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Research Article

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Finite rigid sets of the non-separating curve complex

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Abstract: We prove that the non-separating curve complex of every surface of finite type and genus at least three admits an exhaustion by finite rigid sets.

Keywords: Curve complex, mapping class group, simplicial rigidity

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1 Introduction

Let S be a connected, orientable, finite-type surface. The *curve complex* is the simplicial complex $\mathcal{C}(S)$ whose k-simplices are sets of to $k+1$ distinct isotopy classes of essential simple curves on S that are pairwise disjoint.

The extended mapping class group, denoted by $\mathrm{Mod}^{\pm}(S)$, acts naturally on the set of curves up to isotopy on S. This action preserves disjointness of curves, and therefore extends to an action on the complex $\mathcal{C}(S)$. Via this action, the curve complex works as a combinatorial model to study properties of $\mathrm{Mod}^{\pm}(S)$. For instance, a celebrated theorem of Ivanov in [12] asserts that, for sufficiently complex surfaces, the group $\mathrm{Mod}^{\pm}(S)$ is isomorphic to the group of simplicial automorphisms of the curve complex, a result commonly known as *simplicial rigidity*. In turn, this result is a key ingredient in establishing the isomorphism $\mathrm{Aut}(\mathrm{Mod}^{\pm}(S)) \cong \mathrm{Mod}^{\pm}(S)$.

The curve complex, and its applications to the mapping class group, has motivated the study of similar complexes associated to surfaces. For example, simplicial rigidity has been established for the arc complex [11], the non-separating curve complex [8], the separating curve complex [3, 13], the Hatcher–Thurston complex [10], and the pants graph [14] (see [16] for a survey on complexes associated to surfaces).

Another notion of rigidity which has been of recent interest is that of *finite rigidity:* the simplicial complex $\mathcal{C}(S)$ is said to be finitely rigid if there exists a finite subcomplex X such that any locally injective simplicial map

$$\phi : X \to \mathcal{C}(S)$$

is induced by a *unique* mapping class, that is, there exists a unique $h \in \mathrm{Mod}^{\pm}(S)$ such that the simplicial action $h : \mathcal{C}(S) \to \mathcal{C}(S)$ satisfies $h|_X = \phi$. Such X is called a *finite rigid set* of $\mathcal{C}(S)$ with *trivial pointwise stabilizer*.

The finite rigidity of the curve complex was proven by Aramayona and Leininger in [1], thus answering a question by Lars Louder. Furthermore, they constructed in [2] an exhaustion of $\mathcal{C}(S)$ by finite rigid sets with trivial pointwise stabilizers, thus recovering Ivanov's result [12] on the simplicial rigidity of $\mathcal{C}(S)$.

Following the result of Aramayona and Leininger, finite rigidity has been proven for other complexes: Shinkle proved it for the arc complex [17] and the flip graph [18], Hernández, Leininger and Maungchang proved a slightly different notion for the pants graph [5, 15], and Huang and Tshishiku proved a weaker notion for the separating curve complex [6].

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The main goal of this article is to prove the finite rigidity of the *non-separating curve complex* $\mathcal{N}(S)$, which is the subcomplex of $\mathcal{C}(S)$ spanned by the non-separating curves. To prove the finite rigidity of $\mathcal{N}(S)$, one would like to restrict the finite rigid set of $\mathcal{C}(S)$ in [1] to $\mathcal{N}(S)$; however, it is not clear why this restriction would yield a finite rigid set in $\mathcal{N}(S)$ as their proof uses separating curves in a fundamental way. Below, we construct a different subcomplex of $\mathcal{N}(S)$ and prove its rigidity.

Our main result is compiled in the next theorem.

Theorem 1.1. *Let S be a connected, orientable, finite-type surface of genus $g \geq 3$. There exists a finite simplicial complex $X \subset \mathcal{N}(S)$ such that any locally injective simplicial map*

$$\phi : X \to \mathcal{N}(S)$$

is induced by a unique $h \in \mathrm{Mod}^{\pm}(S)$.

Our second result produces an exhaustion of the non-separating curve complex by finite rigid sets.

Theorem 1.2. *Let S be an orientable finite-type surface of genus $g \geq 3$. There exist subcomplexes*

$$X_1 \subset X_2 \subset \cdots \subset \mathcal{N}(S)$$

such that

$$\bigcup_{i=1}^{\infty} X_i = \mathcal{N}(S)$$

and each X_i is a finite rigid set with trivial pointwise stabilizer.

From Theorem 1.2 we can recover the simplicial rigidity of $\mathcal{N}(S)$ (see [9, Theorem 1.1]).

Corollary 1.3. *Let S be a connected, orientable, finite-type surface of genus $g \geq 3$. Any locally injective simplicial map $\phi : \mathcal{N}(S) \to \mathcal{N}(S)$ is induced by a unique $h \in \mathrm{Mod}^{\pm}(S)$. In particular, this yields an isomorphism between $\mathrm{Mod}^{\pm}(S)$ and the group of simplicial automorphisms of $\mathcal{N}(S)$.*

Plan of the paper. In Section 2, we introduce some basic definitions that will be required. In Section 3, we introduce the notion of finite rigid sets. Sections 4 and 5 deal with the proofs of Theorems 1.1 and 1.2 for closed surfaces. Lastly, Sections 6 and 7 present the proofs of Theorems 1.1 and 1.2 for punctured surfaces.

2 Preliminaries

Let S be a connected, orientable surface without boundary. We will further assume that S has finite type, i.e., $\pi_1(S)$ is finitely generated. As such, S is homeomorphic to $S_{g,n}$, the result of removing n points from a genus g surface. We refer to the removed points as *punctures*. If S has no punctures, we will say that S is *closed*. Otherwise, we will refer to S as a *punctured* surface.

Before fervently jumping into the proofs, we warn the reader that the classification of surfaces, the change of coordinates principle and the Alexander method will be frequently used in proofs, sometimes without mention. For these and other fundamental results on mapping class groups, we refer the reader to [4].

2.1 Curves

By a curve c in S we will mean the isotopy class of an unoriented simple closed curve that does not bound a disk or a punctured disk. Throughout the article, we will make no distinction between curves and their representatives. We will say that c is *non-separating* if any representative γ of c has connected complement in S.

The *intersection number* $i(a, b)$ between two curves a and b is the minimum intersection number between representatives of a and b. If $i(a, b) = 0$, we will say that the curves a and b are *disjoint*. Given two representatives $\alpha \in a$ and $\beta \in b$, we say that they are in *minimal position* if $i(a, b) = |\alpha \cap \beta|$. A fact that will be often used is that for any set of curves we may pick a single representative for each curve such that the representatives are pairwise in minimal position (see [4, Chapter 1.2]).

Given a set of curves $\{c_1, \dots, c_k\}$, consider representatives $\gamma_i \in c_i$ pairwise in minimal position. We will denote a regular neighborhood of $\bigcup \gamma_i$ by $N(\bigcup \gamma_i)$. The set of curves in the boundary of $N(\bigcup \gamma_i)$ will be denoted by

$$\partial(c_1, \dots, c_k).$$

We emphasize that implicit in the definition of $b \in \partial(c_1, \dots, c_k)$ is that b is an isotopy class of a simple closed curve which does not bound a disk or a punctured disk.

2.2 Non-separating curve complex

The *non-separating curve complex* $\mathcal{N}(S)$ is the simplicial complex whose k-simplices are sets of $k+1$ isotopy classes of pairwise disjoint curves.

Note that we can endow $\mathcal{N}(S)$ with a metric by declaring each k-simplex to have the standard euclidean metric and considering the resulting path metric on $\mathcal{N}(S)$.

2.2.1 Pants decompositions

The dimension of $\mathcal{N}(S_{g,n})$ is $3g-3+n$, and the vertex set of a top-dimensional simplex in $\mathcal{N}(S_{g,n})$ is called a *non-separating pants decomposition* of $S_{g,n}$. If $P = \{c_1, \dots, c_k\}$ is a pants decomposition of S, then $S \setminus \bigcup c_i$ is a union of *pairs of pants* (i.e., a union of subsurfaces homeomorphic to $S_{0,3}$).

Let $P = \{c_1, \dots, c_k\}$ be a pants decomposition of the surface S. Two curves $c_i, c_j \in P$ are said to be *adjacent rel to P* if there exists a curve $c_k \in P$ such that c_i, c_j, c_k bound a pair of pants in S; see Figure 1a for an example.

We record the following observation for future use.

Remark 2.1. Let P be a non-separating pants decomposition of $S_{g,n}$, where $g \geq 3$.
- If $n \leq 1$, then every curve in P is adjacent rel to P to at least three other curves.
- If $n > 1$, then every curve is adjacent to at least two other curves.

Consider $A \subset P$, where P is a pants decomposition of S. We say that a set of curves $\tilde{A}$ *substitutes A in P* if

$$(\tilde{A} \cup P) \setminus A$$

is a pants decomposition. In words, we say that $\tilde{A}$ substitutes A in P if both sets have no curves in common and we can replace the curves in A by the curves in $\tilde{A}$ and still get a pants decomposition.

3 Finite rigid sets

For a simplicial subcomplex $X \subset \mathcal{N}(S)$, a map $\phi : X \to \mathcal{N}(S)$ is said to be a *locally injective simplicial map* if ϕ is simplicial and injective on the star of each vertex. A first elementary observation is the following lemma.

Lemma 3.1. *Let $\phi : X \to \mathcal{N}(S)$ be a locally injective simplicial map. If $P \subset X$ is a pants decomposition, then $\phi(P)$ is a pants decomposition.*

Proof. Take a vertex $p \in P$. Since ϕ is injective in the star of p and P is a simplex, ϕ is injective on P. Thus, $\phi(P)$ is a maximal-dimensional simplex, i.e., $\phi(P)$ is a pants decomposition. □

As mentioned in Section 1, the main goal of this article is to construct a finite subcomplex $X \subset \mathcal{N}(S)$ with the following properties.

Definition 3.2 (Finite rigid set). A *finite rigid set* X of $\mathcal{N}(S)$ is a finite subcomplex such that any locally injective simplicial map $\phi : X \to \mathcal{N}(S)$ is induced by a mapping class, i.e., there exists $h \in \mathrm{Mod}^{\pm}(S)$ with $h|_X = \phi$.

In addition, if h is unique, we say that X has *trivial pointwise stabilizer*.

Observe that a subcomplex $X \subset \mathcal{N}(S)$ has trivial pointwise stabilizer if and only if the inclusion $X \hookrightarrow \mathcal{N}(S)$ is induced uniquely by the identity $1 \in \mathrm{Mod}^{\pm}(S)$, hence the name.

Remark 3.3. By the change of coordinates principle (see [4, Chapter 1.3]), every vertex $\{v\} \subset \mathcal{N}(S)$ is a finite rigid set. However, the stabilizer of $\{v\}$ is not trivial.

Remark 3.4. If X is a finite rigid set and $X \subset Y$, then Y may not be a finite rigid set

For example, consider two disjoint curves v_1, v_2 such that $S \setminus \bigcup v_i$ is connected, and two disjoint curves v_1', v_2' such that $S \setminus \bigcup v_i'$ is disconnected. Now, take $X = \{v_1\}$, $Y = \{v_1, v_2\}$ and the locally injective simplicial map $\phi(v_i) = v_i'$. Clearly, ϕ is not induced by a mapping class, and so Y is not a finite rigid set of $\mathcal{N}(S)$. Note that X is a finite rigid set of $\mathcal{N}(S)$ by the remark above.

Following Aramayona and Leininger in [1], we will say that a subcomplex $X \subset \mathcal{N}(S)$ *detects the intersection* of two curves $a, b \in X$ if every locally injective simplicial map $\phi : X \to \mathcal{N}(S)$ satisfies

$$i(a, b) \neq 0 \quad \text{if and only if} \quad i(\phi(a), \phi(b)) \neq 0.$$

4 Finite rigid sets for closed surfaces

In this section, we construct finite rigid sets for closed surfaces and prove their rigidity. This will establish Theorem 1.1 for closed surfaces.

4.1 Constructing the finite rigid set

Let S be a closed surface of genus $g \geq 3$. We will start by defining the curves in the finite rigid set. The reader should keep Figures 1 (a)–(e) in mind throughout the section.

Fix a set $\{p_1, c_1, \dots, p_g, c_g, p_{g+1}\}$ of non-separating curves such that $i(c_i, p_i) = i(c_i, p_{i+1}) = 1$ and the rest of the curves are pairwise disjoint (see Figures 1 (a) and 1 (b)). Such a set of curves is unique up to homeomorphism. Let c_{g+1} be a curve such that $i(p_1, c_{g+1}) = i(p_{g+1}, c_{g+1}) = 1$ and it is disjoint from every other curve in the set above. We define

$$C = \{c_1, \dots, c_{g+1}\}.$$

Notice that $S \setminus \bigcup p_i$ has two connected components $(S \setminus \bigcup p_i)^+$ and $(S \setminus \bigcup p_i)^-$; we will call $(S \setminus \bigcup p_i)^+$ the *top component* and $(S \setminus \bigcup p_i)^-$ the *bottom component*. In the same fashion, $S \setminus \bigcup c_i$ has two connected components $(S \setminus \bigcup c_i)^+$ and $(S \setminus \bigcup c_i)^-$; we will call $(S \setminus \bigcup c_i)^+$ the *front component* and $(S \setminus \bigcup c_i)^-$ the *back component*.

For each $k = 2, \dots, g-1$, the set $\partial(p_1, c_1, \dots, p_k, c_k)$ consists of two curves: one of them in $(S \setminus \bigcup p_i)^+$ and the other one in $(S \setminus \bigcup p_i)^-$. We will call p_k^+ the curve of $\partial(p_1, c_1, \dots, p_k, c_k)$ contained in $(S \setminus \bigcup p_i)^+$, and we will call p_k^- the curve of $\partial(p_1, c_1, \dots, p_k, c_k)$ in $(S \setminus \bigcup p_i)^-$. We set

$$P = \{p_1 \dots, p_{g+1}\} \cup \{p_2^+, p_2^-, \dots, p_{g-1}^+, p_{g-1}^-\}.$$

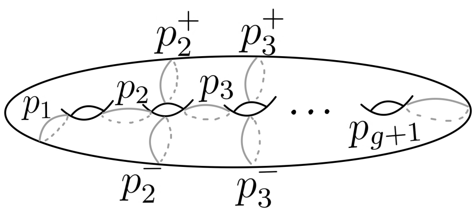
Notice that P is a pants decomposition (see Figure 1 (a)).

For each $k = 2, \dots, g-1$, the set $\partial(p_{k-1}, c_k, p_k)$ has two curves, one in $(S \setminus \bigcup p_i)^+$ and the other one in $(S \setminus \bigcup p_i)^-$. We will denote by u_k the curve in $(S \setminus \bigcup p_i)^+$ and by d_k the curve in $(S \setminus \bigcup p_i)^-$ (see Figure 1 (c)). We set

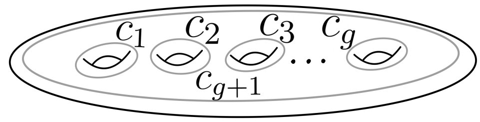
$$U = \{u_2, \dots, u_{g-1}\}$$

and

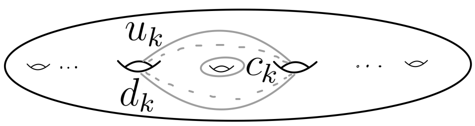
$$D = \{d_2, \dots, d_{g-1}\}.$$



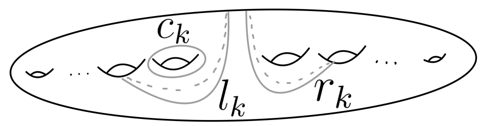
(a) Pants decomposition P of surface S_g. As an example, note that p_1 and p_2 are adjacent rel to P, while p_1 and p_3 are not adjacent rel to P.



(b) Circular curves C.

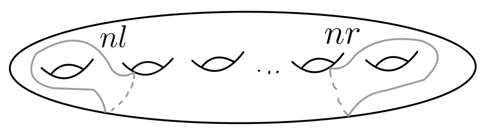


(c) Up and down curves u_k, d_k.



(d) Left and right curves l_k, r_k.

(e) Non-symmetrical down curve nd.



(f) Non-symmetrical left and right curves nl, nr.

Figure 1: Curves in F_R for a closed surface.

Given $k \in \{2, \dots, g-1\}$, the set $\partial(p_k, c_k, p_k^+)$ contains two curves, and only one of them is also a curve in P. We will denote by l_k the curve in $\partial(p_k, c_k, p_k^+)$ not already in P (see Figure 1 (d)). We set

$$L = \{l_2, \dots, l_{g-2}\}.$$

Analogously, let $R = \{r_2, \dots, r_{g-2}\}$ be the set of curves where r_k is the unique curve in $\partial({p_{k+1}}^+, c_{k+1}, p_{k+2})$ that is not in P (see Figure 1 (d)).

The set $\partial(p_2, c_2, \dots, p_{g-1}, c_{g-1}, p_g)$ has two curves, one curve in each component of $S \setminus \bigcup p_i$. Let b be the curve contained in the bottom component $(S \setminus \bigcup p_i)^-$. Then the set $\partial(c_1, b, c_g)$ has exactly two curves, one curve contained in $(S \setminus \bigcup c_i)^+$ and the other in $(S \setminus \bigcup c_i)^-$. Denote by nd the curve in $(S \setminus \bigcup c_i)^+$ (see Figure 1 (e)).

Lastly, consider the torus T_1 that contains p_1 and is bounded by the curves p_2^+, p_2^-. Let nl be the unique curve contained in $T_1 \setminus (nd \cup c_1)$ distinct from c_1 and p_2^+. In the same way, p_{g-1}^+, p_{g-1}^- bound a torus T_g such that $p_g \subset T_g$, let nr be the unique curve in $T_g \setminus (nd \cup c_g)$ distinct from c_g and p_{g-1}^+ (see Figure 1 (f)). We set

$$N = \{nl, nr, nd\}.$$

We set F_R to be the subcomplex of $\mathcal{N}(S)$ spanned by the vertices in

$$P \cup C \cup U \cup D \cup L \cup R \cup N.$$

Remark 4.1. Note that the subcomplex F_R has diameter two. Therefore, any locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ is injective.

4.2 Proving the rigidity of F_R

The first step is to check that the locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ preserves the *non*-adjacency rel to P. To do so, we require the following technical lemma.

Lemma 4.2. *Let S be a finite-type surface, let $F_R \subset \mathcal{N}(S)$ be a subcomplex, let $P \subset F_R$ be a pants decomposition, and let $a, b \in P$ be two curves. Suppose that there exist subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ satisfying the following*

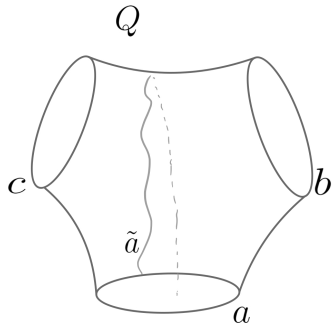


Figure 2: There can be no arc $\tilde{b} \cap Q$ that intersects b and does not intersect $\tilde{a} \cup a$.

assertions:
- $a \in A$ *and* $b \in B$.
- $\tilde{A}$ *substitutes* A *in* P.
- $\tilde{B}$ *substitutes* B *in* P.
- $\tilde{A} \cup \tilde{B}$ *substitutes* $A \cup B$ *in* P.
- $A \cap B = \emptyset$.

Then a, b *are not adjacent rel to* P.

Proof. We will proceed by contradiction. Suppose that the curves a, b are adjacent in a pair of pants Q bound by a, b, c. Since $A \cap B = \emptyset$, either $c \notin A$ or $c \notin B$. Without loss of generality, suppose $c \notin A$.

If $\tilde{A}$ is a substitution of A and $A \cap B = \emptyset$, then there is a curve $\tilde{a} \in \tilde{A}$ such that $i(\tilde{a}, a) \neq 0$, $i(\tilde{a}, b) = 0$ and $i(\tilde{a}, c) = 0$. Now, note that, since $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$ are substitutions, there exists $\tilde{b} \in \tilde{B}$ with $i(\tilde{b}, b) \neq 0$, $i(\tilde{b}, a) = 0$ and $i(\tilde{b}, \tilde{a}) = 0$. However, it is impossible to have arcs $\tilde{a} \cap Q$ and $\tilde{b} \cap Q$ satisfying the intersections above (see Figure 2). □

Now, we prove that ϕ preserves non-adjacency rel to P.

Lemma 4.3. *Let S be a closed surface of genus $g \geq 3$ and let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. If $a, b \in P$ are not adjacent rel to P, then $\phi(a), \phi(b)$ are not adjacent rel to $\phi(P)$.*

Proof. Assume that for two curves $a, b \in P$ we have subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ as in Lemma 4.2. Under these conditions, the lemma ensures that a and b are not adjacent rel to P. Moreover, these properties are carried to the image, that is, the curves $\phi(a), \phi(b) \in \phi(P)$ satisfy the conditions of Lemma 4.2 for the sets $\phi(A), \phi(B) \subset \phi(P)$ and $\phi(\tilde{A}), \phi(\tilde{B})$. As a consequence, we deduce that $\phi(a)$ and $\phi(b)$ are not adjacent rel to $\phi(P)$.

By means of the method above, we are only left to find appropriate subsets $A, \tilde{A}, B, \tilde{B}$ for any non-adjacent curves $a, b \in P$. We will find such subsets for certain $a, b \in P$, as the rest of the cases are similar.

If $a = p_k$ and $b = p_{k+1}$ for $k \in \{3, \ldots, g-3\}$, we can consider

$$A = \{p^-_{k-1}, p_k\}, \qquad \tilde{A} = \{l_{k-1}, d_{k-1}\},$$
$$B = \{p_{k+1}, p^-_{k+1}\}, \qquad \tilde{B} = \{r_k, d_{k+1}\}.$$

It is straightforward to check that these subsets satisfy the conditions of Lemma 4.2, and so $\phi(a), \phi(b)$ are not adjacent rel to $\phi(P)$.

If $a = p_1$, consider $A = \{p_1, p_2\}$ and $\tilde{A} = \{nl, c_1\}$.
- If $b \in \{p_g, p_{g+1}\}$, take $B = \{p_g, p_{g+1}\}$ and $\tilde{B} = \{nr, c_g\}$.
- If $b \in \{p_k, p^-_k\}$ for $k \in \{3, \ldots, g-1\}$, consider $B = \{p_k, p^-_k\}$ and $\tilde{B} = \{c_k, r_{k-1}\}$.
- If $b = p^+_k$ for $k \in \{3, \ldots, g-1\}$, consider $B = \{p^+_k\}$ and $\tilde{B} = \{u_k\}$.

This concludes the proof. □

Using the previous result, we prove that ϕ preserves adjacency rel to P.

Lemma 4.4. *Let S be a closed surface of genus $g \geq 3$ and let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. If $a, b \in P$ are adjacent rel to P, then $\phi(a), \phi(b)$ are adjacent rel to $\phi(P)$.*

Proof. Take $\phi(p_1) \in \phi(P)$. From the non-adjacency rel to $\phi(P)$, it follows that $\phi(p_1)$ has at most three adjacent curves. On the other hand, Remark 2.1 implies that $\phi(p_1)$ has at least three adjacent curves. Thus, we conclude that $\phi(p_1)$ has exactly three adjacent curves, namely $\phi(p_2)$, $\phi(p_2^+)$, $\phi(p_2^-)$. The same argument applies to $\phi(p_2)$, so it is adjacent rel to $\phi(P)$ to exactly three curves, namely $\phi(p_1)$, $\phi(p_2^+)$, $\phi(p_2^-)$.

We now determine the curves adjacent to $\phi(p_2^+)$ and $\phi(p_2^-)$. First, note that the adjacency rel to $\phi(P)$ of $\phi(p_1)$ and $\phi(p_2)$ implies that $\phi(p_2^+)$ and $\phi(p_2^-)$ bound a subsurface homeomorphic to $S_{1,2}$. Since both curves $\phi(p_2^+)$ and $\phi(p_2^-)$ are non-separating, it follows that both have four adjacent curves rel to $\phi(P)$. Finally, considering the non-adjacency rel to $\phi(P)$, it follows that the curves adjacent to $\phi(p_2^+)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_3^+)\}$$

and the curves adjacent to $\phi(p_2^-)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_3^-)\}.$$

In the same style, we can argue inductively to determine the adjacency of each curve in $\phi(P)$. □

As a corollary, we obtain that ϕ preserves the topological type of P.

Corollary 4.5. *Let S be a finite-type surface and let $\phi : F_R \to \mathcal{N}(S)$ be a map that preserves adjacency rel to P. There exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h|_P = \phi|_P$.*

Proof. We construct a homeomorphism h inductively by gluing abstract pairs of pants.

Consider the pairs of pants $Q_1, \dots, Q_k \subset S$ bound by curves in P and denote by $Q'_1, \dots, Q'_k \subset S$ the pairs of pants satisfying $\partial Q'_i = \phi(\partial Q_i)$. Any two pairs of pants are homeomorphic, and thus we can consider homeomorphisms $h_i(Q_i) = Q'_i$. Since ϕ preserves the adjacency rel to P, we can ensure that these homeomorphisms agree on the boundary curves. Then, by gluing the maps h_i, we obtain a homeomorphism h of the surface such that $h|_P = \phi|_P$. □

The next three lemmas prove that F_R detects intersection among certain curves. Recall that F_R detects the intersection between a and b if for any locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ we have that

$$i(a, b) \neq 0 \quad \text{if and only if} \quad i(\phi(a), \phi(b)) \neq 0.$$

Lemma 4.6. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the following intersections for every $k = 2, \dots, g-1$:*
(i) u_k *with* p_k^+.
(ii) d_k *with* p_k^-.

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. We need to check that $i(\phi(u_k), \phi(p_k^+)) \neq 0$ and $i(\phi(d_k), \phi(p_k^-)) \neq 0$.

Seeking a contradiction to case (i), we assume $i(\phi(u_k), \phi(p_k^+)) = 0$. Since ϕ is locally injective, it sends disjoint curves to disjoint curves. Thus, $\phi(u_k)$ is disjoint from every curve in the pants decomposition $\phi(P)$, which implies $\phi(u_k) \in \phi(P)$. However, this contradicts the injectivity of ϕ (see Remark 4.1).

To prove case (ii), the same argument works. □

Remark 4.7. Notice that from the previous lemma we actually know that F_R detects the intersection of u_k with every curve in P. Indeed, u_k is disjoint from any curve in $P \setminus \{p_k^+\}$ and ϕ preserves disjointness. In the same way, F_R detects the intersection of d_k with every curve in P.

Lemma 4.8. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the intersection of c_k with p_k^- and p_k^+ for every $k \in \{2, \dots, g-1\}$.*

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. By Corollary 4.5, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes the pants decomposition P. Observe that detecting intersection is equivalent for ϕ and for $h \circ \phi$, since we have

$$i(\phi(a), \phi(b)) \neq 0 \quad \text{if and only if} \quad i(h \circ \phi(a), h \circ \phi(b)) \neq 0.$$

So, we can rename $h \circ \phi$ to ϕ, and prove the statement assuming that ϕ fixes every $p \in P$.

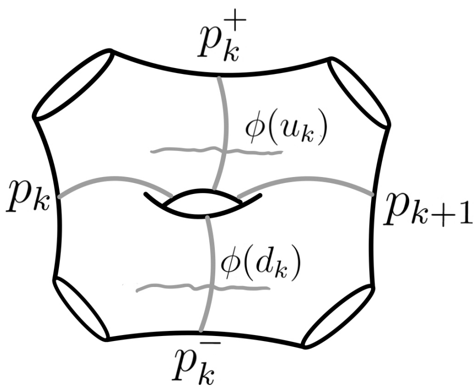


Figure 3: Torus T_k.

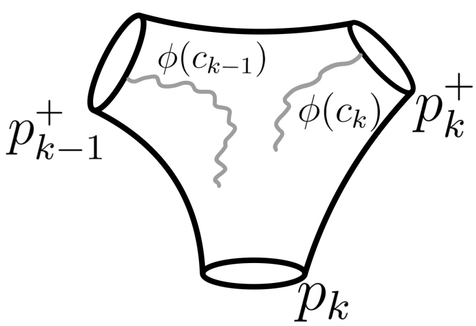


Figure 4: Pair of pants bounded by p_{k-1}^+, p_k and p_k^+.

With the previous simplification, the proof boils down to check that $\phi(c_k)$ intersects p_k^+ and p_k^-. In this direction, consider the torus T_k bounded by the curves p_{k-1}^-, p_{k-1}^+, p_{k+1}^- and p_{k+1}^+ (see Figure 3). Further, define the top of the torus by $T_k^+ = T_k \cap (S \setminus \bigcup p_i)^+$, and the bottom of the torus by $T_k^- = T_k \cap (S \setminus \bigcup p_i)^-$.

By Lemma 4.6, $\phi(u_k)$ is a curve in T_k^+ intersecting p_k^+, and $\phi(d_k)$ is a curve in T_k^- intersecting p_k^- (see Figure 3). Notice that $\phi(c_k)$ is a curve in T_k distinct and disjoint from $\phi(d_k)$ and $\phi(u_k)$. It follows that $\phi(c_k)$ intersects both T_k^+ and T_k^-.

To finish, note that $\phi(c_k) \cap T_k^+$ is disjoint from $\phi(u_k)$, so $\phi(c_k)$ must intersect p_k^+. Indeed, an arc disjoint from a curve in a sphere with four boundary components must intersect every other curve in the sphere. Similarly, $\phi(c_k) \cap T_k^-$ is disjoint from $\phi(d_k)$, so $\phi(c_k)$ must intersect p_k^-. □

Lemma 4.9. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the intersection of $c_k \in C$ with every curve in P.*

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. As in the previous proof, we may assume that ϕ fixes every curve in P.

Now, we start by proving the cases $2 \leq k \leq g-1$. Note that, with the simplification above and Lemma 4.8, we only need to check that $\phi(c_k)$ intersects p_k and p_{k+1}.

To prove that $\phi(c_k)$ intersects p_k, consider the pair of pants Q bounded by the curves p_{k-1}^+, p_k and p_k^+. By Lemma 4.8, there are disjoint arcs $Q \cap \phi(c_{k-1})$ and $Q \cap \phi(c_k)$ with at least one endpoint in p_{k-1}^+ and p_k^+, respectively (see Figure 4). Using that $\phi(c_{k-1})$, p_k^+ are disjoint and $\phi(c_k)$, p_{k-1}^+ are also disjoint, we conclude that any such arc configuration requires $\phi(c_k)$ to intersect p_k. The same argument yields that $\phi(c_k)$ intersects p_{k+1}.

It is left to prove the cases $k \in \{1, g, g+1\}$. First, we prove that $\phi(c_1)$ intersects p_2. Consider the torus T_1 bounded by the curves p_2^+ and p_2^-, and denote by T_1^+ the pair of pants bounded by p_1, p_2 and p_2^+ (see Figure 5). Noting that $\phi(c_1)$ is a curve in T_1 distinct from p_1, p_2, p_2^- and p_2^+, it follows that $\phi(c_1)$ intersects T_1^+. Thus, we have disjoint arcs $\phi(c_1) \cap T_1^+$ and $\phi(c_2) \cap T_1^+$, the latter one having an endpoint in p_2^+ (see Figure 5). Since $\phi(c_1)$, p_2^+ are disjoint and $\phi(c_2)$, p_1 are disjoint, we conclude that $\phi(c_1)$ must intersect p_2. Again, the same argument with slight changes yields that $\phi(c_g)$ intersects p_g.

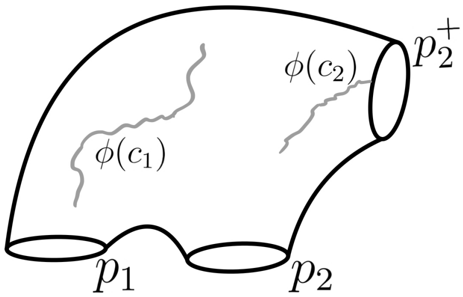


Figure 5: Pair of pants T_1^+.

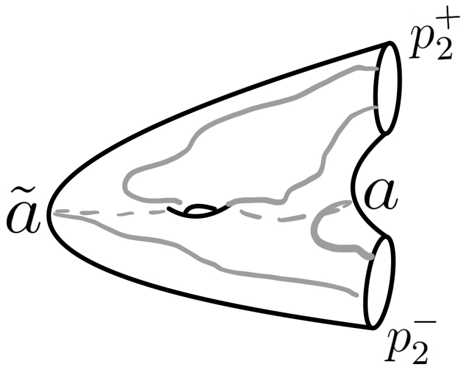


Figure 6: Torus T_1.

Before proving that $\phi(c_1)$ intersects p_1 and $\phi(c_g)$ intersects p_{g+1}, we need to check that $\phi(c_{g+1})$ intersects both p_1, p_g and every p_k^+, p_k^-. Surely, $\phi(c_{g+1})$ intersects at least one of these curves, since otherwise $\phi(c_{g+1})$ would be disjoint and distinct from every curve in the pants decomposition $\phi(P)$. Suppose that $\phi(c_{g+1})$ intersects p_k^+, and consider the pair of pants Q bound by p_{k-1}^+, p_k and p_k^+. Using the intersections above, we deduce that there are disjoint arcs $\phi(c_{k-1}) \cap Q$ and $\phi(c_{g+1}) \cap Q$, so that $\phi(c_{g+1})$ must also intersect p_{k-1}^+. Repeating the argument iteratively, we can detect the intersection of $\phi(c_{g+1})$ with every curve in P.

To finish the proof, we check that $\phi(c_1)$ intersects p_1. Consider the disjoint arcs $a_2 = \phi(c_2) \cap T_1^+$ and $a_{g+1} = \phi(c_{g+1}) \cap T_1^+$, where a_2 has an endpoint in p_2^+ and a_{g+1} has an endpoint in p_1. These two arcs exist by the above intersections. Moreover, a_2 is disjoint from p_1, and a_{g+1} is disjoint from p_2. Recall that $\phi(c_1)$ intersects T_1^+ and, since it is disjoint from both a_1, a_2, it follows that $\phi(c_1)$ intersects p_1. A similar argument yields that $\phi(c_g)$ intersects p_{g+1}. □

So far we have seen that the map $\phi : F_R \to \mathcal{N}(S)$ can be taken to agree with a homeomorphism on $P \subset F_R$, and that it detects some intersections. In the next lemma, we extend it so that ϕ agrees with a homeomorphism on $F_R \setminus N$.

Lemma 4.10. *Let S be a closed surface of genus $g \geq 3$ and let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. There exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h|_{F_R \setminus N} = \phi|_{F_R \setminus N}$.*

Proof. By Corollary 4.5, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes every $p \in P$; we rename $h \circ \phi$ to ϕ. Recall that by Remark 4.1 we know that ϕ is injective.

First, we find a homeomorphism that agrees with ϕ on $c_1 \in C$. Observe that $\phi(c_1)$ is contained in the torus T_1 bounded by p_2^+ and p_2^- (see Figure 6). Moreover, Lemma 4.9 implies that there exist disjoint arcs $a \in \phi(c_2) \cap T_1$ and $\tilde{a} \in \phi(c_{g+1}) \cap T_1$. Notice that both arcs have endpoints in p_2^+ and p_2^-, as otherwise $\phi(c_1)$ would not intersect p_1, contradicting Lemma 4.9. It follows that $\phi(c_1)$ is the curve contained in the annulus $T_1 \setminus (a \cup \tilde{a})$. Even more, there exists a twist $h' \in \mathrm{Mod}^{\pm}(S)$ along p_1 and p_2 such that $h' \circ \phi(c_1) = c_1$. We rename $h' \circ \phi$ to ϕ.

The same argument with minor changes yield homeomorphisms that agree with ϕ on every $c_k \in C \setminus \{c_{g+1}\}$. Thus, we may assume that ϕ fixes every curve in $P \cup C \setminus \{c_{g+1}\} \subset F_R$.

To finish the proof, we are going to check that ϕ is fixing c_{g+1} and every curve in $U \cup D \cup L \cup R$:

- Notice that $\phi(c_{g+1})$ is contained in the torus with boundary $T' = S \setminus \bigcup_{i=2}^{g} p_i$. In fact, $\phi(c_{g+1})$ is the unique curve contained in the annulus $S' \setminus \bigcup_{k=1}^{g} c_k$, i.e., $\phi(c_{g+1}) = c_{g+1}$.
- To prove that ϕ fixes U, consider $u_k \in U$. Observe that $\phi(u_k)$ is contained in the sphere S_k^+ bounded by the curves p_{k-1}^+, p_k, p_{k+1} and p_{k+1}^+. Moreover, $\phi(u_k)$ must be the only curve in the annulus $S_k^+ \setminus (c_k \cup c_{g+1})$, i.e., $\phi(u_k) = u_k$.
- To prove that ϕ fixes D, consider $d_k \in U$. Notice that $\phi(d_k)$ is contained in the sphere S_k^- bounded by the curves p_{k-1}^-, p_k, p_{k+1} and p_{k+1}^-. Now, $\phi(d_k)$ must be the only curve in the annulus $S_k^- \setminus \{c_k, c_{g+1}\}$, i.e., $\phi(d_k) = d_k$.
- To prove that ϕ fixes L, consider the curve $l_k \in L$. The image $\phi(l_k)$ is a curve in the subsurface S_l bounded by $p_{k-1}^-, p_k, p_k^+, p_{k+1}^-$, and p_{k+1}^+. Since $\phi(l_k)$ is disjoint from curves in C, we know that $\phi(l_k)$ is contained in the pair of pants $Q = S_l \setminus (c_k \cup c_{k+1})$. Note that Q has only one boundary component not already in P, and so we conclude $\phi(l_k) = l_k$. Naturally, a similar argument yields that ϕ fixes $r_k \in R$.

Summarizing, we have proven that there exists a mapping class $f \in \mathrm{Mod}^{\pm}(S)$ such that $f \circ \phi$ fixes $F_R \setminus N$. In other words,

$$f^{-1}|_{F_R \setminus N} = \phi|_{F_R \setminus N}.$$

This concludes the proof. □

To finish the proof of Theorem 1.1, we need to check that we can take ϕ to agree with a homeomorphism on $N \subset F_R$, and that such homeomorphism is unique. Before doing so, we require one more definition: The *back-front (orientation reversing) involution* is the unique non-trivial mapping class $\iota \in \mathrm{Mod}^{\pm}(S)$ fixing every curve $c \in P \cup C$.

Proof of Theorem 1.1 for closed surfaces. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. By Lemma 4.10, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes $F_R \setminus N$; we rename $h \circ \phi$ to ϕ.

Consider $nd \in N$ and notice that $\phi(nd)$ is a curve in the genus two surface T bounded by the curves $p_2^+, p_3, p_4 \dots, p_{g-1}, p_{g-1}^+$. Since $\phi(nd)$ is disjoint from C, we have that $\phi(nd)$ is a curve in $Q = T \setminus \bigcup_{c \in C} c$. Note that Q is the disjoint union of two pairs of pants and it has only two boundary components not contained in C, namely nd and $\iota(nd)$, where ι is the back-front involution. It follows that $\phi(nd) \in \{nd, \iota(nd)\}$. Thus, by precomposing by ι if necessary, we may assume that ϕ fixes $F_R \setminus \{nl, nr\}$.

We continue by proving that ϕ fixes nl. Note that $\phi(nl)$ is a curve in the torus T_1 bounded by p_2^- and p_2^+. Even more, $\phi(nl)$ is contained in $Q = T_1 \setminus (c_1 \cup nd)$, which is the union of an annulus and a pair of pants. Note that only one curve in Q is not a curve in $P \cup C$, and so we deduce $\phi(nl) = nl$. An analogous argument leads to the conclusion $\phi(nr) = nr$.

So far, we have found a composition of mapping classes $f \in \mathrm{Mod}^{\pm}(S)$ such that $f \circ \phi$ is the identity in F_R. Therefore, ϕ is induced by the mapping class f^{-1}.

To prove the uniqueness of the inducing mapping class, suppose that $f, \tilde{f} \in \mathrm{Mod}^{\pm}(S)$ both induce the same $\phi : F_R \to \mathcal{N}(S)$. Since $g = \tilde{f} \circ f^{-1}$ fixes the curves $(P \cup C) \subset F_R$, the Alexander method implies that g is either the identity or the back-front orientation reversing involution. However, since g also fixes the curve nd, we have that g is the identity and $f = \tilde{f}$. □

5 Finite rigid exhaustion for closed surfaces

In this section, we prove that, for any closed surface S of genus $g \geq 3$, there exist subcomplexes

$$F_1 \subset F_2 \subset \cdots \subset \mathcal{N}(S)$$

such that each F_i is a finite rigid set with trivial pointwise stabilizer and

$$\bigcup_{i=1}^{\infty} F_i = \mathcal{N}(S).$$

The strategy to produce an exhaustion is to first extend F_R to a larger finite rigid set F_1 with desirable properties. Then the set F_1 will work as base case for an induction that enlarges F_i into F_{i+1}. This method heavily resembles the proof given by Aramayona and Leininger in [2] to produce an exhaustion of the curve complex.

The plan of the proof is summarized in the following lemma (cf. [2, Lemma 3.13]).

Lemma 5.1. *Let S be a finite-type surface, let $F_1 \subset \mathcal{N}(S)$ be a finite rigid set with trivial pointwise stabilizer, and let $\{h_1, \dots, h_k\}$ be a set of generators of $\mathrm{Mod}^{\pm}(S)$. If $F_1 \cup h_j(F_1)$ is a finite rigid set with trivial pointwise stabilizer for every j, then the sets*

$$F_{i+1} = F_i \cup \bigcup_{j=1}^{k} (h_j(F_i) \cup h_j^{-1}(F_i))$$

satisfy that $F_1 \subset F_2 \subset \cdots \subset \mathcal{N}(S)$, $\bigcup F_i = \mathcal{N}(S)$ and every F_i is finite rigid with trivial pointwise stabilizer.

Proof. First, notice that if $F_1 \cup h_j(F_1)$ is a finite rigid set with trivial stabilizer, then the same holds for $F_1 \cup h_j^{-1}(F_1)$.

We will now check that F_2 is finite rigid with trivial stabilizer: Take $\phi : F_2 \to \mathcal{N}(S)$ and observe that by hypothesis there exists $f_j \in \mathrm{Mod}^{\pm}(S)$ such that

$$\phi|_{F_1 \cup h_j(F_1)} = f_j|_{F_1 \cup h_j(F_1)}$$

for every $j \in \{1, \dots, k\}$. Since F_1 has trivial pointwise stabilizer and

$$f_j|_{F_1} = \phi|_{F_1} = f_k|_{F_1},$$

we deduce $f_j = f_k$ for all j, k. Clearly, the same argument holds when considering $F_1 \cup h_j^{-1}(F_1)$. It follows that $\phi = f$ for $f \in \mathrm{Mod}^{\pm}(S)$ and F_2 is a finite rigid set with trivial pointwise stabilizer.

For F_{i+1}, we proceed by induction. Consider $\phi : F_{i+1} \to \mathcal{N}(S)$. By induction, we have $\phi|_{F_i} = f|_{F_i} \in \mathrm{Mod}^{\pm}(S)$ and

$$\phi|_{h_j(F_i)} = f_j|_{h_j(F_i)}$$

for some $f_j \in \mathrm{Mod}^{\pm}(S)$. Observe that $F_1 \subset F_i \cap h_j(F_i)$ and F_1 is trivially stabilized. Thus $f = f_j$ for every j, and so F_{i+1} is finite rigid with trivial stabilizer. □

To produce the finite rigid exhaustion of $\mathcal{N}(S)$, we are going to use the previous lemma. In this direction, we enlarge F_R into a set F_1 and provide a set of generators for $\mathrm{Mod}^{\pm}(S)$.

5.1 Enlarging the finite rigid set

Let $\iota \in \mathrm{Mod}^{\pm}(S)$ be the back-front orientation reversing involution. We define the set of curves $A := A_1 \cup A_2$, where

$$A_1 := \bigcup_{k=1}^{g-1} \partial(c_k, p_{k+1}, c_{k+1}).$$

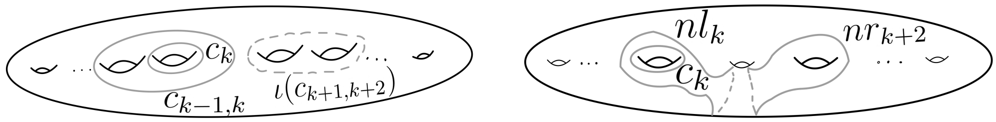
Note that the set $\partial(c_k, p_{k+1}, c_{k+1})$ has two curves, one curve in the front component and one curve in the back component of the surface. We will call $c_{k,k+1}$ the curve in the front component, so

$$\partial(c_k, p_{k+1}, c_{k+1}) = \{c_{k,k+1}, \iota(c_{k,k+1})\}.$$

We proceed to define A_2. Consider the torus T bounded by p_{k-1}^-, p_{k+1}^- and u_k. Now, there is only one curve in the pair of pants $T \setminus (c_{k-1} \cup c_k \cup \iota(c_{k-1,k}))$ that is not already in $P \cup C$; we denote this curve by nl_k (see Figure 7 (a)). Analogously, let nr_k be the unique curve in $T \setminus (c_k \cup c_{k+1} \cup \iota(c_{k,k+1}))$ that is not already a curve in $P \cup C$ (see Figure 7 (b)). We set

$$A_2 = \{nl_k, nr_k \mid k \in \{2, \dots, g-1\}\}.$$

Lemma 5.2. *Let S be a closed surface of genus $g \geq 3$. The subcomplex spanned by $F_R \cup A \subset \mathcal{N}(S)$ is a finite rigid set with trivial pointwise stabilizer.*



(a) Curves in A_1. (b) Curves in A_2.

Figure 7: Curves in A for a closed surface.

Remark 5.3. In the argument below, we sometimes abuse notation using $p_1^+ = p_1^- = p_1$ and $p_g^+ = p_g^- = p_{g+1}$.

Proof. Take a locally injective simplicial map $\phi : F_R \cup A \to \mathcal{N}(S)$. By Theorem 1.1 in the closed case, there is an $h \in \operatorname{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes F_R; we rename $h \circ \phi$ to ϕ. Furthermore, the subcomplex $F_R \cup A$ has diameter two, so ϕ is injective.

First, we check that ϕ fixes A_1. Note that $\phi(\iota(c_{1,2}))$ is contained in the torus T bounded by the curves p_2, p_3^+ and p_3^-. Since $\phi(\iota(c_{1,2}))$ is disjoint from the curves in C, we have that $\phi(\iota(c_{1,2}))$ is contained in the pair of pants $Q = T \setminus (c_1 \cup c_2 \cup c_3)$. Notice that only one curve in Q is non-separating and disjoint from nd, namely $\iota(c_{1,2})$, and so $\phi(\iota(c_{1,2})) = \iota(c_{1,2})$. Now, $\phi(c_{1,2})$ is the unique non-separating curve in Q distinct from $\iota(c_{1,2})$, that is, $\phi(c_{1,2}) = c_{1,2}$.

By an obvious modification of the above argument, we check that

$$\phi(c_{k,k+1}) = c_{k,k+1} \quad \text{and} \quad \phi(\iota(c_{k,k+1})) = \iota(c_{k,k+1})$$

for $k \le g - 1$.

It is left to check that A_2 is also fixed by ϕ. Take $nr_k \in A_2$; we have that $\phi(nr_k)$ is contained in the torus T bounded by the curves p_{k-1}^-, p_{k+1}^- and u_k. Also, $\phi(nr_k)$ is disjoint from c_k, c_{k+1} and $\iota(c_{k,k+1})$, and thus $\phi(nr_k)$ is a curve in the pair of pants

$$Q = T \setminus (c_k \cup c_{k+1} \cup \iota(c_{k,k+1})).$$

Notice that the only curve in Q distinct from curves in $P \cup C$ is the curve nr_k, and thus we have $\phi(nr_k) = nr_k$. For $nl_k \in A_2$, the argument is analogous.

We have proven that if ϕ fixes F_R, then it fixes $F_R \cup A$. The statement follows immediately. □

We define $F_1 := F_R \cup A$ and proceed to prove that it satisfies the hypotheses of Lemma 5.1.

5.2 Constructing the exhaustion for closed surfaces

Let ι be the back-front orientation reversing involution and let δ_α be the right Dehn twist along the curve α. The set of Dehn twists

$$H := \{\delta_{p_1}, \delta_{p_2}, \ldots, \delta_{p_{g+1}}, \delta_{p_2^-}\} \cup \{\delta_{c_1}, \ldots, \delta_{c_k}\}$$

are known as the *Humphries generators*, which are known to generate the group of orientation preserving mapping classes $\operatorname{Mod}(S)$ (see [7]). Hence, $H \cup \{\iota\}$ generates $\operatorname{Mod}^{\pm}(S)$.

Lemma 5.4. *Let $h \in H \cup \{\iota\}$. The set $F_1 \cup h(F_1)$ is a finite rigid set with trivial pointwise stabilizer.*

Proof. First, we prove it for $h = \delta_{p_1}$. Let $\phi : F_1 \cup h(F_1) \to \mathcal{N}(S)$ be any locally injective simplicial map.

Since F_1 is a finite rigid set, we may assume that ϕ fixes F_1 by precomposing with a mapping class. Moreover, since $h(F_1)$ is also finite rigid, there exists a mapping class $f \in \operatorname{Mod}^{\pm}(S)$ such that

$$\phi|_{h(F_1)} = f|_{h(F_1)}.$$

Proving that $F_1 \cup h(F_1)$ is a finite rigid set with trivial pointwise stabilizer boils down to proving that $f = 1 \in \operatorname{Mod}^{\pm}(S)$.

Recall that p_1 is the associated curve to the Dehn twist $h = \delta_{p_1}$. Now, let $S' = S \setminus p_1$ and consider the well-known cutting homomorphism (see [4, Proposition 3.20])

$$1 \to \langle \delta_{p_1} \rangle \to \operatorname{Mod}(S, p_1) \to \operatorname{Mod}(S'),$$

where $\mathrm{Mod}(S') \subset \mathrm{Mod}^{\pm}(S')$ is the subgroup of orientation preserving classes, and $\mathrm{Mod}(S, p_1) \subset \mathrm{Mod}^{\pm}(S)$ is the subgroup of orientation preserving classes that fix p_1. Notice that f fixes all curves in $F_1 \cap h(F_1)$ which fill S'. In particular, f is orientation preserving and fixes p_1. Furthermore, by means of the Alexander method, the image of f by the cutting homomorphism is trivial and the above sequence implies that $f = \delta_{p_1}^k$ for some $k \in \mathbb{Z}$. It is left to see that $k = 0$.

Note that $i(nl, \delta_{p_1}(c_{g+1})) = 0$. As ϕ is locally injective, we know that

$$i(\phi(nl), \phi(\delta_{p_1}(c_{g+1}))) = 0,$$

which is equivalent to

$$i(nl, \delta_{p_1}^{k+1}(c_{g+1})) = 0, \tag{5.1}$$

since $\phi|_{F_1} = \mathrm{id}$ and $\phi|_{h(F_1)} = \delta_{p_1}^k$.

It can be directly checked that equation (5.1) is satisfied if and only if $k = 0$, thus proving the statement of the lemma for $h = \delta_{p_1}$.

The rest of the cases $h \in H$ are proved in exactly the same way, but changing the curves in equation (5.1). We list all cases for completeness:

- If $h = \delta_{p_{g+1}}$, change (nl, c_{g+1}) in (5.1) by (c_{g+1}, nr).
- If $h = \delta_{p_i}$ for $i \in \{2, \dots, g-2\}$, change (nl, c_{g+1}) by $(nl_i, c_{i,i+1})$.
- If $h = \delta_{p_{g-1}}$, change (nl, c_{g+1}) by $(c_{g-2,g-1}, nr_{g-1})$.
- If $h = \delta_{p_2^-}$, change (nl, c_{g+1}) by (c_{g+1}, nr_2).
- If $h = \delta_{c_k}$ for $k \in \{1, \dots, g-1\}$, change (nl, c_{g+1}) by (p_{k+1}, nr_{k+1}).
- If $h = \delta_{c_g}$, change (nl, c_{g+1}) by (nl_{g-1}, p_g).

To finish the proof, we need to consider the case $h = \iota$. In this case, $F_1 \cap h(F_1)$ contains the trivially pointwise stabilized set $P \cup C \cup A_1$, so it follows immediately that $f = 1 \in \mathrm{Mod}^{\pm}(S)$. □

Proof of Theorem 1.2 for closed surfaces. Let S be a closed surface of genus $g \geq 3$. Consider $F_1 := F_R \cup A$ and the set of generators $H \cup \{\iota\}$ of the extended mapping class group $\mathrm{Mod}^{\pm}(S)$. By Lemma 5.1, we have the desired exhaustion, where the hypotheses have been checked in Lemmas 5.2 and 5.4. □

6 Finite rigid sets for punctured surfaces

Let $S = S_{g,n}$ with $n \geq 1$ and $g \geq 3$. In this section, we construct a finite rigid set $F_R' \subset \mathcal{N}(S)$ with trivial pointwise stabilizer. As we will see, the argument to prove the rigidity of F_R' is morally the same as the one in Section 4, although more involved as it requires to manipulate more curves.

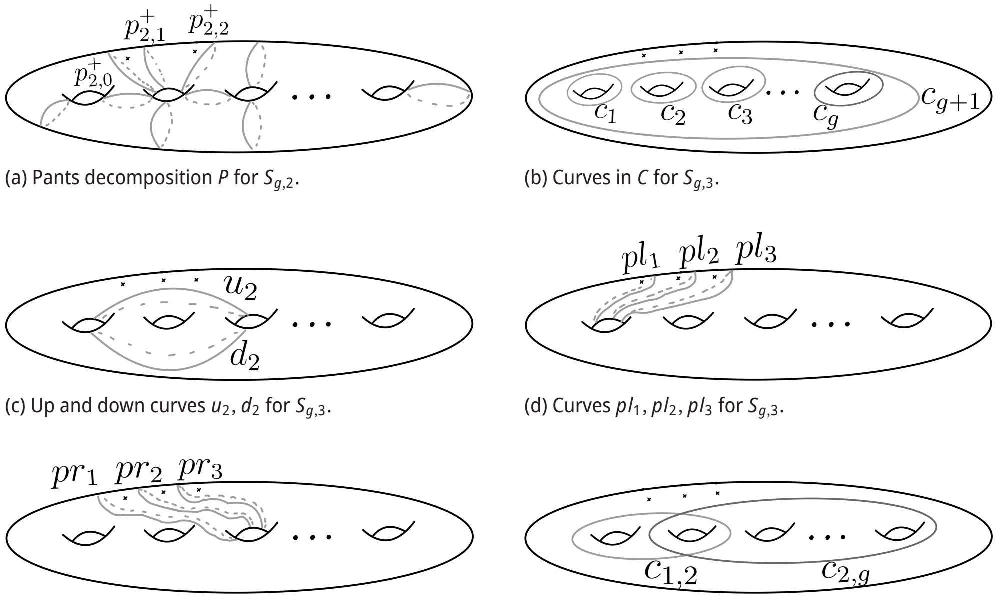
6.1 Constructing the finite rigid set

We start by naming some curves in the punctured surface S.

Consider the closed surface S_g and the pair of pants Q bounded by p_1, p_2, p_2^+. Note that $Q \setminus d_2$ has two connected components; denote by A the connected component intersected by c_{g+1}. Now, $A \setminus (c_1 \cup c_{g+1})$ is the union of two disks, one in the front component and one in the back component of S_g; denote by B the disk in the back component $(S_g \setminus \bigcup c_i)^-$. By removing n points from the interior of B, we obtain the punctured surface S. Via this procedure, we get natural analogues in S of the sets of curves P, C, U, D, L, R, and N.

After this, we define the top of the surface, i.e., $(S \setminus \bigcup p_i)^+$, the bottom $(S \setminus \bigcup p_i)^-$, the front $(S \setminus \bigcup c_i)^+$, and the back $(S \setminus \bigcup c_i)^-$, in the same way we did in Section 4.

The set P is no longer a pants decomposition in S; to fix this we add some curves. First, rename p_2^+ to $p_{2,n}^+$ and consider the pair of pants Q bounded by p_1, p_2 and $p_{2,n}^+$. Observe that $Q \setminus c_1$ is an annulus with n punctures. Now, fix a set of non-separating curves $\{p_{2,0}^+, p_{2,1}^+, \dots, p_{2,n-1}^+, p_{2,n}^+\}$ in $Q \setminus c_1$ satisfying that $p_{2,i}^+$ and $p_{2,i+1}^+$ bound a once punctured annulus. Adding the curves $\{p_{2,0}^+, p_{2,1}^+, \dots, p_{2,n-1}^+, p_{2,n}^+\}$ to the set P makes it a pants decomposition



(a) Pants decomposition P for $S_{g,2}$.

(b) Curves in C for $S_{g,3}$.

(c) Up and down curves u_2, d_2 for $S_{g,3}$.

(d) Curves pl_1, pl_2, pl_3 for $S_{g,3}$.

(e) Curves pr_1, pr_2, pr_3 for $S_{g,3}$.

(f) Curves $c_{1,2}$ and $c_{2,g}$ for $S_{g,3}$.

Figure 8: Curves in F'_R. Punctures are marked as small crosses on the top of the surface.

(see Figure 8 (a)). We will say that the puncture contained in the annulus bounded by $p^+_{2,k-1}$ and $p^+_{2,k}$ is the k-th puncture.

In this setting, we also define the back-front (orientation reversing) involution as the unique non-trivial element $\iota \in \mathrm{Mod}^{\pm}(S)$ that fixes every curve $P \cup C$.

Let pl_1 be the unique curve in the punctured pair of pants bounded by $p_1, p_2, p^+_{2,1}$ that is disjoint from c_2 and is not a curve already in P. Similarly, let pl_k be the unique curve in the punctured pair of pants bounded by $pl_{k-1}, p_2, p^+_{2,k}$ that is disjoint from c_2 and is not already a curve in P (see Figure 8 (d)). We set

$$Pl := \{pl_1, \dots, pl_n\}.$$

Let pr_n be the unique curve in the punctured pair of pants bounded by $p^+_{2,n-1}, p_3, p^+_3$ that is disjoint from c_2 and is not already a curve in P. Inductively, define pr_k to be the unique curve in the punctured pair of pants bounded by $p^+_{2,k-1}, p_3, p^+_3$ that is disjoint from c_2 and not already in P (see Figure 8 (e)). We set

$$Pr := \{pr_1, \dots, pr_n\}.$$

Lastly, we define the set

$$C_* = \partial(c_1, p_2, c_2) \cup \partial(c_2, p_2, \dots, p_g, c_g).$$

Note that the set $\partial(c_1, p_2, c_2)$ has two curves, one curve in the front component which we will denote by $c_{1,g}$, and one curve in the back component which we will denote by $\iota(c_{1,g})$. Also, the set $\partial(c_2, p_2, \dots, p_g, c_g)$ has two curves; we denote by $c_{2,g}$ the curve in the front component and by $\iota(c_{2,g})$ the curve in the back component (see Figure 8 (f)).

The finite rigid set F'_R is the maximal subcomplex of $\mathcal{N}(S)$ spanned by the vertices

$$P \cup C \cup U \cup D \cup L \cup R \cup N \cup Pl \cup Pr \cup C_*.$$

Remark 6.1. Note that the subcomplex F'_R has diameter two. Therefore, any locally injective simplicial map $\phi : F'_R \to \mathcal{N}(S)$ is injective.

The following lemmas prove the finite rigidity of F'_R.

Lemma 6.2. *Let S be a punctured surface of genus $g \geq 3$. For any locally injective simplicial map $\phi : F'_R \to \mathcal{N}(S)$, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h|_P = \phi|_P$.*

Proof. First, notice that we can extend Lemma 4.3 to the case of punctured surfaces, and the proof works with minor changes.

Lemma 4.4 can also be extended to punctured surfaces. The proof for once punctured surfaces is similar to the closed case. Here we prove it for surfaces with $n > 1$ punctures.

Using Remark 2.1, it is straightforward to deduce that the curve $\phi(p^+_{2,i})$ is adjacent rel to P to $\phi(p^-_{2,i-1})$ and $\phi(p_{2,i+1})$ for $i \in \{1, \ldots, n-1\}$. As a consequence of these adjacencies, we deduce that $\phi(p^+_{2,0})$ and $\phi(p^+_{2,n})$ bound a subsurface with two boundary components and n punctures. It follows that $\phi(p^+_{2,0})$ is adjacent rel to P to at least three curves. On the other hand, the non-adjacency rel to P of $\phi(p^+_{2,0})$ implies that it is adjacent to at most three curves. Thus, $\phi(p^+_{2,0})$ is adjacent to exactly three curves, namely $\phi(p^+_{2,1})$, $\phi(p_1)$ and $\phi(p_2)$. The rest of the adjacencies follow just as in the proof of Lemma 4.4.

Once we know that ϕ preserves adjacency rel to P, we can use Corollary 4.5 to produce a mapping class $h \in \mathrm{Mod}^{\pm}(S)$ with $h|_P = \phi|_P$. □

The next lemma proves that we may choose $h \in \mathrm{Mod}^{\pm}(S)$ to coincide with ϕ on all curves of F'_R.

Lemma 6.3. *Let S be a punctured surface of genus $g \geq 3$. For any locally injective simplicial map $\phi : F'_R \to \mathcal{N}(S)$, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $\phi = h|_{F'_R}$.*

Proof. The idea is to progressively detect intersections between curves and, by composing with Dehn twists along P, construct a mapping class that coincides with ϕ on F'_R.

First, by Lemma 6.2, there exists $h \in \mathrm{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes P pointwise; rename $h \circ \phi$ to ϕ.

Next, using the same arguments as in Lemmas 4.6, 4.8 and 4.9, we detect the following intersections:

- Intersections of curves in $U \cup D$ with curves in P.
- Intersections of curves in Pl with curves in P.
- Intersections of curves in Pr with curves in P.
- c_k with curves in P for $k \in \{2, \ldots, g-1\}$.

Notice that we can now use the proof of Lemma 4.10 to produce a mapping class that coincides with ϕ on the curves of U, D and $\{c_2, \ldots, c_{g-1}\}$. Thus, we may assume that these curves are fixed by ϕ.

It follows that ϕ fixes the curves in Pl. To see this, note that $\phi(pl_1)$ is a curve contained in the punctured sphere S' bounded by p_1, u_2, p^+_3 and that it is disjoint from $p^+_{2,i}$ for $i > 0$. Thus, $\phi(pl_1)$ is contained in the once punctured annulus which is a component of $S' \setminus p^+_{2,1}$. As there is only one curve in that annulus that is not already in P, it follows that $\phi(pl_1) = pl_1$. In the same way, $\phi(pl_k)$ is contained in the punctured sphere S'_k bounded by pl_{k-1}, u_2, p^+_3, and it is disjoint from $p^+_{2,i}$ for $i > k-1$. Thus, $\phi(pl_k)$ is contained in the once punctured annulus which is a component of $S'_k \setminus p^+_{2,k}$, and since there is only one curve in that annulus which is not in P, we conclude that $\phi(pl_k) = pl_k$. Naturally, an analogous argument works to prove that ϕ fixes every curve in Pr.

Now, note that ϕ detects the following intersections:

- Curves in C_* with curves in P.
- Curves in $\{c_1, c_g\}$ with curves in P.

For instance, consider $c_{1,2} \in C_*$ and $p \in P$ disjoint from $c_{1,2}$. If $\phi(c_{1,2})$ was disjoint from p, then $\phi(c_{1,2})$ would have to intersect either the curve $\phi(c_1)$, $\phi(c_2)$ or $\phi(\iota(c_{2,g}))$, leading to a contradiction. Thus, $\phi(c_{1,2})$ and p intersect.

Using the above intersections and fixed curves, we now focus on finding a homeomorphism that agrees with ϕ on the curves in C_*.

Observe that $\phi(c_1)$ (resp. $\phi(c_g)$) is contained in the subsurface $S' = S_{1,2}$ bound by the curves p^-_2, p^+_2 (resp. by p^-_{g-1}, p^+_{g-1}). Additionally, since $\phi(c_1)$ is disjoint from the arcs $a_1 = S' \cap \phi(c_{1,2})$ and $a_2 = S' \cap \phi(c_2)$ (resp. $S' \cap \phi(c_{2,g})$ and $S' \cap \phi(c_{g-1})$), we have that $\phi(c_1)$ (resp. $\phi(c_g)$) is contained in the annulus $S' \setminus (a_1 \cup a_2)$. It follows that $\phi(c_1)$ (resp. $\phi(c_g)$) is the unique curve in the annulus. We may again consider a mapping class $h \in \mathrm{Mod}^{\pm}(S)$ that is a composition of twists along curves in P such that $h \circ \phi(c) = c$ for $c \in \{c_1, c_g\}$; we rename $h \circ \phi$ to ϕ (the same argument with more details is given in Lemma 4.10).

For $c_{1,2} \in C_*$, note that $\phi(c_{1,2})$ is contained in a subsurface $S' = S_{2,2}$ bound by pr_1 and p^-_3 (or p_4 if $g = 3$). Even more, $\phi(c_{1,2})$ is contained in the pair of pants $S' \setminus (c_1 \cup c_2 \cup c_3 \cup p_2)$, and therefore it is one of the boundary components. One of the boundaries is a separating curve, so either $\phi(c_{1,2}) = c_{1,2}$ or $\phi(c_{1,2}) = \iota(c_{1,2})$. These two alternatives are related by the involution ι. Thus, by precomposing with ι, we can assume that ϕ fixes $c_{1,2}$.

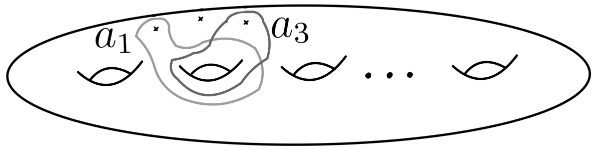


Figure 9: Curves a_1, a_3 in A_3 for $S_{g,3}$.

It is now easy (but lengthy) to check that the curves in

$$\{\iota(c_{1,2}), \iota(c_{2,g}), nl, nr, nd\} \cup L \cup R$$

are fixed by ϕ. Thus, we have found a mapping class $h \in \text{Mod}^{\pm}(S)$ such that $h \circ \phi$ fixes F'_R, and the statement follows. □

We have essentially completed the proof of Theorem 1.1.

Proof of Theorem 1.1 for surfaces with punctures. Let $\phi : F'_R \to \mathcal{N}(S)$ be a locally injective simplicial map. Then Lemma 6.3 provides the mapping class inducing ϕ.

The uniqueness of the mapping class follows as in the closed case. □

7 Finite rigid exhaustion for punctured surface

Let S be a punctured surface of genus $g \geq 3$. In this section, we construct a sequence $F'_1 \subset F'_2 \subset \cdots \subset \mathcal{N}(S)$ such that each F'_i is a finite rigid set with trivial pointwise stabilizer and

$$\bigcup_{i=1}^{\infty} F'_i = \mathcal{N}(S).$$

The strategy to construct the exhaustion is the same as in the closed case (see Section 5). First, we are going to enlarge the finite rigid set F'_R to F'_1, and then use Lemma 5.1 to construct the exhaustion.

7.1 Enlarging the finite rigid set

First, we enlarge F'_R.

Consider the set of curves A_1 and A_2 in the closed surface. Via the same procedure described in Section 6.1, we remove n points from the interior of the closed surface S_g, and so we obtain the punctured surface S. This produces natural analogues of the set of curves A_1 and A_2 in the surface S.

We define the set of curves $A_3 := \{a_1, \dots, a_n\}$, where a_k is the unique curve in the torus bounded by pl_{k-1}, pr_{k+1} and d_2, which is disjoint from c_1, c_2, c_3 and $\iota(c_{1,2})$ (see Figure 9).

We set $A' := A_1 \cup A_2 \cup A_3$.

Lemma 7.1. *The set $F'_R \cup A'$ is finite rigid with trivial pointwise stabilizer.*

Proof. Let $\phi : F'_R \cup A' \to \mathcal{N}(S)$ be a locally injective simplicial map. By precomposing with a mapping class, we may assume that ϕ fixes F'_R pointwise.

First, we prove that A_1 is also fixed by ϕ. The curves $c_{1,2}$, $\iota(c_{1,2}) \in A_1$ are already in $C_* \subset F'_R$, so they are fixed by ϕ. For $c_{2,3} \in A_1$, notice that $\phi(c_{2,3})$ is contained in the sphere S' bounded by p_1, pl_n, c_2, c_3, p_4^+, and p_4^-. Moreover, $\phi(c_{2,3})$ is contained in the pair of pants $S' \setminus c_1 \cup p_3 \cup \iota(c_{2,g})$. But there is only one curve in that pair of pants that is non-separating, i.e., $\phi(c_{2,3}) = c_{2,3}$. Slight modifications yield that $\phi(\iota(c_{2,3})) = \iota(c_{2,3})$. For the rest of A_1, one can proceed as in the closed case (see the proof of Lemma 5.2).

To prove that A_2 is fixed, we can just repeat the argument as in the closed case (see Lemma 5.2).

To finish the proof, we must show that $a_k \in A_3$ is fixed. To check this, note that $\phi(a_k)$ is contained in the torus T bounded by pl_{k-1}, pr_{k+1} and d_2. Note that $\phi(a_k)$ is the unique curve in

$$T \setminus (c_1 \cup c_2 \cup c_3 \cup \iota(c_{1,2}))$$

that is not c_2, that is, $\phi(a_k) = a_k$. Thus, A' is fixed and $F'_R \cup A'$ is a finite rigid set with trivial pointwise stabilizer. □

We define $F'_1 := F'_R \cup A'$.

7.2 Constructing the exhaustion for punctured surfaces

The goal of this section is to construct an exhaustion of $\mathcal{N}(S)$ by finite rigid sets with trivial pointwise stabilizers. In this direction, we will consider a set of generators of $\mathrm{Mod}^{\pm}(S)$ such that the subcomplex F'_1 satisfies the hypotheses of Lemma 5.1. We are going to assume $S = S_{g,n}$ with $n > 0$ punctures.

Let ι be the back-front orientation reversing involution. Consider the usual Humphries generators

$$H' = \{\delta_{p_1}, \dots, \delta_{p_{g-1}}, \delta_{p_2^-}\} \cup \{\delta_{p_{2,i}^+} \mid 0 \le i \le n\} \cup \{\delta_{c_1}, \dots, \delta_{c_k}\}$$

and the half twists

$$\{h_{(k,k+1)} \mid 1 \le k \le n-1\},$$

where $h_{(k,k+1)}$ is the half twist that permutes the puncture k with the puncture $k+1$. It is well known that

$$H' \cup \{\iota\} \cup \{h_{(k,k+1)} \mid 1 \le k \le n-1\}$$

generates $\mathrm{Mod}^{\pm}(S)$ (see [4, Chapter 4.4.4]).

Lemma 7.2. *For every*

$$h \in H' \cup \{\iota\} \cup \{h_{(k,k+1)} \mid 1 \le k \le n-1\},$$

the set $F'_1 \cup h(F'_1)$ is finite rigid with trivial pointwise stabilizer.

Proof. The proof is analogous to the closed case (see Lemma 5.4) and works directly for $h = \iota$, $h = \delta_{p_j}$, $h = \delta_{c_j}$ and $h = \delta p_2{}^-$. We consider here the rest of the cases.

Recall that, given $h = \delta_x$ and $\phi : F'_1 \cup h(F'_1) \to \mathcal{N}(S)$, we can assume that ϕ fixes F'_1 and

$$\phi|_{h(F_1)} = \delta_x^k|_{h(F_1)}.$$

Thus, the proof is a matter of checking that $k = 0$.

For

$$h = \delta_{p_{2,j}^+} \quad \text{with} \quad j \le n-1,$$

we can consider the curves l_j, a_j and plug them into equation (5.1). It follows that this is satisfied if and only if $k = 0$. For $h = \delta_{p_{2,n}^+}$, one uses the curves r_n, a_n in the same equation.

The last case to prove is $h = h_{(k-1,k)}$, which can be proved using the curves a_k, r_k in equation (5.1). □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2 for punctured surfaces. Lemmas 7.1 and 7.2 ensure that F'_1 satisfies the hypothesis of Lemma 5.1, which in turn produces the desired exhaustion. □

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