_l) - \beta(t_{l-1})} = \Xi(\infty, \vec{\xi})(T)
\end{aligned}$$

so that $x_{\vec{g},\infty,\beta} = P_{\infty,\beta}(x)$ and $\vec{\xi}_{\vec{g},\infty,\beta} = \Xi(\infty, \vec{\xi})$. By (P1) and Theorem 3.7, we get (3.4).

To prove (3.5), define $\phi : \mathbb{R}^{\aleph_0} \to \mathbb{R}^{\aleph_0}$ by

$$\phi(\vec{\xi}) = \Big(\xi_0, \frac{\xi_1 - \xi_0}{\sqrt{\beta(t_1) - \beta(t_0)}}, \frac{\xi_2 - \xi_1}{\sqrt{\beta(t_2) - \beta(t_1)}}, \dots\Big) \tag{3.6}$$

for $\vec{\xi} = (\xi_0, \xi_1, \xi_2, \dots) \in \mathbb{R}^{\aleph_0}$, which is a bijective, bicontinuous function. By [4, Corollary 3.6], it suffices to prove that $\Xi(\infty, \phi(\vec{\xi}))(T) = P_{\infty,\beta}(\vec{\xi})(T)$. Indeed, we have

$$\begin{aligned}\Xi(\infty, \phi(\vec{\xi}))(T) &= \xi_0 + \lim_{n\to\infty} \sum_{l=1}^{n} \sqrt{\beta(t_l) - \beta(t_{l-1})} \frac{\xi_l - \xi_{l-1}}{\sqrt{\beta(t_l) - \beta(t_{l-1})}} \\ &= \lim_{n\to\infty} \xi_n = \xi_T = P_{\infty,\beta}(\vec{\xi})(T),\end{aligned}$$

as desired. □

Remark 3.10. Since $\vec{\xi}_{\vec{g},\infty,\beta}(T)$ is the evaluation of $x_{\vec{g},\infty,\beta}(T)$ for $z_j(x) = \xi_j$, $j = 0, 1, 2, \dots$, it follows that the series $\sum_{l=1}^{\infty} \xi_l \sqrt{\beta(t_l) - \beta(t_{l-1})}$ in the proof of Theorem 3.9 converges for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$. Similarly, because $\Xi(\infty, \phi(\vec{\xi}))(T)$ is the evaluation of $x_{\vec{g},\infty,\beta}(T)$ for $x(t_j) = \xi_j$, $j = 0, 1, 2, \dots$, the sequence $\vec{\xi}$ in (3.6) converges for m_{X_∞} a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$. Hence, such a $\vec{\xi}$ for which $GE[F|X_\infty](\vec{\xi})$ exists, belongs to $\mathcal{A}_T$.

4 Applications to the cylinder-type functions

In this section, we apply the simple formulas given in the previous section to obtain the Radon–Nikodym derivatives of various functions, in particular, special types of the Feynman–Kac functional.

Since $\mathcal{P}_{\vec{e},\infty,\beta}$ is an orthogonal projection, it is self-adjoint, that is, $\mathcal{P}^2_{\vec{e},\infty,\beta} = \mathcal{P}_{\vec{e},\infty,\beta}$ and $\mathcal{P}^*_{\vec{e},\infty,\beta} = \mathcal{P}_{\vec{e},\infty,\beta}$, we have the following lemma.

Lemma 4.1. *For $v \in L^2_{0,\beta}[0, T]$, we have*

$$\langle v, \mathcal{P}_{\vec{e},\infty,\beta} v\rangle_{0,\beta} = \|\mathcal{P}_{\vec{e},\infty,\beta} v\|^2_{0,\beta} = \sum_{j=1}^{\infty} \langle v, e_j\rangle^2_{0,\beta}$$

and

$$\|\mathcal{P}^\perp_{\vec{e},\infty,\beta} v\|^2_{0,\beta} = \|v - \mathcal{P}_{\vec{e},\infty,\beta} v\|^2_{0,\beta} = \|v\|^2_{0,\beta} - \|\mathcal{P}_{\vec{e},\infty,\beta} v\|^2_{0,\beta}.$$

Lemma 4.2. *For $v \in L^2_{\alpha,\beta}[0, T]$, we have*

$$\begin{aligned}I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) &= I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},\infty,\beta} v)(x) \\ &= I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},\infty,\beta} v)(I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x))\end{aligned}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0, T]$.

Proof. Using a similar process as in the proof of [4, Lemma 5.1], we can prove the first equality. Since $\mathcal{P}^\perp_{\vec{e},\infty,\beta}$ is an orthogonal projection, we have $(\mathcal{P}^\perp_{\vec{e},\infty,\beta})^2 = \mathcal{P}^\perp_{\vec{e},\infty,\beta}$. Now, replacing v by $\mathcal{P}^\perp_{\vec{e},\infty,\beta} v$ in the first equality, we obtain the second equality, which completes the proof. □

Theorem 4.3. *For $v \in L^2_{\alpha,\beta}[0, T]$, we have*

$$\begin{aligned}I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) &= I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(x) \\ &= I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x))\end{aligned}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0, T]$.

Proof. By Theorem 2.1 and the first equality of Lemma 4.2,

$$\begin{aligned}I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) &= I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\chi_{[0,\cdot]} - \mathcal{P}^\perp_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) \\ &= I_{\alpha,\beta}(v)(x - x(0)) - I_{\alpha,\beta}(v)(\mathcal{P}^\perp_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) \\ &= I_{\alpha,\beta}(v)(x) - I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},\infty,\beta} v)(x) = I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta} v)(x),\end{aligned}$$

which proves the first equality of the theorem. Since $\mathcal{P}^2_{\vec{e},\infty,\beta} = \mathcal{P}_{\vec{e},\infty,\beta}$, the second equality immediately follows from the first equality of the theorem, completing the proof. □

Remark 4.4. Replacing $\mathcal{P}^{\perp}_{\vec{e},n,\beta}$ by $\mathcal{P}_{\vec{e},\infty,\beta}$ in the proof of [4, Lemma 5.1], with aid of Lemma 4.1, we can also prove the first equality of Theorem 4.3.

Example 4.5. Let $v \in L^2_{\alpha,\beta}[0,T]$. If $v \notin V^{\perp}$, then by Theorems 2.2, 2.3 and 4.3, we have

$$\begin{aligned}\int_{C[0,T]} &\exp\{I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x))\}\,dw_{\alpha,\beta;\varphi}(x)\\ &= \Big[\frac{\varphi(\mathbb{R})^2}{2\pi\|\mathcal{P}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\Big]^{\frac12}\int_{\mathbb{R}}\exp\Big\{u - \frac{[u - I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}v)(\alpha)]^2}{2\|\mathcal{P}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\Big\}\,dm_L(u)\\ &= \varphi(\mathbb{R})\exp\Big\{\frac12\|\mathcal{P}_{\vec{e},\infty,\beta}v\|^2_{0,\beta} + I_{\alpha,\beta}(\mathcal{P}_{\vec{e},\infty,\beta}v)(\alpha)\Big\}.\end{aligned}$$

Note that the result holds for $v \in V^{\perp}$.

Example 4.6. Let $v \in L^2_{\alpha,\beta}[0,T]$, let $f : \mathbb{R} \to \mathbb{C}$ be Borel measurable and let $F(x) = f(I_{\alpha,\beta}(v)(x))$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Suppose that f is m_L-integrable. By Theorem 3.7 and Lemma 4.2, for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have

$$\begin{aligned}GE[F|Z_{\vec{e},\infty}](\vec{\xi}) &= \int_{C[0,T]} f(I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) + I_{\alpha,\beta}(v)(\vec{\xi}_{\vec{e},\infty,\beta}))\,dw_{\alpha,\beta;\varphi_0}(x)\\ &= \int_{C[0,T]} f(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(x) + I_{\alpha,\beta}(v)(\vec{\xi}_{\vec{e},\infty,\beta}))\,dw_{\alpha,\beta;\varphi_0}(x).\end{aligned}$$

If $v \notin V$, then, by Theorems 2.2 and 2.3,

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \Big[\frac{1}{2\pi\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\Big]^{\frac12}\int_{\mathbb{R}} f(u)\exp\Big\{-\frac{[u - I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\alpha) - I_{\alpha,\beta}(v)(\vec{\xi}_{\vec{e},\infty,\beta})]^2}{2\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\Big\}\,dm_L(u).$$

Example 4.7. Let $\{v_1,\dots,v_r\}$ be a subset of $L^2_{\alpha,\beta}[0,T]$. Let $f : \mathbb{R}^r \to \mathbb{C}$ be Borel measurable and let

$$F(x) = f(I_{\alpha,\beta}(v_1)(x),\dots,I_{\alpha,\beta}(v_r)(x))$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Suppose that f is m^r_L-integrable. Then for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \int_{C[0,T]} f(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v_1)(x) + I_{\alpha,\beta}(v_1)(\vec{\xi}_{\vec{e},\infty,\beta}),\dots,I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v_r)(x) + I_{\alpha,\beta}(v_r)(\vec{\xi}_{\vec{e},\infty,\beta}))\,dw_{\alpha,\beta;\varphi_0}(x)$$

by Theorem 3.7 and Lemma 4.2. If $v_l \in V$ for $l = 1,\dots,r$, then

$$\begin{aligned}GE[F|Z_{\vec{e},\infty}](\vec{\xi}) &= f(I_{\alpha,\beta}(v_1)(\vec{\xi}_{\vec{e},\infty,\beta}),\dots,I_{\alpha,\beta}(v_r)(\vec{\xi}_{\vec{e},\infty,\beta}))\\ &= f\Big(\sum_{j=1}^{\infty}\langle v_1, e_j\rangle_{0,\beta}\xi_j,\dots,\sum_{j=1}^{\infty}\langle v_r, e_j\rangle_{0,\beta}\xi_j\Big)\end{aligned}$$

for $\vec{\xi} = (\xi_0,\xi_1,\dots) \in \mathbb{R}^{\aleph_0}$. In particular, if $v_l = e_{j_l}$ for $l = 1,\dots,r$, then

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = f(\xi_{j_1},\dots,\xi_{j_r}).$$

If $v_l \in V^{\perp}$ for $l = 1,\dots,r$ and they are orthonormal in $L^2_{0,\beta}[0,T]$, then for $\vec{u} = (u_1,\dots,u_r)$ we have

$$\begin{aligned}GE[F|Z_{\vec{e},\infty}](\vec{\xi}) &= \int_{C[0,T]} f(I_{\alpha,\beta}(v_1)(x) + I_{\alpha,\beta}(v_1)(\vec{\xi}_{\vec{e},\infty,\beta}),\dots,I_{\alpha,\beta}(v_r)(x) + I_{\alpha,\beta}(v_r)(\vec{\xi}_{\vec{e},\infty,\beta}))\,dw_{\alpha,\beta;\varphi_0}(x)\\ &= \Big[\frac{1}{2\pi}\Big]^{\frac r2}\int_{\mathbb{R}^r} f(\vec{u})\exp\Big\{-\frac12\sum_{j=1}^{r}[u_j - I_{\alpha,\beta}(v_j)(\alpha)]^2\Big\}\,dm^r_L(\vec{u})\end{aligned}$$

by Theorems 2.2, 2.3 and Lemma 4.2, since $\langle v_l, e_j\rangle_{0,\beta} = 0$ for $l = 1,\dots,r$, $j = 1,2,\dots$. Note that since $\sum_{j=1}^{\infty}\langle v, e_j\rangle_{0,\beta}\xi_j$ ($v \in L^2_{0,\beta}[0,T]$) is the evaluation of $\sum_{j=1}^{\infty}\langle v, e_j\rangle_{0,\beta}z_j(x)$ for $z_j(x) = \xi_j$ ($j = 1,2,\dots$), the series $\sum_{j=1}^{\infty}\langle v, e_j\rangle_{0,\beta}\xi_j$ converges for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$ by Theorem 3.3.

Example 4.8. Let $v \in L^2_{\alpha,\beta}[0,T]$, let $f : \mathbb{R} \to \mathbb{C}$ be Borel measurable and let

$$F(x) = f(I_{\alpha,\beta}(v)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)))$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Suppose that f is m_L-integrable. Then for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, by Theorem 3.7 and Lemma 4.2, we have

$$\begin{aligned} GE[F|Z_{\vec{e},\infty}](\vec{\xi}) &= GE[f(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\cdot))|Z_{\vec{e},\infty}](\vec{\xi}) \\ &= \int_{C[0,T]} f(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\vec{\xi}_{\vec{e},\infty,\beta}))\,dw_{\alpha,\beta;\varphi_0}(x) \\ &= \int_{C[0,T]} f(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(x))\,dw_{\alpha,\beta;\varphi_0}(x), \end{aligned}$$

since $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\vec{\xi}_{\vec{e},\infty,\beta}) = \sum_{j=1}^{\infty}\langle \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v, e_j\rangle_{0,\beta}\xi_j = 0$, where $\vec{\xi} = (\xi_0, \xi_1, \ldots)$. If $v \notin V$, then, by Theorem 2.3,

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \left[\frac{1}{2\pi\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\right]^{\frac{1}{2}} \int_{\mathbb{R}} f(u)\exp\left\{-\frac{[u - I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v)(\alpha)]^2}{2\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}v\|^2_{0,\beta}}\right\} dm_L(u).$$

Moreover, if $v \in V$, then $GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = f(0)$ for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$.

Remark 4.9. Replacing e_j and $\vec{\xi}_{\vec{e},\infty,\beta}$ by g_j and $\Xi(\infty,\xi)$, respectively, in Examples 4.6, 4.7 and 4.8, we can obtain $GE[F|Z_{\vec{g},\infty}](\vec{\xi})$ from each expression of $GE[F|Z_{\vec{e},\infty}](\vec{\xi})$ by Theorem 3.9. In particular, $GE[F|X_\infty](\vec{\xi})$ can be obtained from the expression of $GE[F|Z_{\vec{g},\infty}](\phi(\vec{\xi}))$, where ϕ is given by (3.6). In this case, $\vec{\xi}_{\vec{e},\infty,\beta}$ is replaced by $P_{\infty,\beta}(\vec{\xi})$.

Theorem 4.10. *Let $v \in L^2_{\alpha,\beta}[0,T]$ and let $F(x) = \exp\{\langle v, x\rangle_{0,\beta}\}$ for $x \in C[0,T]$. Suppose that*

$$\int_{\mathbb{R}} \exp\left\{u\int_0^T v(s)\,d\beta(s)\right\} d\varphi(u) < \infty.$$

Then F is $w_{\alpha,\beta;\varphi}$-integrable and for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} = (\xi_0, \xi_1, \ldots) \in \mathbb{R}^{\aleph_0}$, we have

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \exp\left\{\frac{1}{2}\left[\|h_v\|^2_{0,\beta} - \sum_{j=1}^{\infty}\langle v, c_j\rangle^2_{0,\beta}\right] + \sum_{j=1}^{\infty}\langle v, c_j\rangle_{0,\beta}[\xi_j - z_j(\alpha)] + I_{\alpha,\beta}(h_v)(\alpha) + \xi_0 h_v(0)\right\},$$

where $h_v(u) = \langle v, \chi_{[u,T]}\rangle_{0,\beta}$ for $u \in [0,T]$.

Proof. For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, by the integration by parts formula, we get

$$\langle v, x\rangle_{0,\beta} = h_v(0)x(0) + I_{\alpha,\beta}(h_v)(x).$$

Since

$$I_{\alpha,\beta}(h_v)(x) = \int_0^T h_v(u)\,dx(u)$$

by Theorem 2.1, we can show that z_0 and $I_{\alpha,\beta}(h_v)$ are independent with respect to $w_{\alpha,\beta;\varphi_0}$, using the same process as in the proof of Theorem 3.5. From this fact we get

$$\begin{aligned} \int_{C[0,T]} F(x)\,dw_{\alpha,\beta;\varphi}(x) &= \varphi(\mathbb{R})\left[\int_{C[0,T]} \exp\{h_v(0)x(0)\}\,dw_{\alpha,\beta;\varphi_0}(x)\right]\left[\int_{C[0,T]} \exp\{I_{\alpha,\beta}(h_v)(x)\}\,dw_{\alpha,\beta;\varphi_0}(x)\right] \\ &= \left[\frac{1}{2\pi\|h_v\|^2_{0,\beta}}\right]^{\frac{1}{2}} \int_{\mathbb{R}^2} \exp\left\{h_v(0)u_0 + u - \frac{[u - I_{\alpha,\beta}(h_v)(\alpha)]^2}{2\|h_v\|^2_{0,\beta}}\right\} dm_L(u)\,d\varphi(u_0) \\ &= \exp\left\{\frac{1}{2}\|h_v\|^2_{0,\beta} + I_{\alpha,\beta}(h_v)(\alpha)\right\}\int_{\mathbb{R}} \exp\{h_v(0)u_0\}\,d\varphi(u_0) < \infty \end{aligned}$$

if $\|h_v\|_{0,\beta} \neq 0$. If $\|h_v\|_{0,\beta} = 0$, then $\|h_v\|_{\alpha,\beta} = 0$ so that F is $w_{\alpha,\beta;\varphi}$-integrable. By Theorem 3.7, Lemma 4.2 and Example 4.6, for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} = (\xi_0, \xi_1, \dots) \in \mathbb{R}^{\aleph_0}$, we have

$$\begin{aligned} GE[F|Z_{\vec{e},\infty}](\vec{\xi}) &= \int_{C[0,T]} F(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x) + \vec{\xi}_{\vec{e},\infty,\beta})\, dw_{\alpha,\beta;\varphi_0}(x) \\ &= \int_{C[0,T]} \exp\{h_v(0)[I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,0]})(x) + \vec{\xi}_{\vec{e},\infty,\beta}(0)] \\ &\qquad + I_{\alpha,\beta}(h_v)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,\cdot]})(x)) + I_{\alpha,\beta}(h_v)(\vec{\xi}_{\vec{e},\infty,\beta})\}\, dw_{\alpha,\beta;\varphi_0}(x) \\ &= \exp\Big\{\frac{1}{2}\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}h_v\|^2_{0,\beta} + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}h_v)(\alpha) + \xi_0 h_v(0) + I_{\alpha,\beta}(h_v)(\vec{\xi}_{\vec{e},\infty,\beta})\Big\}. \end{aligned}$$

By the integration by parts formula,

$$\langle h_v, e_j\rangle_{0,\beta} = c_j(u)h_v(u)|_0^T + \int_0^T c_j(u)v(u)\, d\beta(u) = \langle c_j, v\rangle_{0,\beta}.$$

Therefore, by (3.2) and Lemma 4.1, we obtain

$$\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}h_v\|^2_{0,\beta} = \|h_v\|^2_{0,\beta} - \sum_{j=1}^{\infty}\langle v, c_j\rangle^2_{0,\beta}, \quad I_{\alpha,\beta}(h_v)(\vec{\xi}_{\vec{e},\infty,\beta}) = \sum_{j=1}^{\infty}\langle v, c_j\rangle_{0,\beta}\xi_j$$

and

$$I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}h_v)(\alpha) = I_{\alpha,\beta}(h_v)(\alpha) - \sum_{j=1}^{\infty}\langle v, c_j\rangle_{0,\beta} z_j(\alpha).$$

Consequently, we have

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \exp\Big\{\frac{1}{2}\Big[\|h_v\|^2_{0,\beta} - \sum_{j=1}^{\infty}\langle v, c_j\rangle^2_{0,\beta}\Big] + \sum_{j=1}^{\infty}\langle v, c_j\rangle_{0,\beta}[\xi_j - z_j(\alpha)] + I_{\alpha,\beta}(h_v)(\alpha) + \xi_0 h_v(0)\Big\},$$

completing the proof. □

Corollary 4.11. *Under the assumptions as in Theorem 4.10, for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, $GE[F|Z_{\vec{g},\infty}](\vec{\xi})$ can be expressed by the right-hand side of the equality in Theorem 4.10 with*

$$z_j(\alpha) = \frac{\alpha(t_j) - \alpha(t_{j-1})}{\sqrt{\beta(t_j) - \beta(t_{j-1})}}$$

and

$$\langle v, c_j\rangle_{0,\beta} = \frac{1}{\sqrt{\beta(t_j) - \beta(t_{j-1})}}\int_{t_{j-1}}^{t_j} v(s)[\beta(s) - \beta(t_{j-1})]\, d\beta(s) + h_v(t_j)\sqrt{\beta(t_j) - \beta(t_{j-1})}.$$

In particular, for m_{X_∞} a.e. $\vec{\xi} = (\xi_0, \xi_1, \dots) \in \mathbb{R}^{\aleph_0}$, $GE[F|X_\infty](\vec{\xi})$ is given by the expression of $GE[F|Z_{\vec{g},\infty}](\vec{\xi})$ replacing ξ_j by $\frac{\xi_j - \xi_{j-1}}{\sqrt{\beta(t_j) - \beta(t_{j-1})}}$.

Proof. It is not difficult to show that for $u \in [0, T]$,

$$c_j(u) = \frac{\beta(u) - \beta(t_{j-1})}{\sqrt{\beta(t_j) - \beta(t_{j-1})}}\chi_{[t_{j-1},t_j)}(u) + \chi_{[t_j,T]}(u)\sqrt{\beta(t_j) - \beta(t_{j-1})}.$$

Since

$$z_j(\alpha) = \frac{\alpha(t_j) - \alpha(t_{j-1})}{\sqrt{\beta(t_j) - \beta(t_{j-1})}},$$

the first result immediately follows from Theorem 4.10. The second result follows from the fact that

$$GE[F|X_\infty](\vec{\xi}) = GE[F|Z_{\vec{g},\infty}](\phi(\vec{\xi})),$$

where ϕ is given by (3.6). □

Letting $v \equiv 1$ in Theorem 4.10, we have the following corollary which is one of our main results.

Corollary 4.12. *Suppose that* $\int_{\mathbb{R}} \exp\{u[\beta(T)-\beta(0)]\}\,d\varphi(u) < \infty$. *For* $\xi_0 \in \mathbb{R}$, *let*

$$K(\xi_0) = \exp\Big\{\frac{1}{6}[\beta(T)-\beta(0)]^3 + \beta(T)[\alpha(T)-\alpha(0)] - I_{\alpha,\beta}(\beta)(\alpha) + \xi_0[\beta(T)-\beta(0)]\Big\}.$$

Then, for $m_{Z_{\vec{e},\infty}}$ *a.e.* $\vec{\xi} = (\xi_0, \xi_1, \ldots) \in \mathbb{R}^{\aleph_0}$, *we have*

$$GE\Big[\exp\Big\{\int_0^T x(u)\,d\beta(u)\Big\}\,\Big|\,Z_{\vec{e},\infty}\Big](\vec{\xi}) = K(\xi_0)\exp\Big\{-\frac{1}{2}\sum_{j=1}^{\infty}\Big[\int_0^T c_j(u)\,d\beta(u)\Big]^2 + \sum_{j=1}^{\infty}[\xi_j - z_j(\alpha)]\int_0^T c_j(u)\,d\beta(u)\Big\}.$$

Proof. Let $v \equiv 1$ in Theorem 4.10. Since $h_v(u) = \int_u^T 1\,d\beta(u) = \beta(T)-\beta(u)$, we have

$$\|h_v\|_{0,\beta}^2 = \int_0^T [\beta(T)-\beta(u)]^2\,d\beta(u) = \frac{1}{3}[\beta(T)-\beta(0)]^3$$

and

$$I_{\alpha,\beta}(h_v)(\alpha) = \int_0^T [\beta(T)-\beta(u)]\,d\alpha(u) = \beta(T)[\alpha(T)-\alpha(0)] - I_{\alpha,\beta}(\beta)(\alpha).$$

Moreover, we have $h_v(0) = \beta(T)-\beta(0)$ and $\langle v, c_j\rangle_{0,\beta} = \int_0^T c_j(u)\,d\beta(u)$. Now, the corollary follows from Theorem 4.10. □

Corollary 4.13. *Under the assumptions as in Corollary 4.12, for* $m_{Z_{\vec{g},\infty}}$ *a.e.* $\vec{\xi} = (\xi_0, \xi_1, \ldots) \in \mathbb{R}^{\aleph_0}$, *we have*

$$\begin{aligned} GE\Big[\exp\Big\{\int_0^T x(u)\,d\beta(u)\Big\}\,\Big|\,Z_{\vec{g},\infty}\Big](\vec{\xi}) = K(\xi_0)\exp\Big\{&-\frac{1}{8}\sum_{j=1}^{\infty}[\beta(t_j)-\beta(t_{j-1})][2\beta(T)-\beta(t_j)-\beta(t_{j-1})]^2 \\ &+ \frac{1}{2}\sum_{j=1}^{\infty}[\xi_j - z_j(\alpha)]\sqrt{\beta(t_j)-\beta(t_{j-1})}[2\beta(T)-\beta(t_j)-\beta(t_{j-1})]\Big\}. \end{aligned}$$

In particular, $GE[F|X_\infty](\vec{\xi})$ *is given by the right-hand side of the above equality replacing* ξ_j *by* $\frac{\xi_j-\xi_{j-1}}{\sqrt{\beta(t_j)-\beta(t_{j-1})}}$.

Proof. By Corollary 4.11,

$$\begin{aligned} \int_0^T c_j(u)\,d\beta(u) &= \frac{[\beta(t_j)-\beta(t_{j-1})]^2}{2\sqrt{\beta(t_j)-\beta(t_{j-1})}} + \sqrt{\beta(t_j)-\beta(t_{j-1})}[\beta(T)-\beta(t_j)] \\ &= \frac{1}{2}\sqrt{\beta(t_j)-\beta(t_{j-1})}[2\beta(T)-\beta(t_j)-\beta(t_{j-1})]. \end{aligned}$$

Now, the validity of the corollary follows from Theorem 3.9, Corollaries 4.11 and 4.12. □

Remark 4.14. For the conditioning $Z_{\vec{e},\infty}$, the Radon–Nikodym derivatives of the general type of cylinder functions and the functions in a Banach algebra given in [4] can be expressed by similar forms in [4] with $(e_1, e_2, \ldots, e_n)$ replaced by $(e_1, e_2, \ldots)$.

5 Applications to the time integrals

In this section, we apply the simple formulas given in Section 3 to various functions containing the time integral.

Example 5.1. For $m \in \mathbb{N}$ and $t \in [0, T]$, let $F_t(x) = [x(t)]^m$ for $x \in C[0, T]$, and suppose that $\int_{\mathbb{R}} |u|^m\,d\varphi(u) < \infty$. By Theorems 2.1, 2.2, 3.7 and [5, Theorem 7], for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have

$$GE[F_t|Z_{\vec{e},\infty}](\vec{\xi}) = \sum_{k=0}^{[\frac{m}{2}]}\frac{m!}{2^k k!(m-2k)!}[\vec{\xi}_{\vec{e},\infty,\beta}(t) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,t]})(\alpha)]^{m-2k}\|\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,t]}\|_{0,\beta}^{2k}, \tag{5.1}$$

where $[\cdot]$ denotes the greatest integer function. In addition, by (P1), (P2) and Theorem 3.9,

$$GE[F_t|Z_{\vec{g},\infty}](\vec{\xi}) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!}[\Xi(\infty,\vec{\xi})(t) + \alpha(t) - P_{\infty,\beta}(\alpha)(t)]^{m-2k}[\Phi_j(t,t)]^k \equiv G_1(t,\vec{\xi}) \tag{5.2}$$

for $t \in [t_{j-1}, t_j)$ and for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$. Furthermore, if $t = T$, then $\|\mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,T]}\|^2_{0,\beta} = 0$ since $1 = \chi_{[0,T]} \in V$ which is generated by the g_j. Consequently, we have

$$GE[F_T|Z_{\vec{g},\infty}](\vec{\xi}) = \Big[\xi_0 + \sum_{l=1}^{\infty} \xi_l\sqrt{\beta(t_l) - \beta(t_{l-1})}\Big]^m$$

for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} = (\xi_0, \xi_1, \dots) \in \mathbb{R}^{\aleph_0}$. In particular, by Theorem 3.9, we have

$$GE[F_T|X_\infty](\vec{\xi}) = \xi_T^m$$

for m_{X_∞} a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, where $\lim_{n\to\infty} \xi_n = \xi_T$.

Now, we can obtain the following example by Example 5.1.

Example 5.2. For $m \in \mathbb{N}$, let $F(x) = \int_0^T [x(t)]^m \, d\lambda(t)$ for $x \in C[0,T]$, where λ is a finite complex measure on the Borel class of $[0,T]$, and suppose that $\int_{\mathbb{R}} |u|^m \, d\varphi(u) < \infty$. Then for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, $GE[F|Z_{\vec{e},\infty}](\vec{\xi})$ is given by

$$GE[F|Z_{\vec{e},\infty}](\vec{\xi}) = \int_0^T GE[F_t|Z_{\vec{e},\infty}](\vec{\xi}) \, d\lambda(t),$$

where $GE[F_t|Z_{\vec{e},\infty}](\vec{\xi})$ is expressed by (5.1). In addition, for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} = (\xi_0, \xi_1, \dots) \in \mathbb{R}^{\aleph_0}$, we have

$$GE[F|Z_{\vec{g},\infty}](\vec{\xi}) = \xi_T^m \lambda(\{T\}) + \sum_{j=0}^{\infty} [\Xi(\infty,\vec{\xi})(t_j)]^m \lambda(\{t_j\}) + \sum_{j=1}^{\infty} \int_{(t_{j-1},t_j)} G_1(t,\vec{\xi}) \, d\lambda(t), \tag{5.3}$$

where $G_1(t,\vec{\xi})$ is given by the right-hand side of (5.2). We note that $GE[F|X_\infty](\vec{\xi})$ can be obtained from (5.3) by Theorem 3.9 replacing $\Xi(\infty,\vec{\xi})(t_j)$ by ξ_j. In particular, if $\alpha(t) = P_{\infty,\beta}(\alpha)(t)$ and $d\lambda(t) = d\beta(t)$ for $t \in [0,T]$, then, by Theorem 3.9 and [3, Corollary 3.9], we have

$$GE[F|Z_{\vec{g},\infty}](\vec{\xi}) = \sum_{j=1}^{\infty} \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{m-2k} \frac{m!(l+k)![\beta(t_j) - \beta(t_{j-1})]^{\frac{l}{2}+k+1}[\Xi(\infty,\vec{\xi})(t_{j-1})]^{m-2k-l}\xi_j^l}{2^k l!(m-2k-l)!(l+2k+1)!}.$$

Under the conditions stated above, $GE[F|X_\infty](\vec{\xi})$ is given by the above equality replacing $\frac{l}{2}+k+1$, $\Xi(\infty,\vec{\xi})(t_{j-1})$, ξ_j by $k+1$, ξ_{j-1} and $\xi_j - \xi_{j-1}$, respectively.

Example 5.3. Let $s_1, s_2 \in [0,T]$ and let $G(s_1, s_2, x) = x(s_1)x(s_2)$ for $x \in C[0,T]$. Then $G(s_1, s_2, \cdot)$ is $w_{\alpha,\beta;\varphi}$-integrable by [5, Theorem 5] so that for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have

$$\begin{aligned}GE[G(s_1,s_2,\cdot)|Z_{\vec{e},\infty}](\vec{\xi}) &= \int_{C[0,T]} [I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_1]})(x) + \vec{\xi}_{\vec{e},\infty,\beta}(s_1)][I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_2]})(x) + \vec{\xi}_{\vec{e},\infty,\beta}(s_2)] \, dw_{\alpha,\beta;\varphi_0}(x) \\ &= \langle \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} + [\vec{\xi}_{\vec{e},\infty,\beta}(s_1) \\ &\quad + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_1]})(\alpha)][\vec{\xi}_{\vec{e},\infty,\beta}(s_2) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_2]})(\alpha)]\end{aligned}$$

by Theorems 2.2 and 3.7.

Lemma 5.4. *Let $s_1, s_2 \in [0,T]$. Then we have the following:*

(1) *If $s_1 \in [t_{j-1}, t_j] \cup \{T\}$, $s_2 \in [t_{k-1}, t_k] \cup \{T\}$ for $1 \le j < k$, then*

$$\langle \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = 0.$$

(2) *If $s_1, s_2 \in [t_{j-1}, t_j]$, then*

$$\langle \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = \Phi_j(s_2, s_1).$$

Proof. If $s_1 \in [t_{j-1}, t_j]$ and $s_2 \in [t_{k-1}, t_k] \cup \{T\}$, the proof of (1) is similar to the proof of [4, (1) in Lemma 4.6] with aid of (P2). Since $\chi_{[0,T]} \in V$, which is induced by the g_j, we have $\mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,T]} = 0$ so we have

$$\langle \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,T]}, \mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,T]}\rangle_{0,\beta} = \|\mathcal{P}^{\perp}_{\vec{g},\infty,\beta}\chi_{[0,T]}\|^2_{0,\beta} = 0$$

if $s_1 = s_2 = T$, completing the proof of (1). The proof of (2) is similar to the proof of [4, (2) in Lemma 4.6]. □

Example 5.5. Let $s_1, s_2 \in [0, T]$ and let $G(s_1, s_2, x) = x(s_1)x(s_2)$ for $x \in C[0, T]$. Then, by Theorem 3.9, Example 5.3 and Lemma 5.4, we have the following:

(1) If $s_1 \in [t_{j-1}, t_j] \cup \{T\}$, $s_2 \in [t_{k-1}, t_k] \cup \{T\}$ for $1 \le j < k$, then, for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$,

$$GE[G(s_1, s_2, \cdot)|Z_{\vec{g},\infty}](\vec{\xi}) = [\Xi(\infty, \vec{\xi})(s_1) + \alpha(s_1) - P_{\infty,\beta}(\alpha)(s_1)][\Xi(\infty, \vec{\xi})(s_2) + \alpha(s_2) - P_{\infty,\beta}(\alpha)(s_2)].$$

In particular, $GE[G(s_1, s_2, \cdot)|X_\infty](\vec{\xi})$ is given by the above equality replacing $\Xi(\infty, \vec{\xi})$ by $P_{\infty,\beta}(\vec{\xi})$.

(2) If $s_1, s_2 \in [t_{j-1}, t_j]$, then for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$,

$$GE[G(s_1, s_2, \cdot)|Z_{\vec{g},\infty}](\vec{\xi}) = [\Xi(\infty, \vec{\xi})(s_1) + \alpha(s_1) - P_{\infty,\beta}(\alpha)(s_1)][\Xi(\infty, \vec{\xi})(s_2) + \alpha(s_2) - P_{\infty,\beta}(\alpha)(s_2)] + \Phi_j(s_2, s_1).$$

In particular, $GE[G(s_1, s_2, \cdot)|X_\infty](\vec{\xi})$ is given by the above equality replacing $\Xi(\infty, \vec{\xi})$ by $P_{\infty,\beta}(\vec{\xi})$.

We now have the following theorem from [3, Theorem 3.3], based on Theorem 3.7 and Example 5.3.

Theorem 5.6. *For $x \in C[0, T]$, let*

$$G_3(x) = \left[\int_0^T x(t)\, d\lambda(t)\right]^2,$$

where λ is a finite complex measure on the Borel class of $[0, T]$. Suppose that

$$\int_0^T [\alpha(t)]^2\, d|\lambda|(t) < \infty \quad \text{and} \quad \int_{\mathbb{R}} u^2\, d\varphi(u) < \infty.$$

Then, for $m_{Z_{\vec{e},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have

$$GE[G_3|Z_{\vec{e},\infty}](\vec{\xi}) = \int_0^T\int_0^T \langle \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s_2]}\rangle_{0,\beta}\, d\lambda^2(s_1, s_2) + \left[\int_0^T [\vec{\xi}_{\vec{e},\infty,\beta}(s) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},\infty,\beta}\chi_{[0,s]})(\alpha)]\, d\lambda(s)\right]^2.$$

Using the same method as in the proof of [3, Theorem 3.3] with aid of Lemma 5.4, Example 5.5 and Theorem 5.6, we can prove the following corollary.

Corollary 5.7. *Let the assumptions be as in Theorem 5.6. Then, for $m_{Z_{\vec{g},\infty}}$ a.e. $\vec{\xi} \in \mathbb{R}^{\aleph_0}$, we have*

$$GE[G_3|Z_{\vec{g},\infty}](\vec{\xi}) = \int_0^T\int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2)\, d\lambda^2(s_1, s_2) + \left[\int_0^T [\Xi(\infty, \vec{\xi})(s) + \alpha(s) - P_{\infty,\beta}(\alpha)(s)]\, d\lambda(s)\right]^2,$$

where

$$\Lambda(s, t) = \sum_{j=1}^{\infty} \chi_{[t_{j-1},t_j]^2}(s, t)\Phi_j(s, t)$$

for $(s, t) \in [0, T]^2$, $s_1 \vee s_2 = \max\{s_1, s_2\}$ and $s_1 \wedge s_2 = \min\{s_1, s_2\}$. In particular, $GE[G_3|X_\infty](\vec{\xi})$ is given by the above equality replacing $\Xi(\infty, \vec{\xi})$ by $P_{\infty,\beta}(\vec{\xi})$.

Remark 5.8. We note that the range of the conditioning function $Z_{\vec{e},\infty}$ in this paper is infinite dimensional while the conditioning function $Z_{\vec{e},n}$ in [4] is vector-valued but its range is finite dimensional. Although the expressions of the formulas in Theorem 3.7 and in other results are similar to those in [4], their proofs in this paper are different from the proofs in [4]. We also note that the topology on $\mathbb{R}^{\aleph_0}$ is the product topology, so that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\aleph_0})$ is induced by this topology.

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