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Research Article

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Uniform boundedness of oscillatory singular integrals with rational phases

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Abstract: We prove the uniform boundedness of oscillatory singular integrals with singular kernels $|x|^{-n}\Omega(\frac{x}{|x|})$ and rational phases of the form $P(x) + \frac{1}{Q(x)}$ for arbitrary real-valued polynomials P and Q. Our main result shows that the condition $Q(0) = 0$ imposed in [M. Folch-Gabayet and J. Wright, An estimation for a family of oscillatory integrals, *Studia Math.* **154** (2003), no. 1, 89–97] is superfluous, which answers a question left open in that paper. As a secondary improvement of existing results, we also extend the space for $\Omega(\cdot)$ from $L\log L(\mathbb{S}^{n-1})$ to the strictly larger space $H^1(\mathbb{S}^{n-1})$.

Keywords: Oscillatory integrals, singular integrals, Calderón–Zygmund kernels, Hardy spaces

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1 Introduction

The investigation of oscillatory singular integrals has a rich and enduring history ([1, 4, 6–8, 10, 11]). For oscillatory singular integrals in dimensions higher than 1, boundedness does not hold for general rational phases. In [5], the authors obtained some very interesting estimates for oscillatory singular integrals with phase functions of the form $P(x) + \frac{1}{Q(x)}$, where $P(x)$ and $Q(x)$ are real-valued polynomials in n variables. To describe their results, we let $n \geq 2$, $K(x)$ be a Calderón–Zygmund kernel given by

$$K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}, \tag{1.1}$$

where $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ is integrable over the unit sphere $\mathbb{S}^{n-1}$ with respect to the induced Lebesgue measure σ and satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0. \tag{1.2}$$

Let $d \in \mathbb{N} \cup \{0\}$, and let $\mathcal{P}_{n,d}$ denote the space of polynomials in n variables whose coefficients are real and whose degrees do not exceed d. The following is a result from [5].

Theorem 1.1 (Folch-Gabayet and Wright [5]). *Let $K(x)$ be a Calderón–Zygmund kernel given by* (1.1)–(1.2). *Let $P(x), Q(x) \in \mathcal{P}_{n,d}$ such that $Q(0) = 0$ and $\Omega \in L\log L(\mathbb{S}^{n-1})$. Then*

$$\left|\text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+\frac{1}{Q(x)})} K(x)\, dx\right| \leq B, \tag{1.3}$$

where B may depend on $\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}$, n and d but not otherwise on the coefficients of P and Q.

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This naturally led to the following question:

Question. Would the conclusion of Theorem 1.1 hold if the condition $Q(0) = 0$ is removed?

For the case $\deg(Q) = 1$, the authors of [5] answered the above question in the affirmative:

Theorem 1.2 ([5]). *Let $K(x)$ be a Calderón–Zygmund kernel given by* (1.1)–(1.2). *Let $P(x) \in \mathcal{P}_{n,d}$ and $Q(x) = a + v \cdot x$, where $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Suppose that $\Omega \in L\log L(\mathbb{S}^{n-1})$. Then*

$$\Bigg|\text{p.v.} \int\limits_{\mathbb{R}^n} e^{i(P(x)+\frac{1}{Q(x)})} K(x)\,dx\Bigg| \le B, \tag{1.4}$$

where B may depend on $\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}$, n and d but not otherwise on a, v and the coefficients of P.

The main purpose of this paper is to give a complete answer to the question stated above by showing that the condition $Q(0) = 0$ in Theorem 1.1 can be dropped irrespective of the degree of $Q(x)$.

In addition to the improvement of Theorem 1.1 by lifting the vanishing condition on $Q(0)$, we shall also expand the class of $K(x)$ in Theorem 1.1 by allowing $\Omega(\cdot)$ to be in $H^1(\mathbb{S}^{n-1})$, the Hardy space over the unit sphere. It is well known that the space $L\log L(\mathbb{S}^{n-1})$ is a proper subspace of $H^1(\mathbb{S}^{n-1})$. We state our result as follows.

Theorem 1.3. *Let $K(x)$ be a Calderón–Zygmund kernel given by* (1.1)–(1.2). *Let $P(x), Q(x) \in \mathcal{P}_{n,d}$ and $\Omega \in H^1(\mathbb{S}^{n-1})$. Then*

$$\Bigg|\text{p.v.} \int\limits_{\mathbb{R}^n} e^{i(P(x)+\frac{1}{Q(x)})} K(x)\,dx\Bigg| \le B\|\Omega\|_{H^1(\mathbb{S}^{n-1})}, \tag{1.5}$$

where B may depend on n and d but not otherwise on the coefficients of P and Q.

The proof of Theorem 1.3 will appear in Section 3.

Boundedness results such as (1.5) can be used together with Plancherel's Theorem to obtain the L^2 boundedness of corresponding singular integral operators defined by polynomial mappings. We refer the readers to [5] for more details.

In the rest of the paper we shall use $A \lesssim B$ ($A \gtrsim B$) to mean that $A \le cB$ ($A \ge cB$) for a certain constant c whose actual value is not essential for the relevant arguments to work. We shall also use $A \approx B$ to means "$A \lesssim B$ and $B \lesssim A$".

2 A few lemmas

In order to prove Theorem 1.3, one of the tools we shall need is the following lemma:

Lemma 2.1. *Let $A > 1$, $d \in \mathbb{N}$ and*

$$q(t) = \sum_{j=1}^{d} q_j t^j,$$

where $q_1, \dots, q_d \in \mathbb{R}$ and $q_d \neq 0$. Then there are m ($m \le d$) disjoint subintervals $G_1 = (L_1, R_1), \dots, G_m = (L_m, R_m)$ of $(0, \infty)$ such that

(i) $0 = L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m = \infty$,

(ii) *for each $l \in \{1, \dots, m\}$, there exists a $k_l \in \{1, \dots, d\}$ such that*

$$|q_{k_l} t^{k_l}| > A \cdot \max\{|q_k t^k| : k \in \{1, \dots, d\} \backslash \{k_l\}\}$$

for all $t \in G_l$,

(iii) *for every $\xi \in \{L_2, \dots, L_m, R_1, \dots, R_{m-1}\}$, there exists a pair of $j, k \in \{1, \dots, d\}$ such that $j \neq k$ and $\xi \approx |\frac{q_k}{q_j}|^{\frac{1}{j-k}}$,*

(iv) *for $1 \le l \le m-1$,*

$$\frac{L_{l+1}}{R_l} \le A^{\frac{d(d-1)}{2}}.$$

The above lemma can be viewed as a"strengthening" of [5, Lemma 2.1]. For the proof, instead of employing the method of induction as done in [5], we shall use a more direct approach.

Proof. Let $\Lambda = \{j : 1 \le j \le d \text{ and } q_j \ne 0\}$. Since (i)–(iv) hold trivially when $|\Lambda| = 1$, we may assume that $|\Lambda| \ge 2$. For every $j \in \Lambda$, let

$$S_j = \{t \in (0,\infty) : |q_j t^j| > A \cdot \max\{|q_k t^k| : k \in \Lambda \setminus \{j\}\}\}.$$

Then either $S_j = \emptyset$ or $S_j = (a_j, b_j)$ where

$$a_j = \max\Big(\Big\{\Big(A\Big|\frac{q_k}{q_j}\Big|\Big)^{\frac{1}{j-k}} : k \in \Lambda \text{ and } k < j\Big\} \cup \{0\}\Big),$$

$$b_j = \min\Big(\Big\{\Big(A\Big|\frac{q_k}{q_j}\Big|\Big)^{\frac{1}{j-k}} : k \in \Lambda \text{ and } k > j\Big\} \cup \{\infty\}\Big),$$

and $a_j < b_j$. By $A > 1$,

$$S_j \cap S_{j'} = \emptyset$$

for any $j, j' \in \Lambda$ satisfying $j \ne j'$. Let $G_1, \dots, G_m$ denote all the nonempty S_j's, arranged from left to right and let $G_l = (L_l, R_l)$ for $1 \le l \le m$. Clearly, (i)–(iii) are satisfied.

Let $l \in \{1, \dots, m-1\}$. Then there exist an integer s satisfying $1 \le s \le \frac{d(d-1)}{2}$ and a partition

$$R_l = \zeta_0 < \zeta_1 < \dots < \zeta_s = L_{l+1},$$

such that

$$|q_j t^j| \ne |q_k t^k|$$

for all distinct j, k in Λ and $t \in [R_l, L_{l+1}] \setminus \{\zeta_0, \zeta_1, \dots, \zeta_s\}$. For each $\nu \in \{1, \dots, s\}$, there are $j_\nu, k_\nu \in \Lambda$ such that $j_\nu \ne k_\nu$ and

$$\max\{|q_k t^k| : k \in \Lambda \setminus \{j_\nu\}\} = |q_{k_\nu} t^{k_\nu}| < |q_{j_\nu} t^{j_\nu}| \tag{2.1}$$

for all $t \in (\zeta_{\nu-1}, \zeta_\nu)$. Since

$$(\zeta_{\nu-1}, \zeta_\nu) \subseteq (R_l, L_{l+1}) \subseteq (0,\infty) \setminus \bigcup_{j \in \Lambda} S_j,$$

we have

$$|q_{j_\nu} t^{j_\nu}| \le A \cdot \max\{|q_k t^k| : k \in \Lambda \setminus \{j_\nu\}\} = A |q_{k_\nu} t^{k_\nu}| \tag{2.2}$$

for all $t \in (\zeta_{\nu-1}, \zeta_\nu)$. It follows from (2.1) and (2.2) that, for $\nu = 1, \dots, s$,

$$\frac{\zeta_\nu}{\zeta_{\nu-1}} \le A^{\frac{1}{|j_\nu - k_\nu|}} \le A,$$

which implies (iv). □

Next we recall the classical van der Corput's lemma.

Lemma 2.2. (i) *Let ϕ be a real-valued C^k function on $[a,b]$ satisfying $|\phi^{(k)}(x)| \ge 1$ for every $x \in [a,b]$. Suppose that $k \ge 2$, or that $k = 1$ and ϕ' is monotone on $[a,b]$. Then there exists a positive constant c_k such that*

$$\Big|\int_a^b e^{i\lambda\phi(x)}\,dx\Big| \le c_k |\lambda|^{-\frac{1}{k}}$$

for all $\lambda \in \mathbb{R}$. The constant c_k is independent of λ, a, b and ϕ.

(ii) *Let ϕ and c_k be the same as in* (i)*. If $\psi \in C^1([a,b])$, then*

$$\Big|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx\Big| \le c_k |\lambda|^{-\frac{1}{k}} (\|\psi\|_{L^\infty([a,b])} + \|\psi'\|_{L^1([a,b])})$$

holds for all $\lambda \in \mathbb{R}$.

Below is an easy consequence of van der Corput's lemma which will be needed in our proof of Theorem 1.3.

Lemma 2.3. *Let $\Phi(t,u)$ be a real-valued C^∞ function on $[a,b]\times U$, where U is an open set in $\mathbb{R}^m$. Suppose that $d\in\mathbb{N}\cup\{0\}$ and for every $(t,u)\in[a,b]\times U$, there exists an integer $k=k(t,u)>d$ such that*

$$\frac{\partial^k\Phi(t,u)}{\partial t^k}\neq 0.$$

Then, for every compact subset W of U, there exist two positive constants $\rho=\rho(d,m,a,b,\Phi,W)$ and $C=C(d,m,a,b,\Phi,W)$ such that

$$\left|\int_J e^{i[R(t)+\lambda\Phi(t,u)]}\psi(t)\,dt\right|\leq C|\lambda|^{-\rho}(\|\psi\|_{L^\infty(J)}+\|\psi'\|_{L^1(J)})$$

holds for all subintervals J of $[a,\ b]$, $\psi\in C^1(J)$, $\lambda\in\mathbb{R}$, $u\in W$ and $R(\cdot)\in\mathcal{P}_{1,d}$.

To prove the above lemma, one first uses Lemma 2.2 (or a direct integration by parts when $d=0$ and $k=1$) locally and then finish with a compactness argument. Details are omitted.

Another result we shall need is the following lemma from Stein [9, p. 331].

Lemma 2.4. *Let $q(x)$ be a homogeneous polynomial of degree d on $\mathbb{R}^n$. Write*

$$m_q=\int_{\mathbb{S}^{n-1}}|q(\omega)|\,d\sigma(\omega).$$

Then

$$\int_{\mathbb{S}^{n-1}}\left|\ln\left(\left|\frac{q(\omega)}{m_q}\right|\right)\right|\,d\sigma(\omega)\leq B_d,$$

where B_d is independent of $q(\cdot)$.

3 Proof of Theorem 1.3

We are now ready to present the proof of Theorem 1.3. Initially we will assume that $P,Q\in\mathcal{P}_{n,d}$, $\Omega\in L^\infty(\mathbb{S}^{n-1})$ and satisfies the vanishing mean value condition (1.2). We will prove that there exists an $B_{n,d}>0$ independent of Ω and the coefficients of the polynomials P and Q such that

$$\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+\frac{1}{Q(x)})}\frac{\Omega(x)}{|x|^n}\,dx\right|\leq B_{n,d}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.1}$$

Since the case $Q(0)=0$ is already covered by the result of Folch-Gabayet and Wright (see Theorem 1.1), we shall assume that

$$Q(x)=\eta\Big(1+\sum_{1\leq|\alpha|\leq d}a_\alpha x^\alpha\Big), \tag{3.2}$$

where $\eta=Q(0)\neq 0$. For $1\leq k\leq d$, let

$$q_k(x)=\sum_{|\alpha|=k}a_\alpha x^\alpha.$$

Then, for each $k\in\{1,\dots,d\}$, either $q_k(\cdot)\equiv 0$ or $q_k(\omega)\neq 0$ for a.e. $\omega\in\mathbb{S}^{n-1}$.

Let

$$\Lambda=\Lambda_Q=\{k:1\leq k\leq d\text{ and }q_k(\cdot)\not\equiv 0\}.$$

Then, for all $\omega\in\mathbb{S}^{n-1}$ and $t>0$,

$$Q(t\omega)=\eta\Big(1+\sum_{k\in\Lambda}q_k(\omega)t^k\Big). \tag{3.3}$$

We will present our argument for the more general case of $|\Lambda|>1$, while omitting the discussion of the case when $|\Lambda|=1$. However, it can be treated in a similar but simpler manner.

Let $A = 2d^2$. For each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$, let

$$G_j(\omega) = \{t \in (0,\infty) : |q_j(\omega)|t^j > A|q_k(\omega)|t^k \text{ for all } k \in \Lambda\backslash\{j\}\}.$$

Thus,

$$G_j(\omega) = \Big(\bigcap_{k\in\Lambda_j^+}\Big(0, \Big(\frac{A|q_k(\omega)|}{|q_j(\omega)|}\Big)^{\frac{1}{j-k}}\Big)\Big) \cap \Big(\bigcap_{k\in\Lambda_j^-}\Big(\Big(\frac{A|q_k(\omega)|}{|q_j(\omega)|}\Big)^{\frac{1}{j-k}}, \infty\Big)\Big),$$

where we used

$$\Lambda_j^+ = \{k \in \Lambda : k > j\}, \quad \Lambda_j^- = \{k \in \Lambda : k < j\}$$

and the convention that

$$\bigcap_{S\in\emptyset} S = (0,\infty).$$

For each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$, let

$$\begin{aligned} G_j^{(1)}(\omega) &= G_j(\omega) \cap \big(0, 4^{-\frac{d}{j}}|q_j(\omega)|^{-\frac{1}{j}}\big),\\ G_j^{(2)}(\omega) &= G_j(\omega) \cap \big[4^{-\frac{d}{j}}|q_j(\omega)|^{-\frac{1}{j}}, 4^{\frac{d}{j}}|q_j(\omega)|^{-\frac{1}{j}}\big],\\ G_j^{(3)}(\omega) &= G_j(\omega) \cap \big(4^{\frac{d}{j}}|q_j(\omega)|^{-\frac{1}{j}}, \infty\big). \end{aligned}$$

For $t \in G_j^{(3)}(\omega)$, we have

$$Q(t\omega) \approx q_j(\omega)t^j \tag{3.4}$$

and

$$\frac{d}{dt}(Q(t\omega)) \approx \frac{d}{dt}(q_j(\omega)t^j). \tag{3.5}$$

It follows from the arguments in the proof of [5, Theorem 1.1] and (3.4)–(3.5) that

$$\bigg|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\bigg(\sum_{j\in\Lambda}\int_{G_j^{(3)}(\omega)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\bigg)d\sigma(\omega)\bigg| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.6}$$

Trivially,

$$\bigg|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\bigg(\sum_{j\in\Lambda}\int_{G_j^{(2)}(\omega)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\bigg)d\sigma(\omega)\bigg| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.7}$$

Therefore, in order to prove (3.1), it suffices to prove that

$$\bigg|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\bigg(\sum_{j\in\Lambda}\int_{G_j^{(1)}(\omega)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\bigg)d\sigma(\omega)\bigg| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \tag{3.8}$$

and

$$\bigg|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\bigg(\sum_{j\in\Lambda}\int_{(0,\infty)\backslash\bigcup_{j\in\Lambda}G_j(\omega)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\bigg)d\sigma(\omega)\bigg| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.9}$$

Since (3.9) follows from Lemma 2.1 (iv) easily, we will focus our attention on the proof of (3.8). By applying Lemma 2.3 with $[a,\ b] = [0, \frac{1}{4}]$, $u = (u_1, \dots, u_d)$,

$$U = \Big\{u \in \mathbb{R}^d : \frac{4}{5} < \max_{1\le j\le d}|u_j| < \frac{6}{5}\Big\}, \quad W = \Big\{u \in \mathbb{R}^d : \max_{1\le j\le d}|u_j| = 1\Big\},$$

and

$$\Phi(t,u) = \bigg(1 + \sum_{k=1}^{d} u_k t^k\bigg)^{-1}, \tag{3.10}$$

there exist two positive constants $\rho = \rho(d)$ and C_d such that

$$\bigg|\int_J e^{i[R(t)+\lambda\Phi(t,u)]}\psi(t)\,dt\bigg| \le C_d|\lambda|^{-\rho}(\|\psi\|_{L^\infty(J)} + \|\psi'\|_{L^1(J)}) \tag{3.11}$$

holds for all subintervals J of $[0, \frac{1}{4}]$, $\psi \in C^1(J)$, $\lambda \in \mathbb{R}$, $u \in W$ and $R(\cdot) \in \mathcal{P}_{1,d}$.

Let $N = [\rho^{-1}] + 1$, $\delta = \min\{|\eta|^{\frac{1}{N}}, 1\}$. For each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$, let

$$Y_j(\omega,\delta) = G_j^{(1)}(\omega) \cap (|q_j(\omega)|^{-\frac{1}{j}}\delta^{\frac{1}{j}}, \infty),$$
$$H_j(\omega,\delta) = G_j^{(1)}(\omega) \cap \Big[\frac{1}{6}|q_j(\omega)|^{-\frac{1}{j}}\delta^{\frac{1}{j}}, |q_j(\omega)|^{-\frac{1}{j}}\delta^{\frac{1}{j}}\Big],$$
$$Z_j(\omega,\delta) = G_j^{(1)}(\omega) \cap \Big(0, \frac{1}{6}|q_j(\omega)|^{-\frac{1}{j}}\delta^{\frac{1}{j}}\Big).$$

We also let $\nu_\omega \in \Lambda$ such that

$$|q_{\nu_\omega}(\omega)|^{\frac{1}{\nu_\omega}} = \max\{|q_k(\omega)|^{\frac{1}{k}} : k \in \Lambda\}.$$

For each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$, if $Y_j(\omega,\delta) \neq \emptyset$ and $t \in Y_j(\omega,\delta)$, then

$$\delta^{\frac{1}{j}} < |q_j(\omega)|^{\frac{1}{j}} t \leq |q_{\nu_\omega}(\omega)|^{\frac{1}{\nu_\omega}} t = (|q_{\nu_\omega}(\omega)|t^{\nu_\omega})^{\frac{1}{\nu_\omega}} \leq (|q_j(\omega)|t^j)^{\frac{1}{\nu_\omega}} < 4^{-\frac{d}{\nu_\omega}} \leq \frac{1}{4}.$$

It is thus clear that, in this case, $\delta = |\eta|^{\frac{1}{N}} < 1$ and the set

$$|q_{\nu_\omega}(\omega)|^{\frac{1}{\nu_\omega}} Y_j(\omega,\delta) = \{|q_{\nu_\omega}(\omega)|^{\frac{1}{\nu_\omega}} t : t \in Y_j(\omega,\delta)\}$$

is a subinterval of $(\delta^{\frac{1}{j}}, \frac{1}{4})$. Let

$$\tilde{P}_\omega(t) = P(|q_{\nu_\omega}(\omega)|^{-\frac{1}{\nu_\omega}} t\omega),$$

$u = (u_1, \dots, u_d)$, where

$$u_k = \begin{cases} q_k(\omega)|q_{\nu_\omega}(\omega)|^{-\frac{k}{\nu_\omega}} & \text{if } k \in \Lambda, \\ 0 & \text{if } k \notin \Lambda \end{cases}$$

and let $\Phi(\cdot,\cdot)$ be given as in (3.10). By (3.11),

$$\begin{aligned}\Big|\int_{Y_j(\omega,\delta)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\Big| &= \Big|\int_{|q_{\nu_\omega}(\omega)|^{\frac{1}{\nu_\omega}} Y_j(\omega,\delta)} e^{i(\tilde{P}_\omega(t)+\eta^{-1}\Phi(t,u))}\frac{dt}{t}\Big| \\ &\leq C_d|\eta|^\rho\Big(\frac{1}{\delta^{\frac{1}{j}}} + \int_{\delta^{\frac{1}{j}}}^\infty \frac{dt}{t^2}\Big) = 2C_d|\eta|^{\rho-\frac{1}{Nj}} \lesssim 1,\end{aligned}$$

which implies that

$$\Big|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big(\sum_{j\in\Lambda}\int_{Y_j(\omega,\delta)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\Big)d\sigma(\omega)\Big| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.12}$$

Trivially we have

$$\Big|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big(\sum_{j\in\Lambda}\int_{H_j(\omega,\delta)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\Big)d\sigma(\omega)\Big| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.13}$$

Next we shall prove that

$$\Big|\int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big(\sum_{j\in\Lambda}\int_{Z_j(\omega,\delta)} e^{i(P(t\omega)+\frac{1}{Q(t\omega)})}\frac{dt}{t}\Big)d\sigma(\omega)\Big| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}, \tag{3.14}$$

which, together with (3.12)–(3.13), would give us (3.8).

For each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$, if $t \in Z_j(\omega,\delta)$, then

$$\Big|\sum_{k\in\Lambda} q_k(\omega)t^k\Big| \leq (1 + (d-1)A^{-1})|q_j(\omega)|t^j \leq 6^{-j}(1 + (d-1)A^{-1})\delta < \frac{\delta}{4} \leq \frac{1}{4}.$$

Let

$$P_{N,\eta}(x) = P(x) + \eta^{-1}\sum_{s=0}^{N-1}\Big(-\sum_{k\in\Lambda} q_k(x)\Big)^s.$$

Then $P_{N,\eta}(x)$ is a polynomial whose degree does not exceed Nd and, for each $j \in \Lambda$ and a.e. $\omega \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\int\limits_{Z_j(\omega,\delta)} \left|e^{i(P(t\omega)+\frac{1}{Q(t\omega)})} - e^{iP_{N,\eta}(t\omega)}\right| \frac{dt}{t} &\le \frac{4}{3}|\eta|^{-1} \int\limits_{Z_j(\omega,\delta)} \left|\sum_{k\in\Lambda} q_k(\omega)t^k\right|^N \frac{dt}{t} \\ &\le \frac{4}{3}d^N|\eta|^{-1}\max\{(|q_j(\omega)|t^j)^N : t \in Z_j(\omega,\delta)\} \\ &\le \frac{4}{3}d^N|\eta|^{-1}\delta^N \lesssim 1.\end{aligned} \tag{3.15}$$

Recall that Stein established in [9] that, for $K(\cdot)$ given as in (1.1) with an $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $\{\gamma_\alpha : |\alpha| \le M\} \subset \mathbb{R}$,

$$\left|\int\limits_{\varepsilon_1\le|x|\le\varepsilon_2} e^{i\left(\sum_{|\alpha|\le M}\gamma_\alpha x^\alpha\right)}K(x)\,dx\right| \le A_M,$$

where A_M is independent of ε_1, ε_2 and γ_α (see [9, pp. 334–335]). It follows that, for all $h > 0$,

$$\left|\text{p.v.}\int\limits_{|x|\le h} e^{iP_{N,\eta}(x)}K(x)\,dx\right| \le B_d\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \tag{3.16}$$

holds with a B_d independent of h and the coefficients of $P_{N,\eta}$. Observe that each $Z_j(\omega,\delta)$ is the intersection of no more than $(d+1)$ intervals in the form of either $(0, (|q(\omega)/\tilde{q}(\omega)|^\gamma)$ or $((|q(\omega)/\tilde{q}(\omega)|^\gamma, \infty)$, where $q(\cdot)$, $\tilde{q}(\cdot)$ are homogeneous polynomials of degrees not exceeding d, and $|\gamma| \le 1$. Thus, one may use (3.16) and Lemma 2.4 to get

$$\left|\int\limits_{\mathbb{S}^{n-1}} \Omega(\omega)\left(\sum_{j\in\Lambda}\int\limits_{Z_j(\omega,\delta)} e^{iP_{N,\eta}(t\omega)}\frac{dt}{t}\right)d\sigma(\omega)\right| \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{3.17}$$

By combining (3.17) and (3.15), we obtain (3.14). Then (3.8) follows, as does (3.1).

We will end our proof of Theorem 1.3 by providing a brief explanation of how one can derive (1.5) from (3.1).

Let $\beta : \mathbb{S}^{n-1} \to \mathbb{C}$ be an arbitrary regular H^1 atom on $\mathbb{S}^{n-1}$, i.e. $\beta(\cdot)$ is supported in $\mathbb{S}^{n-1} \cap B(p,h)$ for some $p \in \mathbb{S}^{n-1}$ and $h > 0$, has mean-value zero over $\mathbb{S}^{n-1}$, and satisfies $\|\beta\|_\infty \le h^{1-n}$.

Suppose that $h < \frac{1}{4}$. Let M be an orthogonal matrix which has p as its first row vector. Define the linear transformation L on $\mathbb{R}^n$ by

$$L(x_1, x_2, \dots, x_n) = (x_1, hx_2, \dots, hx_n)M,$$

and let

$$\tilde{\Omega}(x) = \frac{\det(L)|x|^n\beta(\frac{Lx}{|Lx|})}{|Lx|^n}$$

for $x \in \mathbb{R}^n\backslash\{0\}$. Then $\tilde{\Omega}$ is homogeneous of degree 0, has mean-value zero over $\mathbb{S}^{n-1}$ and satisfies $\|\tilde{\Omega}\|_\infty \lesssim 1$. By the uniform nature of (3.1) which allows it to be applied with $P(x)$ and $Q(x)$ replaced by $P(Lx)$ and $Q(Lx)$, we get

$$\left|\text{p.v.}\int\limits_{\mathbb{R}^n} e^{i(P(x)+\frac{1}{Q(x)})}\frac{\beta(\frac{x}{|x|})}{|x|^n}\,dx\right| = \left|\text{p.v.}\int\limits_{\mathbb{R}^n} e^{i(P(Lx)+\frac{1}{Q(Lx)})}\frac{\tilde{\Omega}(x)}{|x|^n}\,dx\right| \le B_{n,d}\|\tilde{\Omega}\|_{L^\infty(\mathbb{S}^{n-1})} \lesssim 1 \tag{3.18}$$

for all $h \in (0, \frac{1}{4})$. The above inequality also holds when $h \ge \frac{1}{4}$ as it follows directly from (3.1) and $\|\beta\|_\infty \le h^{1-n} \lesssim 1$. Finally, by the atomic decomposition of $H^1(\mathbb{S}^{n-1})$ (see [2, 3]) and (3.18), one obtains (1.5). The proof of Theorem 1.3 is now complete.

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