DE GRUYTER Forum Math. 2025; 37(6): 1913–1915

Corrigendum

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Paley inequality for the Weyl transform and its applications

https://doi.org/10.1515/forum-2024-0402
Received August 27, 2024; revised January 3, 2025

Corrigendum to: R. Singhal and N. S. Kumar, Paley inequality for the Weyl transform and its applications, Forum Math. 37 (2025), no. 1, 309–323 (https://doi.org/10.1515/forum-2023-0302)

Communicated by: Guozhen Lu

(1) Throughout our article [3], one can work with $\{S_T(n)\}$ instead of $\{S_T^*(n)\}$ as the singular value sequence associated to any compact operator is always decreasing.
(2) In Section 3, Theorem 3.1 and Theorem 3.2 of [3], the Marcinkiewicz interpolation theorem cannot be applied directly to the map $T \mapsto \{S_T(n)\}_{n\in\mathbb{N}}$ and $T \mapsto \{\frac{S_T(n)}{\phi(n)}\}_{n\in\mathbb{N}}$ as the mappings are not sublinear. Hence, an alternate proof of the theorems is provided.

Theorem 3.1. *For $1 < p \le 2$, if $f \in L^{p,p'}(G \times \widehat{G})$, then $W(f) \in \mathcal{B}_{p'}(L^2(G))$ and there exists $C > 0$ such that*

$$\|W(f)\|_{\mathcal{B}_{p'}(L^2(G))} \le C\|f\|_{p,p'}.$$

Proof. Define an operator T on $L^1 + L^{2,1}(G \times \widehat{G})$ by $T(f) = S_{W(f)}(n)$. Since the singular value sequence is a decreasing sequence, using [4, Corollary 1.35], $0 \le S_{A+B}(2n) \le S_{A+B}(2n-1) \le S_A(n) + S_B(n)$ for $n \in \mathbb{N}$ and compact operators A and B. Therefore, for all $n \in \mathbb{N}$, we have

$$|T(f+g)(n)| \le |T(f)(\lceil \tfrac{n}{2} \rceil)| + |T(g)(\lceil \tfrac{n}{2} \rceil)|,$$

where $\lceil \cdot \rceil$ is the least integer function. Now, it can be shown that

$$d_{T(f+g)}(\alpha_1 + \alpha_2) \le 2\big(d_{T(f)}(\alpha_1) + d_{T(g)}(\alpha_2)\big)$$

for $\alpha_1, \alpha_2 > 0$.

Since the Weyl transform maps $L^1(\mathbb{R}^{2n})$ to $\mathcal{B}_\infty(L^2(\mathbb{R}^n))$ and $L^2(\mathbb{R}^{2n})$ to $\mathcal{B}_2(L^2(\mathbb{R}^n))$, similar to classical Marcinkiewicz interpolation theorem ([1, Theorem 4.13]), we can show that T maps $L^{p,p'}(G \times \widehat{G})$ to $l^{p'}(\mathbb{N})$ continuously and

$$\|\{S_{W(f)}(n)\}\|_{l^{p'}} \lesssim \|f\|_{p,p'}.$$

Since $\|W(f)\|_{\mathcal{B}_{p'}(L^2(\mathbb{R}^n))} = \|\{S_{W(f)}(n)\}\|_{l^{p'}}$, we get the desired result. □

Theorem 3.2. *Consider a positive function $\phi \in l^{1,\infty}(\mathbb{N})$. For $1 < p \le 2$ and $f \in L^p(G \times \widehat{G})$, we have*

$$\Big(\sum_{n\in\mathbb{N}} S_{W(f)}(n)^p \phi(n)^{2-p}\Big)^{\frac{1}{p}} \lesssim \|\phi\|_{l^{1,\infty}(\mathbb{N})}^{\frac{2-p}{p}} \|f\|_p.$$

Proof. Consider the measure ν on $\mathbb{N}$ given by

$$\nu(\{n\}) := \phi^2(n). \tag{1}$$

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For $1 < p \leq \infty$, we let $l^p(\mathbb{N}, \nu)$ denote the space of all complex-valued sequences $x = (x_n)_{n\in\mathbb{N}}$ such that

$$\|x\|_p^p = \sum_{n\in\mathbb{N}} |x_n|^p \phi^2(n) < \infty.$$

We now claim that if $f \in L^p(G \times \widehat{G})$, then $\{\frac{S_{W(f)}(n)}{\phi(n)}\}_{n\in\mathbb{N}} \in l^p(\mathbb{N}, \nu)$. We will denote this correspondence by T and show that T is a bounded map. Our strategy here is to use the techniques involved in the Marcinkiewicz interpolation theorem. To do this, we first define sequence $\{P(f)(n)\}_{n\in\mathbb{N}}$ as

$$P(f)(n) := \frac{S_{W(f)}(\lceil \frac{n}{2} \rceil)}{\phi(n)},$$

where $\lceil \cdot \rceil$ is the least integer function.

Similar to the previous theorem, for all $n \in \mathbb{N}$, we have

$$|T(f+g)(n)| = \left|\frac{S_{W(f)+W(g)}(n)}{\phi(n)}\right| \leq \left|\frac{S_{W(f)}(\lceil \frac{n}{2} \rceil)}{\phi(n)}\right| + \left|\frac{S_{W(g)}(\lceil \frac{n}{2} \rceil)}{\phi(n)}\right|.$$

Thus

$$|T(f+g)(n)| \leq |P(f)(n)| + |P(g)(n)|. \tag{2}$$

Now, we claim P is both weak type $(2, 2)$ and $(1, 1)$ with respect to measure ν.

The distribution function, in this case, is given by

$$d_{P(f)}(y) = \nu(\{n \in \mathbb{N} : |P(f)(n)| > y\}).$$

To show that P is of weak type $(1, 1)$, we prove that

$$\|P(f)\|_{1,\infty} \lesssim \|\phi\|_{l^{1,\infty}(\mathbb{N})} \|f\|_1.$$

Since for all $n \in \mathbb{N}$,

$$S_{W(f)}(n) \leq \|W(f)\| \leq \|f\|_1,$$

we have

$$\nu(\{n \in \mathbb{N} : |P(f)(n)| > y\}) \leq \nu\Big(\Big\{n \in \mathbb{N} : \frac{\|f\|_1}{\phi(n)} > y\Big\}\Big).$$

Hence

$$\sum_{\substack{n\in\mathbb{N} \\ y<|P(f)(n)|}} \phi^2(n) \leq \sum_{\substack{n\in\mathbb{N} \\ y<\frac{\|f\|_1}{\phi(n)}}} \phi^2(n).$$

Now, let $w = \frac{\|f\|_1}{y}$. Then

$$\begin{aligned}
\sum_{\substack{n\in\mathbb{N} \\ \phi(n)<w}} \phi^2(n) &= \sum_{\substack{n\in\mathbb{N} \\ \phi(n)<w}} \int_0^{\phi^2(n)} d\tau = \int_0^{w^2} d\tau \sum_{\substack{n\in\mathbb{N} \\ \sqrt{\tau}<\phi(n)<w}} 1 \\
&= \int_0^{w} 2s\, ds \sum_{\substack{n\in\mathbb{N} \\ s<\phi(n)<w}} 1 \leq \int_0^{w} 2\Big(s \sum_{\substack{n\in\mathbb{N} \\ s<\phi(n)}} 1\Big)\, ds \\
&\leq \int_0^{w} 2\|\phi\|_{l^{1,\infty}(\mathbb{N})}\, ds = 2w\|\phi\|_{l^{1,\infty}(\mathbb{N})} = \frac{2\|\phi\|_{l^{1,\infty}(\mathbb{N})}}{y}\|f\|_1.
\end{aligned}$$

Thus, for $y > 0$, we have

$$y d_{P(f)}(y) = y \sum_{\substack{n\in\mathbb{N} \\ y<|P(f)(n)|}} \phi^2(n) \lesssim \|\phi\|_{l^{1,\infty}(\mathbb{N})}\|f\|_1.$$

Also, by using the Plancherel theorem for Weyl transform, it can be seen that T maps $L^2(G \times \widehat{G})$ continuously to $l^2(\mathbb{N}, \nu)$ since

$$\sum_{n\in\mathbb{N}} |P(f)(n)|^2 \phi^2(n) = \sum_{n\in\mathbb{N}} |S_{W(f)}(\lceil \tfrac{n}{2} \rceil)|^2 = 2\sum_{n\in\mathbb{N}} |S_{W(f)}(n)|^2 = 2\|W(f)\|^2_{\mathcal{B}_2(L^2(G))} \lesssim \|f\|_2^2.$$

This shows that P is weak type (2, 2). Now, for $f \in L^p(G \times \widehat{G})$, we define

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \delta\alpha, \\ 0 & \text{for } |f(x)| \leq \delta\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| \leq \delta\alpha, \\ 0 & \text{for } |f(x)| > \delta\alpha, \end{cases}$$

for suitable $\delta > 0$, so that $f = f_0^\alpha + f_1^\alpha$, and by (2), we have

$$|T(f)| \leq |P(f_0^\alpha)| + |P(f_1^\alpha)|.$$

Now the proof follows similar to classical Marcinkiewicz interpolation theorem [2, Theorem 1.3.2] and we have

$$\|T(f)\|_p \lesssim \|\phi\|_{l^{1,\infty}(\mathbb{N})}^{(\frac{2-p}{p})} \|f\|_p$$

or

$$\Big(\sum_{n\in\mathbb{N}} s_{W(f)}(n)^p \phi(n)^{2-p}\Big)^{\frac{1}{p}} \lesssim \|\phi\|_{l^{1,\infty}(\mathbb{N})}^{\frac{2-p}{p}} \|f\|_p.$$

Hence the proof. □

Acknowledgment: We thank Dr. Kanat Tulenov for pointing out this mistake and discussing it afterwards.

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