Research Article

Gonzalo García-Ros*, Iván Alhama, and Manuel Cánovas

Use of discriminated nondimensionalization in the search of universal solutions for 2-D rectangular and cylindrical consolidation problems

https://doi.org/10.1515/geo-2018-0016
Received Feb 12, 2018; accepted May 04, 2018

Abstract: The solution to the 2-D consolidation problem, both for rectangular and cylindrical domains, has been widely studied in the scientific literature, reporting the most precise solutions in the form of analytical expressions difficult to handle for the engineer due to the high number of parameters involved. In this paper, after introducing a precise definition of the characteristic time, both this magnitude and the average degree of consolidation are obtained in terms of the least number of dimensionless groups that rule the problem. To do this, the groups are firstly derived from the dimensionless governing equations deduced from the mathematical model, following a discriminated nondimensionalization procedure which provides new groups that cannot be obtained by classical nondimensionalization. By a large number of numerical simulations, the dependences of the characteristic time and the average degree of consolidation on the new dimensionless groups have allowed to represent these unknowns graphically in the form of universal curves. This allows these quantities to be read with the least mathematical effort. A case study is solved to demonstrate the reliability and accuracy of the results.

Keywords: anisotropy; discriminated nondimensionalization; characteristic time; average degree of consolidation

1 Introduction

The theory of linear consolidation of soils has been broadly studied in recent decades and is well established in many books [1, 2]. All of them present a complete study of 1-D rectangular and radial consolidation with universal curves of the average degree of consolidation, $\overline{U}(t)$, from which the settlements due to surface loads are derived. The solution of more complex scenarios has been carried out in recent years by analytical methods. These scenarios include multilayered soils, assuming drain resistance and smearing (Battaglio et al. [3]), multi-stage loading (Lu *el at.* [4]), vacuum assisted preloading (Indraratna *el at.* [5]) and other non-linear problems [6–8]. Numerical solutions have also been provided by other authors for scenarios with drains [9, 10].

As for the consolidation scenarios in 2-D domains, both in rectangular (use of infinite strip drains, of theoretical application) and cylindrical coordinates (use of vertical drains with radial geometry), the complexity of the problem, even in linear scenarios where the soil properties are assumed constant, has led to the emergence of a variety of solutions. On the one hand, the separate analysis of the vertical and horizontal (or radial) components of the process, with solutions in the form of series, and their subsequent superposition (Carrillo [11]) is the method generally taught in civil engineering schools [12] and that is also frequently used in the professional field. On the other hand, it is also possible to obtain closed-form solutions for the two-dimensional problem [13, 14], or even to reach analytical or semi-analytical expressions from spectral or approximate methods [15, 16]. All these formulations are presented as a very good approach for the solution of consolidation problems, but their handling is quite complicated since they require the use of numerous parameters, which usually leads to cumbersome calculations.

This paper focuses on the search for the average degree of consolidation in its most universal form, as a function of the lowest number of parameters of the problem, in 2-D anisotropic rectangular and cylindrical domains. For

***Corresponding Author: Gonzalo García-Ros:** Civil Engineering Department, Technical University of Cartagena, Cartagena, Spain; Email: gonzalo.garcia@upct.es, Tel.: +34 678437060
Iván Alhama: Civil Engineering Department, Technical University of Cartagena, Cartagena, Spain
Manuel Cánovas: Metallurgical and Mining Engineering Department, Universidad Católica del Norte, Antofagasta, Chile

this purpose we i) derive the dimensionless groups that rule this problem by the discriminated nondimensionalization of the governing equations, using the most recent concepts involved in this technique, and ii) numerically verify the dependence of the average degree of consolidation on the above groups.

The concepts involved in the application of discriminated nondimensionalization will be explained in detail in the next section. Before this, let us make a historical revision of the time factor concept (by means of which most authors define the dimensionless time), its introduction in the context of consolidation and the connection between the reference used to make the time dimensionless in the time factor and the characteristic time of the consolidation process, both in 1-D and 2-D scenarios.

Regardless of the geometric domain, soil consolidation is an asymptotic process and, as such, there is no characteristic or particular value of time that can be assigned to its duration. However, for each percentage of fall in the excess pore pressure at a given point of the domain (or its average value in the whole domain, which defines the average degree of consolidation) there is only one time value. Most authors make no reference to a characteristic time to be used in the definition of the dimensionless time of the process, but solve the governing equation in finite 1-D domains analytically and give the name of time factor to the term $c_v t/H^2$ that emerges as argument in the solution. In this way, Terzaghi *el at.* [17] say "*$c_v t/H^2$ is a pure number called the time factor*", while Berry and Reid [18] write "*t' is the dimensionless vertical time factor defined by t' = $c_v t/H^2$*" and Atkinson [1] makes no reference to the subject. Later, Azizi [19] says "*… $c_v t/H^2$ representing a time factor. … Because the (dimensionless) time factor is time dependent, the solution yields the precise dissipation with time of excess water pressure u at any depth z*." Finally, Juárez and Rico [2] write that the factors z/H and $c_v t/H^2$ that appear in the solution of soil consolidation are dimensionless quantities, calling the second as the time factor.

Somewhat more explicit as regards the subject are Sagaseta *el at.* [20] and Muir Wood [21]. After deriving the consolidation equation, the first authors introduce the dimensionless variables z' = z/H, t' = $c_v t/H^2$ and the consolidation degree (as they call) u' = $(u_o - u)/u_o$. With these new variables they finally obtain a universal (normalized) equation. The second authors, in relation to the consolidation equation, affirm: "*We can make this equation more universal by replacing our variables u, z and t by non-dimensional variables $u' = u/u_o$ (u_o being a reference pore pressure, typically the initial uniform excess pore pressure), $z' = z/H$ and $t' = c_v t/H^2$, where t′ emerges as a dimensionless time factor*." In other words, the objective of these authors is to obtain a dimensionless governing equation following a correct procedure of nondimensionalization.

In contrast with the above references, Scott [22] is the only author to follow a formal nondimensionalization procedure for 1-D rectangular scenarios, similar to that followed in this paper. After comparing heat conduction and soil consolidation equations, establishing the similarity between the parameters that rule these processes (consolidation coefficient and thermal diffusivity), he says: "*Because of the linearity of the equation, it is appropriate to normalize the various parameters to make the equation nondimensional. In this way, the solution obtained in terms of dimensionless parameters is in a more suitable form for general applications. The normalization in one-dimensional terms, for example, is accomplished by relating the variables to the characteristic constants of the system as follows: An arbitrary constant value of pore pressure* u_o *is chosen and a dimensionless pressure variable* (u′) *is defined such that* $u' = u/u_o$. *Next, a characteristic length H in the system is selected to give a dimensionless length variable*, $z' = z/H$. *Finally, by choosing an arbitrary time constant t_o, we are able to obtain a dimensionless time variable*, $t' = t/t_o$. *Substituting these dimensionless variables in the equation of consolidation gives* $\frac{1}{t_o}\frac{\partial u'}{\partial t'} = \frac{c_v}{H^2}\frac{\partial^2 u'}{\partial z'^2}$. *It is apparent that one of the characteristic constants can be selected for convenience to make* $t_o = H^2/c_v$ *and therefore writes the former equation as* $\frac{\partial u'}{\partial t'} = \frac{\partial^2 u'}{\partial z'^2}$. t′, *usually called time factor is redefined to be* $t' = c_v t/H^2$." In this way, Scott first looks for three references (u_o, H and t_o, the last an unknown) to define the dimensionless variables u′, z′ and t′. Then, he substitutes these variables in the governing equation to derive its dimensionless form. Finally, 'for convenience' (what is to make unity the only dimensionless parameter that emerges in the equation), the unknown reference t_o is deduced.

The procedure applied in this paper to determine the consolidation time (t_o) is apparently the same as the one followed by Scott. However, the assignment of an order of magnitude unity to the dimensionless normalized variables (and their changes) provides a clear physical meaning for the reference time. Moreover, making use of the properties of homogeneous functions, the proposed procedure applied to 2-D scenarios provides more precise solutions for the unknown t_o than those that are possible to find with classical nondimensionalization.

This work takes the following form: After explaining the discriminated nondimensionalization procedure to determine the discriminated groups that derive from it, the technique is applied to the soil consolidation process in 2-D rectangular and cylindrical scenarios, looking for the dependence of the average degree of consolidation on these groups. Numerical simulations show this dependence by means of suitable abacuses. Then, following the steps involved in discriminated nondimensionalization, the precise concept of time of consolidation is introduced, obtaining its dependence on the parameters of the problem, based on the mathematical theory of homogeneous functions. Therefore, using this new concept and the dependence of the average degree of consolidation on the time of consolidation and other parameters, numerical simulations will allow the representation of this unknown through universal curves. Finally, a case study is addressed in order to compare the curves given in the manuscript to classical methods. In the last section, the main contributions of this work are summarized.

2 Discriminated nondimensionalization procedure

The discriminated nondimensionalization technique combines the scale analysis of Bejan [23] with the most general concepts of spatial discrimination introduced by Alhama and Madrid [24]. The first turns the dimensional governing equations into their dimensionless form from which the dimensionless groups are deduced, while the second, which applies both to governing equations and boundary conditions (turning these also into their dimensionless form), provides the smallest number of independent dimensionless groups that rule the problem. The steps involved in the application of this technique are:

(i) Choose the references for both dependent and independent variables in order to define the dimensionless forms of them in such a way that they extend to the interval [0,1]. Discrimination forces the variables of vector character (coordinates, velocities…) to have, in general, a different reference for each spatial component. Furthermore, references are generally explicit in the statement of the problem but they may not be. In the last case, they are introduced as unknowns whose order of magnitude is found once the nondimensionalization procedure has finished.

(ii) Derive the dimensionless governing equations by replacing the old variables with the new dimensionless ones. Each term of these new equations is separated into two factors. The first, dimensionless, is formed by the normalized variables and their changes (derivatives), while the second, non-dimensionless, is a grouping of geometric and physical parameters.

(iii) Assuming that the first factor of ii) is of order of magnitude unity – a reasonable hypothesis in linear problems –, the other factors (also called coefficients) must be of the same order of magnitude (not necessarily unity) so providing a correct balance in the equation.

(iv) The independent ratios formed with these coefficients are the discriminated dimensionless groups that are sought. There are as many groups as coefficients minus one; they must be dimensionless and of order of magnitude unity.

(v) Based on the theory of homogeneous functions (Buckingham pi theorem, [25]), the solution of any unknown of the problem – correctly expressed in its dimensionless form – can be written in the form $\psi(\pi_1, \pi_2 \ldots, \pi_n)$, being $\pi_{i(1\leq i\leq n)}$ the dimensionless groups derived from discriminated nondimensionalization and ψ an arbitrary function. This allows to determine the orders of magnitude of both the average degree of consolidation and the characteristic time.

This proposed method has been applied successfully in different engineering problems, such as fluid mechanics (Madrid and Alhama [26]), fluid flow and solute transport (Manteca *el at.* [27]), geothermics (Cánovas *el at.* [28, 29]) and mechanical engineering (Pérez *el at.* [30]), among others.

3 The governing equations of soil consolidation

Two physical models are considered: i) a 2-D rectangular anisotropic domain that drains towards the soil surface and towards the left boundary, due to the existence of a drain, Figure 1, and ii) a 2-D cylindrical anisotropic domain which drains towards the upper and inner boundary, also due to a drainage condition, Figure 1. The assumed hypotheses are: i) the soil is laterally confined and the drainage occurs in vertical and horizontal directions, ii) the excess pore pressure is generated by an external load constant in time, iii) Darcy´s law is applicable to the movement of the water through the soil, iv) the soil is fully saturated and the pore water and the soil particles behave as an incompressible medium with respect to the soil skeleton, v) the weight of the soil, grains and water can be ignored,

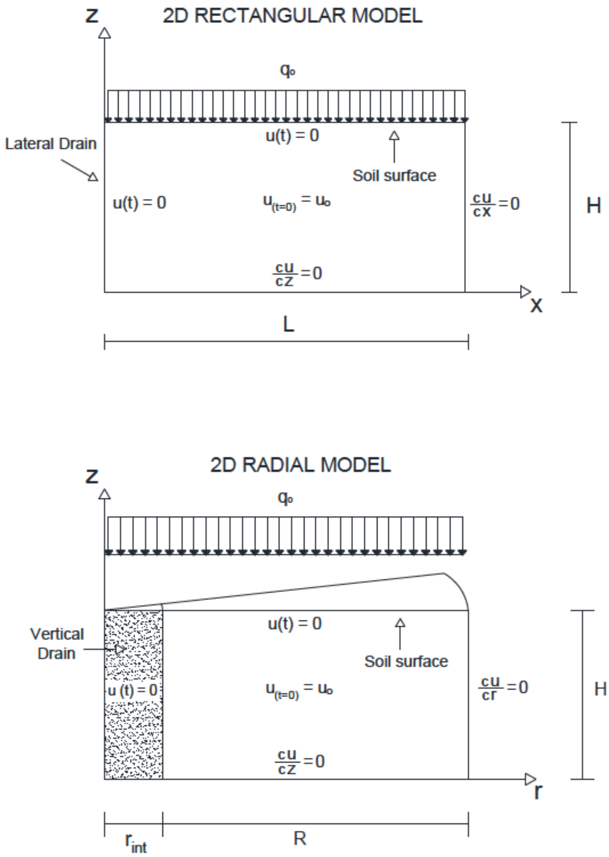

Figure 1: 2-D physical models for rectangular (top) and radial (bottom) soil consolidation scenarios.

vi) the deformation of the soil is free, without redistribution of stresses vii) the soil skeleton does not creep under a constant effective stress, and viii) the applied load increment produces only small strains, so that the total volume of the soil can be assumed to be constant. Under these hypothesis, the soil consolidation processes is ruled by the equations and boundary conditions (1-4) for rectangular coordinates and equations and boundary conditions (5-8) for cylindrical coordinates:

Rectangular coordinates:

$$\frac{\partial u}{\partial t} = c_{v,z}\frac{\partial^2 u}{\partial z^2} + c_{v,x}\frac{\partial^2 u}{\partial x^2} u_{(x=0,z,t)} = u_{(x,z=H,t)} \tag{1}$$

$$u_{(x=0,z,t)} = u_{(x,z=H,t)} = 0 \tag{2}$$

$$\frac{\partial u}{\partial x}_{(x=L,z,t)} = \frac{\partial u}{\partial z}_{(x,z=0,t)} = 0 \tag{3}$$

$$u_{(x,z,t=0)} = u_o \tag{4}$$

Radial coordinates:

$$\frac{\partial u}{\partial t} = c_{v,z}\frac{\partial^2 u}{\partial z^2} + c_{v,r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{5}$$

$$u_{(r=r_{in},z,t)} = u_{(r_{in}<r<r_{out},z=H,t)} = 0 \tag{6}$$

$$\frac{\partial u}{\partial r_{(r=r_{out},z,t)}} = \frac{\partial u}{\partial z_{(r_{in}<r<r_{out},z=0,t)}} = 0 \tag{7}$$

$$u_{(r_{in}<r<r_{out},z,t=0)} = u_o \tag{8}$$

These equations, and their simplifications for 1-D cases, can be solved analytically. However, their solutions [2, 17], obtained in the form of series of functions, seem to be quite complex to manage.

4 The search for dimensionless parameters

4.1 Rectangular domains

The references for the variables x, z and u are explicit in the statement of the problem; these are L, H and u_o, respectively. As regards the time, the common references are $H^2/c_{v,z}$ or $L^2/c_{v,x}$ (in rectangular coordinates) and $H^2/c_{v,z}$ or $r_{out}^2/c_{v,r}$ (in cylindrical). So, substituting

$$x' = \frac{x}{L},\quad Z' = \frac{z}{H},\quad u' = \frac{u}{u_o},\quad t' = \frac{t c_{v,z}}{H^2}, \tag{9}$$

in equation (1) and simplifying, yields

$$\left(\frac{c_{v,z}}{H^2}\right)\frac{\partial u'}{\partial t'} = \left(\frac{c_{v,z}}{H^2}\right)\frac{\partial^2 u'}{\partial z'^2} + \left(\frac{c_{v,x}}{L^2}\right)\frac{\partial^2 u'}{\partial x'^2} \tag{10}$$

This dimensionless governing equation shows that its solution does not depend on the initial excess pore pressure u_o. Assuming that the changes in u' at the whole domain are of order of magnitude unity, the coefficients of the terms of the equation – whose unit of measure is the inverse of time – are of the same order of magnitude. The only independent ratio formed from these coefficients is a dimensionless group that rules the solution of the problem:

$$\pi_1 = \left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right) \tag{11}$$

The dimensionless form of the unknowns of the problem will always depend on the group π_1, as well as on the group $\pi_2 = \left(\frac{t\, c_{v,z}}{H^2}\right)$ if the unknowns are time dependent, and on x' and z' if they are a function of a given location. For example, the average degree of settlement ($\overline{U}$) will depend on π_1 and π_2.

$$\overline{U}\,(t) = \psi\left\{\left(\frac{t\, c_{v,z}}{H^2}\right), \left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right)\right\} \tag{12}$$

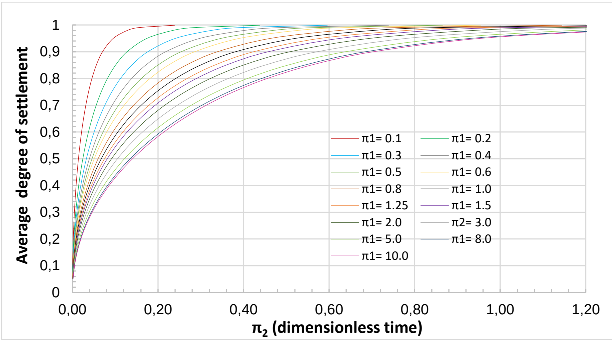

Figure 2a: Average degree of settlement $\overline{U}(t)$ as a function of π_1 and π_2. π_1 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1, 1.25, 1.5, 2, 3, 5, 8 and 10.

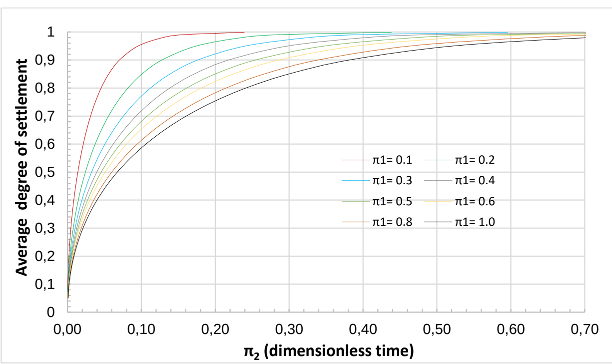

Figure 2b: Average degree of settlement $\overline{U}(t)$ as a function of π_1 and π_2. π_1 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, and 1.

while the local degree of settlement (U) will depend on π_1, π_2, x′ and z′

$$U(x,z,t) = \psi\left\{\left(\frac{t\,c_{v,z}}{H^2}\right),\left(\frac{x}{L}\right),\left(\frac{z}{H}\right),\left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right)\right\} \tag{13}$$

By means of expression (12), it is possible to represent the average degree of settlement as an abacus in the form $\overline{U}(t) = \psi(c_{v,z}t/H^2)$, using π_1 as parameter. This dependence has been solved numerically and the results are presented in Figure 2a. Figure 2b is a detail of this dependence for a better reading. The use of this abacus is immediate: starting from the parameters of the problem (L, H, $c_{v,x}$, $c_{v,z}$) and the time t for which $\overline{U}(t)$ is sought, dimensionless groups π_1 and π_2 are determined; entering with these groups in the abacus, $\overline{U}(t)$ can be read.

As shown, there are significant deviations in the average degree of settlement for a given time and different scenarios (or values of π_1). Note that when $\pi_1 > 1$, which means that $(H^2/c_{v,z}) < (L^2/c_{v,x})$, the reference for dimensionless time is smaller than the characteristic time for excess pore pressure to be released by draining the water horizontally. This means that small values of π_2 do not allow pressure to be released rapidly by vertical or horizontal drainage, which results in small values of the average degree of consolidation. The opposite occurs when $\pi_1 < 1$ or $(H^2/c_{v,z}) > (L^2/c_{v,x})$; in this case, the small values of π_2 do not allow pressure to be released by vertical drainage but do so by horizontal drainage, giving higher values of the average degree of consolidation.

The alternative choice of the reference $L^2/c_{v,x}$ to make time dimensionless (making $\pi_2 = c_{v,x}t/L^2$) provides an abacus quite similar to that of Figures 2a and 2b, but substituting the parameter of the curves π_1 = 0.1 by π_1 = 10, 0.2 by 5, 0.5 by 2 and so on.

The value π_1 = 10 may be considered as a limit curve that provides a good approximation of the 1-D vertical consolidation, for which $c_{v,x} = 0$. Therefore, π_1 (whose theoretical value tends to infinity in this case) does not emerge as a dimensionless group in the nondimensionalization process of the governing equation.

4.2 Cylindrical domains

In the general anisotropic case, the problem is defined by the geometrical parameters r_{in}, r_{out} and H, and the consolidation coefficients $c_{v,r}$ and $c_{v,z}$. From this scenario, a direct dimensionless group based on the boundary conditions or the domain geometry can be written: $\pi_3 = r_{out}/r_{in}$.

As in the rectangular case, references for r, z and u are also explicit: r_{out}, H and u_o, respectively, while for time the reference $H^2/c_{v,z}$ is assumed. Therefore, the dimensionless variables are

$$r' = \frac{r}{r_{out}} \quad Z' = \frac{z}{H} \quad u' = \frac{u}{u_o} \quad t' = \frac{tc_{v,z}}{H^2} \tag{14}$$

These, by introducing them into equation (5), give rise to the dimensionless governing equation whose solution, as in the rectangular case, does not depend on the initial excess pore pressure

$$\left(\frac{c_{v,z}}{H^2}\right)\frac{\partial u'}{\partial t'} = \left(\frac{c_{v,z}}{H^2}\right)\frac{\partial^2 u'}{\partial z'^2} + \left(\frac{c_{v,r}}{r_{out}^2}\right)\left(\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}\right) \tag{15}$$

The coefficients of this equation, $\left(\frac{c_{v,z}}{H^2}\right)$ and $\left(\frac{c_{v,r}}{r_{out}^2}\right)$, provide one dimensionless group

$$\pi_1 = \left(\frac{c_{v,z}r_{out}^2}{c_{v,r}H^2}\right) \tag{16}$$

The solution for the average degree of consolidation for a given time will depend on the dimensionless form of t,

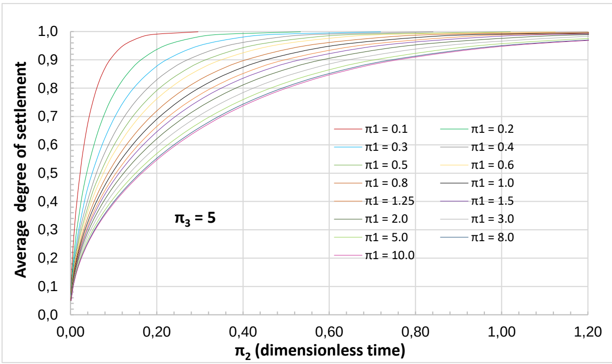

Figure 3a: Average degree of settlement $\overline{U}(t)$ for $\pi_3 = 5$. $\pi_1 = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1, 1.25, 1.5, 2, 3, 5, 8 and 10.

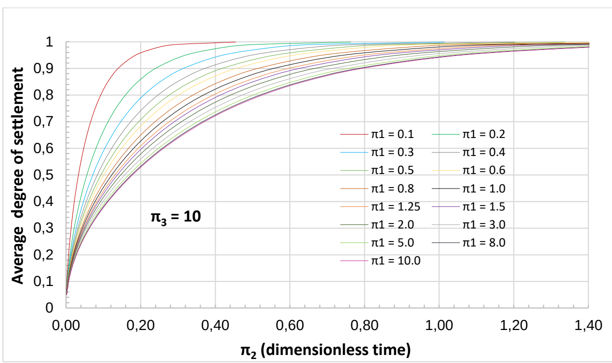

Figure 3b: Average degree of settlement $\overline{U}(t)$ for $\pi_3 = 10$. $\pi_1 = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1, 1.25, 1.5, 2, 3, 5, 8 and 10.

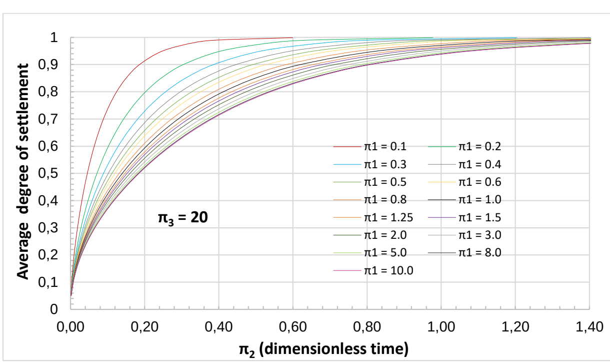

Figure 3c: Average degree of settlement $\overline{U}(t)$ for $\pi_3 = 20$. $\pi_1 = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1, 1.25, 1.5, 2, 3, 5, 8 and 10.

$\pi_2 = \left(\frac{t\, c_{v,z}}{H^2}\right)$, and the groups π_1 and π_3. This is,

$$\overline{U}(t) = \psi\left\{\left(\frac{t\, c_{v,z}}{H^2}\right), \left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{17}$$

The local degree of consolidation depends, in addition, on the dimensionless coordinates

$$U(r,z,t) = \psi\left\{\left(\frac{r}{r_{out}}\right), \left(\frac{z}{H}\right), \left(\frac{t\, c_{v,z}}{H^2}\right), \left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{18}$$

The dependence of the average degree of consolidation may now be represented by a set of abacuses, each of them for a given value of the group π_3. As in the rectangular case, within each abacus $\overline{U}(t) = \psi(c_{v,z}t/H^2) = \psi(\pi_2)$ and each curve is related to a value of π_1. Figures 3a to 3c show four abacuses, each one for the values $\pi_3 = 5$, 10 and 20, respectively, covering a wide range of real cases.

Starting from the parameters of the problem (r_{in}, r_{out}, H, $c_{v,r}$, $c_{v,z}$) and the time for which $\overline{U}(t)$ is sought, the values of the groups π_1, π_2 and π_3 are determined. With π_1 and π_3, the corresponding abacus and curve – interpolating if required – are chosen; finally, introducing π_2 in the curve, $\overline{U}(t)$ can be obtained. Similar comments to those made for rectangular domains may apply to these abacuses, regarding the aspect of the curves and the evolution of the average degree of consolidation. However, the slopes of the curves are reduced as π_3 increases, providing a lower degree of consolidation for the same time and the same value of π_1. Note that an increase in π_3, while the rest of the parameters of the problem remain constant, leads to a smaller drainage surface inside the cylinder and a lower degree of consolidation for the same time.

The above results are simplified for 1-D radial coordinates, which supposes that drainage only occurs at the boundary $r = r_{in}$. Group π_1 disappears and the average degree of consolidation only depends on π_2 and π_3.

$$\overline{U}(t) = \psi\left\{\left(\frac{t\, c_{v,z}}{H^2}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{19}$$

5 Characteristic time of consolidation, t_o

5.1 Rectangular domains

Following Alhama and Madrid [24], it is possible to assume an unknown reference – not explicit in the statement of the problem –, instead of the ratios $H^2/c_{v,z}$ or $L^2/c_{v,x}$ formed by parameters of the problem, to make time dimensionless in the governing equations. The order of magnitude of this reference, once it has been well defined and introduced in the process of nondimensionalization, can be obtained as a function of the rest of the dimensionless groups.

The reference time, called characteristic time (t_o), is of the order of the time required to reduce the initial value of the excess pore pressure (u_o) to a certain (small) percentage, at a given point of the domain (far from the drainage boundaries). In this way, t_o is associated to the time for which the average degree of consolidation is sufficiently high. As a consequence, the range of values of $t' = t/t_o$ can be approached to [0,1], as occurs with the other dimensionless variables.

By substituting x', z', u' and t' in (1), the resulting dimensionless governing equation is

$$\left(\frac{1}{t_o}\right)\frac{\partial u'}{\partial t'} = \left(\frac{c_{v,z}}{H^2}\right)\frac{\partial^2 u'}{\partial z'^2} + \left(\frac{c_{v,x}}{L^2}\right)\frac{\partial^2 u'}{\partial x'^2} \tag{20}$$

The two independent ratios formed from the coefficients of this equation are the dimensionless groups sought:

$$\pi_1 = \left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right) \quad \pi_{2,t_o} = \left(\frac{t_o c_{v,z}}{H^2}\right) \tag{21}$$

These groups have a clear physical meaning, as occurs with all those derived by discrimination. The first group, written in the form $\pi_1 = \left(\frac{L^2}{c_{v,x}}\right)/\left(\frac{H^2}{c_{v,z}}\right)$, is the ratio between two characteristic times: i) the time that takes the process to reach a significant reduction in the excess pore pressure along the horizontal axis, and ii) the time needed to reach a significant reduction in the excess pore pressure along the vertical axis. The second group, $\pi_2 = t_o/\left(\frac{H^2}{c_{v,z}}\right)$, compares the characteristic time needed to reach a significant reduction in the excess pore pressure in the whole domain with the reference ii). Note that this last group could have been presented in the form $t_o/\left(\frac{L^2}{c_{v,x}}\right)$, comparing the characteristic time of the whole domain with the reference i). As we will see later, both ways reach the same universal curves for the average degree of consolidation.

From the resulting groups (21), the value of t_o is deduced from the solution $\pi_{2,t_o} = F(\pi_1)$,

$$t_o = \left(\frac{H^2}{c_{v,z}}\right)\psi\left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right) \tag{22}$$

where F is an arbitrary unknown function of the argument π_1. This solution depends on the location where the decrease in excess pore pressure is measured, as well as on the percentage of falling selected. So, we have chosen the coordinates of the domain $x' = 0.9$ and $z' = 0.1$ and fixed the percentage of falling at different values: 50, 80, 90 and 95%. Numerical simulations were carried out for values of π_1 within the interval [0.01,15], in order to cover the totality of practical cases, changing $c_{v,x}$ or L since these parameters only appear in π_1. For each simulation, t_o is read and π_2 deduced from $\pi_{2,t_o} = \left(\frac{t_o c_{v,z}}{H^2}\right)$, providing in this way one point (π_1, π_{2,t_o}) of the dependence $\pi_{2,t_o} = F(\pi_1)$, shown in Figure 4. The curves show a notable change in t_o for π_1 values below 4, but a slow increase for higher values. This behaviour is coherent, since if we maintain

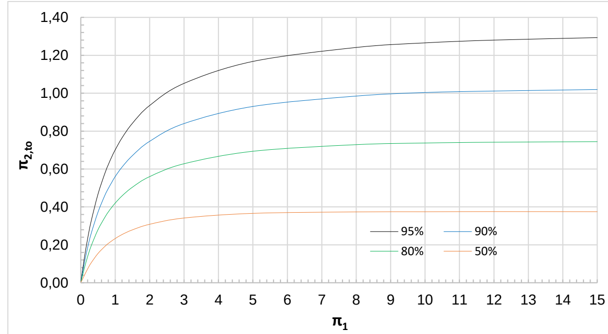

Figure 4: Universal curves $\pi_{2,t_o} = \psi(\pi_1)$ for 2-D rectangular anisotropic domains. Reference location: $x' = 0.9$, $z' = 0.1$. Percentage of fall in the excess pore pressure: 50, 80, 90 and 95%.

$\left(H^2/t_o c_{v,z}\right)$ constant, being $\left(L^2/t_o c_{v,x}\right) >> \left(H^2/t_o c_{v,z}\right)$ – which means that water drainage essentially takes place vertically –, an increase in $\left(L^2/t_o c_{v,x}\right)$ hardly influences the consolidation time t_o; however, if $\left(H^2/t_o c_{v,z}\right)$ is maintained constant and $\left(L^2/t_o c_{v,x}\right) << \left(H^2/t_o c_{v,z}\right)$ – which means that water drainage essentially occurs in a horizontal direction –, an increase in $\left(L^2/t_o c_{v,x}\right)$ produces a significant rise in the consolidation time t_o.

The way in which these curves are used is as follows: Starting from the parameters $c_{v,x}$, $c_{v,z}$, L and H, and a given percentage of excess pore pressure fall, π_1 is evaluated from $\pi_1 = \left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right)$ and π_{2,t_o} from the dependence $\pi_{2,t_o} = \psi(\pi_1)$, Figure 4. Finally t_o is determined using equation (21).

The reduction of these results to the 1-D case is immediate. Then, equation (10) only contains two addends, whose coefficients give rise to one dimensionless group, $\pi_{2,t_o} = \left(\frac{t_o c_{v,z}}{H^2}\right)$ – of order of magnitude unity –, which provides the solution for t_o.

$$t_o \sim \left(\frac{H^2}{c_{v,z}}\right) \tag{23}$$

Again, by giving the same physical meaning for t_o, one numerical simulation for each percentage of fall in excess pore pressure provides the constant that changes the above expression to an equality, $t_o = C_o\left(\frac{H^2}{c_{v,z}}\right)$. Table 1 shows the values of this constant for different fall percentages, at the measurement point $z' = 0.1$.

5.2 Cylindrical domains

Again, assuming t_o as the time required to dissipate the initial pore pressure to a negligible value in a location far from the drainage boundaries, the dimensionless govern-

Table 1: Vertical 1-D consolidation. Values of the constant C_o that relate t_o and $H^2/c_{v,z}$ for different falls in excess pore pressure. Measurement point: z′ = 0.1.

	u = 0.1 u_o (90% consolidation)	u = 0.2 u_o (80% consolidation)	u = 0.3 u_o (70% consolidation)	u = 0.5 u_o (50% consolidation)
C_o	1.026	0.7452	0.5809	0.3737

ing equation is written in the form

$$\left(\frac{1}{t_o}\right)\frac{\partial u'}{\partial t'} = \left(\frac{c_{v,z}}{H^2}\right)\frac{\partial^2 u'}{\partial z'^2} + \left(\frac{c_{v,r}}{r_{out}^2}\right)\left(\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}\right) \quad (24)$$

where t′ = t/t_o. In addition to the aforementioned π_3, the dimensionless groups that can be formed from this equation are

$$\pi_1 = \left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}\right) \quad \pi_{2,t_o} = \left(\frac{t_o c_{v,z}}{H^2}\right) \quad (25)$$

and the solution of t_o, obtained from $\pi_{2,t_o} = \psi(\pi_1, \pi_3)$, results

$$t_o = \left(\frac{H^2}{c_{v,z}}\right)\psi\left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}, \frac{r_{out}}{r_{in}}\right) \quad (26)$$

To represent this dependence it is necessary to fix a measurement point and a percentage of fall for the excess pore pressure. Numerical simulations were carried out to read t_o for a set of values of π_1 within the interval [0.01,15], π_3= 5, 10, 20 and 50, and a 90% excess pore pressure fall at z′=0.1, r′=0.9. Figure 5 shows the abacus of this dependence using the group π_3 as parameter. The curves reflect the notable influence of π_1 and π_3 on t_o, but for π_1 >> 1, which means that vertical consolidation prevails against horizontal, t_o only depends on H and $c_{v,z}$, doing constant (≈ 1) the group π_{2,t_o} whatever the value of π_3. As in rectangular domains, the use of these curves starts from the geometrical and physical parameters of the problem: $c_{v,r}$, $c_{v,z}$, H, r_{in} and r_{out}. From these, π_1 and π_3 are determined and, by using Figure 5, the corresponding value of π_{2,t_o} can be read or interpolated. Finally, t_o is determined from equation (25).

To obtain the 1-D radial coordinates dependences, which means deleting the vertical consolidation, the characteristic time simplifies to the expression $\pi_{2,t_o} = \psi(\pi_3)$, where $\pi_{2,t_o} = \left(\frac{t_o c_{v,r}}{r_{out}^2}\right)$, or

$$t_o = \left(\frac{r_{out}^2}{c_{v,r}}\right)\psi\left(\frac{r_{out}}{r_{in}}\right) \quad (27)$$

Figure 6 shows this dependence for a 90% fall measured at r’ = 0.9.

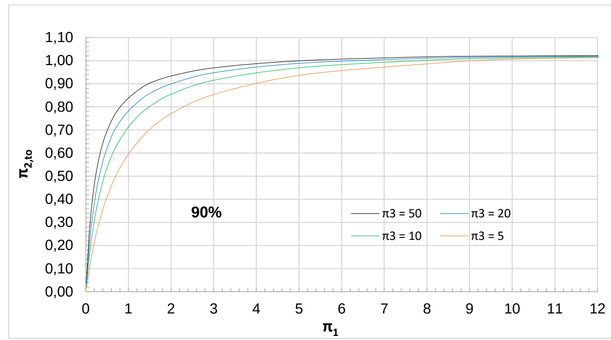

Figure 5: Universal curves $\pi_{2,t_o} = \psi(\pi_1, \pi_3)$ for 2-D cylindrical anisotropic domains. Reference location: z′ = 0.1, r′ = 0.9. π_3= 5, 10, 20 and 50. Percentage of fall in the excess pore pressure: 90%.

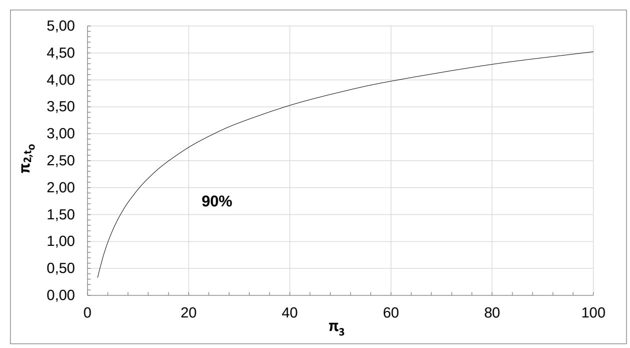

Figure 6: Radial 1-D domains. Universal curves $\pi_{2,t_o} = \psi(\pi_3)$. Reference location: r’ =0.9, percentage of fall in the excess pore pressure: 90%.

6 Universal curves based on the characteristic consolidation time t_o

The introduction of the quantity t_o provides the opportunity to investigate more universal curves of $\overline{U}(t)$ based on this time. Hereinafter, we assume that t_o defines the time required to reduce the initial excess pore pressure to the value $0.1u_o$, at the point z′ = 0.1 and x′ = 0.9 (z′ = 0.1 and r′ = 0.9 in cylindrical coordinates), so covering a significant release of the excess pore pressure in the whole domain. The first step to obtain these new curves is to find t_o using the curves presented in the above section. With the reference

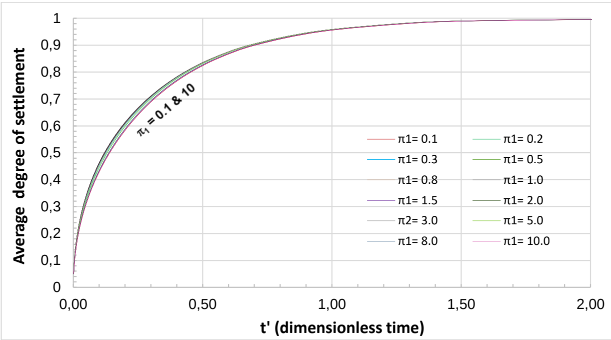

Figure 7a: 2-D rectangular domains. $\overline{U}(t)$ as a function of $t' = \frac{t}{t_o}$ and $\pi_1 = \frac{c_{v,z}L^2}{c_{v,x}H^2}$.

t′ = t/t$_o$ (instead of t′ = $c_{v,z}t/H^2$), the resulting expressions of the average degree of consolidation are:
Rectangular 2-D coordinates:

$$\overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{c_{v,z}L^2}{c_{v,x}H^2}\right)\right\} \tag{28}$$

Cylindrical 2-D coordinates:

$$\overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{c_{v,z}r_{out}^2}{c_{v,r}H^2}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{29}$$

Radial 1-D coordinates:

$$\overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{30}$$

A significant number of simulations were carried out to represent the dependences of the above expressions. For the 2-D rectangular case,Figure 7a shows the results for a range of values of $\pi_1 = \frac{c_{v,z}L^2}{c_{v,x}H^2}$ that sufficiently covers real scenarios, $\pi_1 = 0.1$-10 (an order of magnitude above and below unity). Note that the curves $\pi_1 = 0.1$ and $\pi_1 = 10$ are the same, and also $\pi_1 = 0.2$ and $\pi_1 = 5$ (in general the curves of parameters $\pi_1 = C_o$ and $\pi_1 = C_o^{-1}$). As shown, the curves are so close that they allow any of them to read $\overline{U}(t)$ with an error below 10%. Taking the curve $\pi_1 = 1$ (or $\pi_1 = 0.1$) to read $\overline{U}(t)$ for all π_1 values, the reading is in the side of unsafety (safety), since real values of $\overline{U}(t)$ are below (above) the readings.

For 2-D cylindrical domains, equation (29), the existence of three dimensionless groups requires a group of abacuses in order to obtain $\overline{U}(t)$, one for each value of the group $\pi_3 = r_{out}/r_{in}$. Figures 7b to 7d show the abacus for π_3 = 5, 10 and 20, respectively. Note that the curves $\pi_1 \geq 0.8$ are so close that they can be considered an only one.

For radial 1-D scenarios, π_1 does not influence the solution and, from numerical simulations, the dependence (26) is shown in Figure 7e.

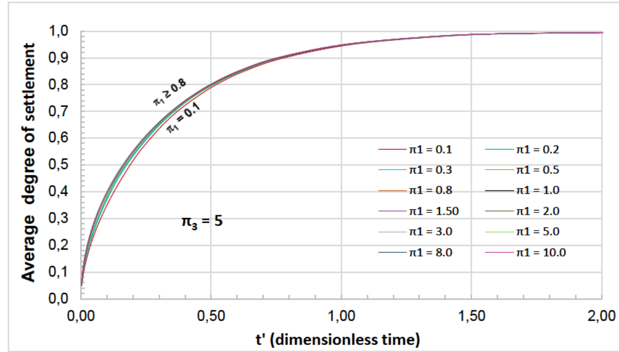

Figure 7b: 2-D radial domains. $\overline{U}(t)$ as a function of $t' = \frac{t}{t_o}$ and $\pi_1 = \frac{c_{v,z}r_{out}^2}{c_{v,r}H^2}$. $\pi_3 = 5$. π_1 = 0.1, 0.2, 0.3, 0.5, 0.8, 1, 1.5, 2, 3, 5, 8 and 10.

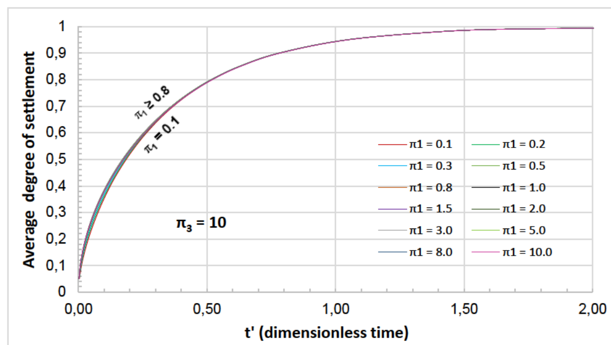

Figure 7c: 2-D radial domains. $\overline{U}(t)$ as a function of $t' = \frac{t}{t_o}$ and $\pi_1 = \frac{c_{v,z}r_{out}^2}{c_{v,r}H^2}$. $\pi_3 = 10$. π_1 = 0.1, 0.2, 0.3, 0.5, 0.8, 1, 1.5, 2, 3, 5, 8 and 10.

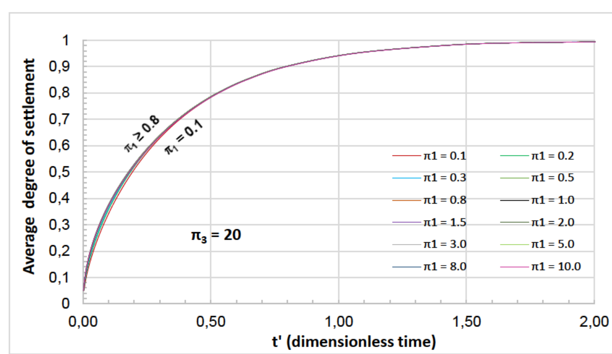

Figure 7d: 2-D radial domains. $\overline{U}(t)$ as a function of $t' = \frac{t}{t_o}$ and $\pi_1 = \frac{c_{v,z}r_{out}^2}{c_{v,r}H^2}$. $\pi_3 = 20$. π_1 = 0.1, 0.2, 0.3, 0.5, 0.8, 1, 1.5, 2, 3, 5, 8 and 10.

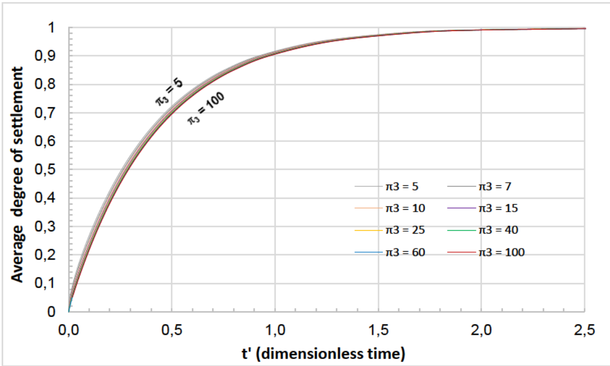

Figure 7e: 1-D radial domain. $\overline{U}(t)$ as a function of $t' = \frac{t}{t_o}$ and $\pi_3 = \frac{r_{out}}{r_{in}}$.

7 Discriminated nondimensionalization dimensionless groups versus classical groups

It should be mentioned that, compared with discriminated nondimensionalization, the classical procedure leads to poorer results. If the same reference, as in the classical method, is used to make horizontal and vertical coordinates dimensionless, the values of their dimensionless forms and their derivatives are not constrained in the interval [0,1] but only one of them. This means that the coefficients of the terms of the dimensionless governing equations cannot be of the same order of magnitude (even if measured in the same unity) and, consequently, the final dimensionless groups are not of the order of magnitude unity. In addition, classical nondimensionalization generally leads to a large number of dimensionless groups, particularly in anisotropic scenarios, which leads to a less precise result. For example, in 2-D radial domains, the solution for t_o provided by classical nondimensionalization (which is the same as that obtained by dimensional analysis) is

$$t_o = \left(\frac{H^2}{c_{v,z}}\right)\psi\left(\frac{c_{v,z}}{c_{v,r}}, \frac{r_{out}}{r_{in}}, \frac{H}{r_{out}}\right) \tag{31}$$

while for $\overline{U}(t)$

$$\overline{U}(t) = \psi\left\{\left(\frac{t\,c_{v,z}}{H^2}\right), \left(\frac{c_{v,z}}{c_{v,r}}\right), \left(\frac{r_{out}}{r_{in}}\right), \left(\frac{H}{r_{out}}\right)\right\} \tag{32}$$

$$\text{or } \overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{c_{v,z}}{c_{v,r}}\right), \left(\frac{r_{out}}{r_{in}}\right), \left(\frac{H}{r_{out}}\right)\right\}$$

Note that while the ratio r_{out}/r_{in} emerges in both classical and discriminated nondimensionalization, the ratios $c_{v,z}/c_{v,r}$ and H/r_{out} are not assumed as dimensionless when discrimination is applied but join in a new group to provide a kind of corrected conductivity ratio. For radial 1-D domains, the above solutions simplify to others less precise than those obtained by discrimination.

$$t_o = \left(\frac{{r_{out}}^2}{c_{v,r}}\right)\psi\left(\frac{r_{out}}{r_{in}}\right) \tag{33}$$

$$\overline{U}(t) = \psi\left\{\left(\frac{t c_{v,r}}{r_{out}^2}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\} \tag{34}$$

$$\text{or } \overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{r_{out}}{r_{in}}\right)\right\}$$

Finally, in rectangular coordinates, the classical method leads to solutions

$$t_o = \left(\frac{H^2}{c_{v,z}}\right)\psi\left(\frac{c_{v,z}}{c_{v,x}}, \frac{H}{L}\right) \tag{35}$$

$$\overline{U}(t) = \psi\left\{\left(\frac{t c_{v,z}}{H^2}\right), \left(\frac{c_{v,z}}{c_{v,x}}\right), \left(\frac{H}{L}\right)\right\} \tag{36}$$

$$\text{or } \overline{U}(t) = \psi\left\{\left(\frac{t}{t_o}\right), \left(\frac{c_{v,z}}{c_{v,x}}\right), \left(\frac{H}{L}\right)\right\}$$

whereby all functions depend on one more argument than discriminated solutions.

Table 2: Geotechnical properties of the soil.

parameter	value	units
$c_{v,z}$	1.2	m^2/yr
H	4	m
$c_{v,r}$	3	m^2/yr
r_{in}(drain radius)	0.2	m
r_{out}(influence radius)	4	m

8 Case study

In this section a case study is addressed to compare the solutions given in the manuscript with those provided by the classical methods. In the problem to be analyzed, a fully saturated clay soil is subjected to a wide extension load of constant value and permanent over time, giving rise to a consolidation phenomenon. In order to accelerate this process, vertical drains of circular section are installed in the soil before applying the load, following a scheme similar to that of Figure 1 (bottom). The geotechnical properties of the soil to be studied are summarized in Table 2.

The study focuses on the analysis of the evolution of the average degree of settlement, ($\overline{U}$), in order to determine the duration of the consolidation process. To get the value of $\overline{U}$, the solutions in the form of series provided by the classical methods can be used, which treat the problem by superposing the vertical ($\overline{U}_v$) and radial ($\overline{U}_r$) solutions [11], equations (37) and (38), respectively.

$$\overline{U}_v = 1 - \sum_{n=0}^{n=\infty} \frac{2}{M^2} e^{\left(-M^2 \cdot T_v\right)} \tag{37}$$

with $M = \frac{\pi}{2}(2n+1)$

$$\overline{U}_r = 1 - e^{\left(-\frac{2T_r}{F(n)}\right)} \tag{38}$$

with $F(n) = \frac{n^2}{n^2-1}\ln(n) - \frac{3n^2-1}{4n^2}, \quad n = \frac{r_{out}}{r_{in}}$

where (T_v) and (T_r) are, respectively, the vertical and radial time factors, whose expressions are:

$$T_v = \frac{c_{v,z}\, t}{H^2} \text{ and } T_r = \frac{c_{v,r} t}{r_{out}^2} \tag{39}$$

The above solutions lead to the time-dependent average degree of consolidation given by:

$$(1-\overline{U}) = (1-\overline{U}_v)(1-\overline{U}_r) \tag{40}$$

This means that, in order to determine the time for which the consolidation process can be considered finished, it will be necessary to test different values of t until the characteristic time is found, which means a laborious task.

A value of t = 2.7 years can be initially taken. By means of equations (37) and (38), $\overline{U}_v$= 0.52 and $\overline{U}_r$ = 0.37 are obtained, finally resulting $\overline{U}$ = 0.70. At this point, it is worth mentioning that there are graphic solutions in the form of abacuses for expressions (37) and (38) [18], which saves the cumbersome task of calculating the series of addends. However, this method requires: i) obtaining two time factors, (T_v = 0.20, T_r= 0.51) and the parameter n (of value 20), ii) the use of two abacuses to determine $\overline{U}_v$ and $\overline{U}_r$, and iii) use the expression (40) to get $\overline{U}$.

On the other hand, the use of the solutions proposed in this article is, by comparison, much simpler and requires fewer steps. First, we must obtain the values of π_1, π_2 and π_3, the dimensionless groups that govern the solution pattern of the proposed problem. These are:

$$\pi_1 = \left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}\right) = 0.4 \tag{41}$$

$$\pi_2 = \left(\frac{t c_{v,z}}{H^2}\right) = 0.20$$

$$\text{and } \pi_3 = \left(\frac{r_{out}}{r_{in}}\right) = 20$$

From these values, the average degree of consolidation is read directly using Figure 3c, obtaining the same result as with the previous method ($\overline{U}$ = 0.70).

To determine the characteristic time of the problem, it would be necessary to repeat the previous procedure (testing with different times) as many times as necessary until reaching, for example, a high average degree of settlement. However, with the characteristic time curves provided in this paper, the process is greatly simplified. As can be deduced from expression (26)

$$t_o = \left(\frac{H^2}{c_{v,z}}\right)\psi\left(\frac{c_{v,z} r_{out}^2}{c_{v,r} H^2}, \frac{r_{out}}{r_{in}}\right) \tag{26}$$

the characteristic time, expressed in a dimensionless form π_{2,t_o}, is a function of the groups π_1 and π_3. In this way, based on the values π_1 = 0.4 and π_3 = 20, the value of π_{2,t_o} is obtained directly from Figure 5, resulting π_{2,t_o} = 0.6. Finally, from the expression $\pi_{2,t_o} = \left(\frac{t_o c_{v,z}}{H^2}\right)$ the value of t_o = 8 years is obtained, a time for which a fall in the excess pore pressure of 90% is reached at a point in the domain that is far away from both the surface drainage boundary and the vertical drain (z′ = 0.1, r′ = 0.9). This criterion of defining the characteristic time ensures high values for the average degree of consolidation; indeed, when we calculate this for t_o = 8 years – applying again equations (37-40) or Figure 3c – we obtain a value of $\overline{U}$ = 0.95. A result that, on the other hand, can also be obtained from Figure 7d for a value of t′ = 1.

9 Conclusions

The nondimensionalization of governing equations, in conjunction with the concept of spatial discrimination in anisotropic media, provides expressions for the average degree of consolidation as a function of the smallest number of dimensionless groups that enable the standard representation of this quantity by an abacus, for rectangular coordinates, and by a group of abacuses for cylindrical coordinates. This technique is applied in the paper to the consolidation process in 2-D, rectangular and cylindrical anisotropic scenarios.

Compared to the classical nondimensionalization, the use of discrimination, in its wider meaning, has two advantages. Firstly, as regards spatial directions, discrimination forces us to separate the references of the coordinate variables, deleting the form factor – or geometrical aspect ratio – of the domain as independent group and joining it with the ratio of the vertical and horizontal consolidation coefficients. The reduction from three classical independent groups to two discriminated groups makes it easier to represent the dependence of the average degree of consolidation on these groups by abacuses. Secondly, the introduction of the quantity 'characteristic time', defined as the time at which the excess pore pressure is reduced to a small fraction of its initial value – at a given point of the domain –, allows the expression of this quantity to be determined as a function of the other dimensionless groups of the problem.

When both the different references for spatial coordinates and characteristic time are used to define the dimensionless variables of the problem, the resulting dimensionless discriminated governing equations provide groups on the basis of which the new universal curves of the average degree of consolidation can be represented. These curves differ very little from each other, to such an extent that one of them can be used to read the average degree of consolidation with an acceptable percentage of error for engineering purposes.

The dependences of the characteristic time and the average degree of consolidation are solved numerically by the network method and shown finally in the form of universal abacuses.

A case study has been addressed to show that the solutions provided by the proposed curves are as precise as those obtained by classical methods, but they present two important advantages: i) the procedure to obtain the solutions is much quicker and easier, with no mathematical manipulation, and ii) the determination of the characteristic consolidation time is direct, and not through successive trials.

Acknowledgement: We are grateful to Francisco Alhama for all the knowledge he has infused us throughout our training as researchers and, in particular, for this work.

Nomenclature

$\mathbf{c}_{v,z}$ vertical consolidation coefficient (m^2/s)
$\mathbf{c}_{v,x}$ horizontal consolidation coefficient (m^2/s)
$\mathbf{c}_{v,r}$ radial consolidation coefficient (m^2/s)
$\mathbf{C}_o$ numerical constants (dimensionless)
H depth of the domain up to impermeable condition (m)
L length of the rectangular domain (m)
n ratio between the drain influence radius and the drain radius
r radial coordinate (m)
$\mathbf{r}_{in}$ drain radius or inner radius (m)
$\mathbf{r}_{out}$ drain influence radius or outer radius (m)
t time (s)
$\mathbf{t}_o$ characteristic time of the consolidation process (s)
u excess pore pressure (Pa)
$\mathbf{u}_o$ initial excess pore pressure (Pa)
U local degree of consolidation or settlement (dimensionless)
$\overline{U}$ average degree of consolidation or settlement (dimensionless)
x, z spatial coordinates, horizontal and vertical, (m)
π dimensionless number (or group)
ψ arbitrary mathematical function

Subscripts

1,2... denote different numbers

Superscripts

′ denote dimensionless quantity

References

[1] Atkinson J., The mechanics of soils and foundations. Mc Graw-Hill, 1993
[2] Juárez Badillo E., Rico Rodríguez A., Mecánica de suelos, Tomo 1. Limusa, México D.F., 2009
[3] Battaglio M., Bonzani I., Campolo D., Nonlinear consolidation models of clay with time dependant drainage properties. Mathematical and computer modelling, 2005, 42(5-6), 613-620
[4] Lu M., Xie K., Wang S., Consolidation of vertical drain with depth-varying stress induced by multi-stage loading. Computers and Geotechnics, 2011, 38(8), 1096-1101
[5] Indraratna B., Rujikiatkamjorn C., Balasubramaniam A., McIntosh G., Soft ground improvement via vertical drains and vacuum assisted preloading. Geotextiles and Geomembranes, 2012, 30, 16-23
[6] Cornetti P., Battaglio M., Nonlinear consolidation of soil modeling and solution techniques. Mathematical and computer modelling, 1994, 20(7), 1-12
[7] Bonzani I., Lancellotta R., Some remarks on nonlinear consolidation models. Applied Mathematics Letters, 2005, 18, 811-815
[8] Abbasi N., Rahimi H., Javadi A., Fakher A., Finite difference approach for consolidation with variable compressibility and permeability. Computers and Geotechnics, 2007, 34, 41-52
[9] Chai J., Miura N., Kirekawa T., Hino T., Optimum PVD installation depth for two-way drainage deposit. Geomechanics and Engineering, an International Journal, 2009, 1(3), 179-192.
[10] Menéndez C., Nieto P., Ortega F., Bello A., Non-linear analysis of the consolidation of an elastic saturated soil with incompressible fluid and variable permeability by FEM. Applied Mathematics and Computation, 2010, 216, 458-476
[11] Carrillo, N., Simple two and three dimensional case in the theory of consolidation of soils. Studies in Applied Mathematics, 1942, 21(1-4), 1-5

[12] Braja D., Advanced soil mechanics. Taylor and Francis, New York and London, 2008
[13] Tang X., Niu B., Cheng G., Shen H., Closed-form solution for consolidation of three-layer soil with a vertical drain system. Geotextiles and Geomembranes, 2013, 36, 81-91
[14] Di Francesco R., Exact solution of Terzaghi's consolidation equation and extension to two/three-dimensional cases. Applied Mathematics, 2013, 4, 713-717
[15] Walker R., Indraratna B., Consolidation analysis of a stratified soil with vertical and horizontal drainage using the spectral method. Géotechnique, 2009, 59(5), 439-449.
[16] Walker R., Vertical drain consolidation analysis in one, two and three dimensions. Computers and Geotechnics, 2011, 38(8), 1069-1077
[17] Terzaghi K., Peck R., Mesri G., Soil Mechanics in engineering practice. John Wiley and Sons, New York, 1996.
[18] Berry P., Reid D., An Introduction to Soil Mechanics. McGraw-Hill, London, 1988
[19] Azizi F., Applied analysis in geotechnics. E and FN Spon, New York, 2000
[20] Sagaseta C., Cañizal J., da Costa A., Geotecnia I. Propiedades del terreno. Ed. ETS Caminos, Canales y Puertos, Santander, 2007
[21] Muir Wood D., Soils Mechanics (A one-dimensional introduction). Cambridge, 2009
[22] Scott R., Principles of soil mechanics. Addison-Wesley Publishing Company, Inc., 1963
[23] Bejan A., Convection heat transfer. 4^{th} ed. Wiley, New York, 2013
[24] Alhama F, Madrid C., Análisis dimensional discriminado. Aplicación a problemas avanzados de dinámica de fluidos y transferencia de calor. Ed. Reverté, Barcelona, 2012
[25] Buckingham, E., On physically similar systems: illustrations of the use of dimensional equations. Phys. Rev., 1914, 4(4), 345-376
[26] Madrid C., Alhama F., Study of the laminar natural convection problem along an isothermal vertical plate based on discriminated dimensional analysis. Chemical Engineering Communications, 2008, 195(12), 1524-1537
[27] Manteca I., Alcaraz M., Trigueros E., Alhama, F., Dimensionless characterization of salt intrusion benchmark scenarios in anisotropic media. Applied Mathematics and Computation, 2014, 247, 1173-1182
[28] Cánovas M., Alhama I., Trigueros E., Alhama F., Numerical simulation of Nusselt-Rayleigh correlation in Bénard cells. A solution based on the network simulation method. International Journal of Numerical Methods for Heat and Fluid Flow, 2015, 25(5), 986-997
[29] Cánovas M., Alhama I., Trigueros E., Alhama, F., A review of classical dimensionless numbers for the Yusa problem based on discriminated non-dimensionalization of the governing equations. Hydrological Processes, 2016, 30(22), 4101-4112
[30] Pérez J., Conesa M., Alhama I., Solving ordinary differential equations by electrical analogy: a multidisciplinary teaching tool. European Journal of Physics, 2016, 37(6), 065703