Corrigendum

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Corrigendum to: Separable ultrametric spaces and their isometry groups

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Krzysztof Krupiňski pointed out to me that there is a mistake in the statement of [1, Lemma 4.9], and a gap in the proof of [1, Theorem 4.13]. In this corrigendum, the right version of the lemma and a correct proof of the theorem are presented. We use most of the definitions and notation conventions from [1] without explicitly formulating them.

First of all, in the statement of Lemma 4.9 there should be an additional assumption that balls B_0, B_1 are disjoint. Then the lemma is true, its original proof goes through without any changes, and it still can be used in the proof of Theorem 4.13, which is the only place in [1], where Lemma 4.9 is applied.

Lemma 1 ([1, Lemma 4.9]). *Let B_0, B_1 be disjoint 'open' balls in an ultrametric space X. Suppose that the equivalence classes they determine in E_{B_0}, E_{B_1}, respectively, are agreeable. If $x_0 \in B_0$, $x_1 \in B_1$, then*

$$d(x_0, x_1) = \operatorname{dist}([x_0], [x_1]).$$

Proof. Suppose that $d(x_0, x_1) > \operatorname{dist}([x_0], [x_1])$, and put $r = d(x_0, x_1)$, $C_0 = B_{<r}(x_0)$, $C_1 = B_{<r}(x_1)$. Then there exists $\phi \in \operatorname{Iso}(X)$ such that $\phi[C_0] = C_1$, so $E_{C_0} = E_{C_1}$, and $E_{B_0}, E_{B_1} \leq E_{C_0}$. However, $C_0 \neq C_1$, and $B_0 \leq C_0$, $B_1 \leq C_1$. Therefore B_0, B_1 cannot be agreeable. □

In the proof of Theorem 4.13, it is claimed that if B is an agreeable family satisfying the boundedness condition, then, for every chain L of balls coming from equivalence classes in B, either L is bounded from below in B, or it must contain a sequence of balls with diameters converging to 0. This is not true in general, so the presented construction of a d-transversal $Y \in \bigcap B$ is not quite correct. The remaining part of the corrigendum is devoted to a correct proof of this theorem.

Let X be a W-space with metric d. Recall that every ball B in X gives rise to an equivalence relation E_B on the set S_X of all d-transversals in X, defined by

$$Y_0 \cong Y_1 \ (\mathrm{mod}\ E_B) \iff \exists \phi \in \operatorname{Iso}(X)(Y_0 \cap \phi[B] \neq \emptyset \text{ and } Y_1 \cap \phi[B] \neq \emptyset).$$

Suppose that D is an equivalence class of an equivalence relation E_B defined as above by an 'open' ball B. Then D determines a unique ball in X, so we will identify such equivalence classes with the balls determined by them. In particular, by an agreeable family of balls (or a family of balls satisfying the boundedness condition), we will mean a family of balls determined by an agreeable family of equivalence classes (or a family of classes satisfying the boundedness condition.)

In order to prove the theorem, we need two auxiliary results.

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Lemma 2. *Let be X a W-space, and let B be an agreeable family of balls in X which satisfies the boundedness condition. Let $C \in B$. Then one of the following must hold:*
(i) *for every $x'_C \in C$, and every ball $D \in B$ with $D \subseteq C$ there is a ball $E \in B$ such that $E \subseteq D$, and $E \cap [x'_C] = \emptyset$,*
(ii) *there is a decreasing sequence $\{C_n\}$ of balls in B such that $C_n \subseteq C$, and $\mathrm{diam}(C_n) \to 0$,*
(iii) *there is $D \in B$ with $D \subseteq C$ which is inclusion-minimal in B.*

Proof. Suppose that (i) does not hold, and fix $x'_C \in C$ and $D \in B$ witnessing it. Suppose that (ii) and (iii) do not hold either. We will construct a strictly decreasing sequence $\{C_\alpha\}_{\alpha<\omega_1}$ of balls in B. As this is obviously not possible (because B is countable), (ii) or (iii) must hold.

Put $C_0 = D$. Suppose that we have constructed C_β for $\beta < \alpha$. Suppose that $\alpha = \gamma + 1$. Since (iii) does not hold, C_γ is not inclusion-minimal in B, and there is $E \in B$ with $E \subsetneq C_\gamma$. Put $C_\alpha = E$.

Suppose now that α is a limit ordinal. Find a cofinal sequence $\{D_n\}$ in $\{C_\beta\}_{\beta<\alpha}$. As (ii) does not hold, the radii of D_n are bounded from below by some $r > 0$. Observe that since X is locally non-rigid, r can be chosen so that, for the balls $F = B_{\leq r}(x'_C)$, $F' = B_{<r}(x'_C)$, the relation $E_{F'}$ strictly refines the relation E_F, that is, $\delta = (E_F, E_{F'})$ is a covering pair in Δ_X.

Fix n. Since $D_n \cap [x'_C] \neq \emptyset$, and $r < \mathrm{diam}(D_n)$, there exists $\phi_n \in \mathrm{Iso}(X)$ such that $\phi_n[F'] \subseteq D_n$, that is, $E_{F'}$ refines E_{D_n}. In other words, δ witnesses that covering pairs coming from $\{D_n\}$ are bounded from below. The boundedness condition implies that there is a ball $G \in B$ with $G \subseteq D_n$ for every n. Put $C_\alpha = G$. This finishes the inductive construction. □

Lemma 3. *Let X be a W-space, and let B be an agreeable family of balls in X. Suppose that $C \in B$ and $x'_C \in C$ are such for every ball $D \in B$ with $D \subseteq C$ there is a ball $E \in D$ such that $E \subseteq D$, and $E \cap [x'_C] = \emptyset$. Then there is $x_C \in C$ such that $d(x_C, y) = \mathrm{dist}([x_C], [y])$ for every $D \in B$ with $x_C \notin D$, and for every $y \in D$.*

Proof. Fix a maximal pairwise disjoint family $B' \subseteq B$ of balls E such that $E \subseteq C$, and $E \cap [x'_C] = \emptyset$. Fix $y_E \in E$ for every $E \in B'$. Because B is agreeable, Lemma 1 implies that $d(y_E, y_{E'}) = \mathrm{dist}([y_E], [y_{E'}])$ for every $E, E' \in B'$. As X is a W-space, Lemma 4.10 in [1] implies that there exists $x_C \in [x'_C]$ such that $d(x_C, y_E) = \mathrm{dist}([x_C], [y_E])$ for $E \in B'$. Clearly, $x_C \in C$, and $x_C \notin E$ for every $E \in B'$.

Let $D \in B$ be a ball such that $x_C \notin D$. Fix $y \in D$. If $D \cap C = \emptyset$, then Lemma 1 implies $d(x_C, y) = \mathrm{dist}([x_C], [y])$. Otherwise $D \subseteq C$, so there is $E \in B'$ such that $D \subseteq E$ or $E \subseteq D$. In both cases $x_C \notin D \cup E$, so $d(y, y_E) < d(y, x_C)$. Suppose that there is $x \in [x_C]$ such that $d(y, x) < d(y, x_C)$. But then the ultrametric triangle inequality implies that $d(y_E, x) < d(y_E, x_C)$, which is impossible. Thus, $d(y, x_C) = \mathrm{dist}(y, [x_C])$. □

Theorem 4 ([1, Theorem 4.13]). *Let X be a W-space, and let S_X, Δ_X, G_δ for $\delta \in \Delta_X$ be defined as above. Then Δ_X is plenary, $\mathrm{Iso}(X)$ acts transitively and faithfully on S_X, and all elements of $\mathrm{Iso}(X)$ respect Δ_X. Thus, it can be regarded as a transitive permutation group $(\mathrm{Iso}(X), S_X, \Delta_X)$. Moreover,*

$$\mathrm{Iso}(X) \cong Wr_{\delta \in \Delta_X} G_\delta.$$

Proof. Consider the action of $\mathrm{Iso}(X)$ on S_X given by

$$\phi.Y = \phi[Y]$$

for $\phi \in \mathrm{Iso}(X)$, $Y \in S_X$.

By [1, Lemma 4.7], Δ_X is a countable plenary family, and every relation in Δ_X has countably many classes. By [1, Lemmas 4.10, 4.11 and 4.12], the action defined above is faithful, and $(\mathrm{Iso}(X), S_X, \Delta)$ is a transitive maximal permutation group respecting Δ_X. Therefore, to prove the theorem, we only need to check condition (3) of Theorem 3.5 in [1].

Suppose that B is an agreeable family of balls in X which satisfies the boundedness condition. Let

$$B_0 = \{C \in B : \text{there exists } D \in B \text{ such that } D \text{ is inclusion-minimal in } B,\ D \subseteq C\},$$
$$B_1 = \{C \in B : \text{there exists } \{C_n\} \subseteq B \text{ such that } \{C_n\} \text{ is decreasing},\ \mathrm{diam}(C_n) \to 0,\ C_n \subseteq C\},$$

and let $B_2 = B \setminus (B_0 \cup B_1)$.

Let A_0 be a selector for the family of all inclusion-minimal balls in B. Let A_1 be the set of all points $x \in X$ of the form $\{x\} = \bigcap C_n$, where $\{C_n\}$ is a sequence of balls in B with $\mathrm{diam}(C_n) \to 0$. Then A_0 intersects every ball in B_0, A_1 intersects every ball in B_1, and by Lemma 1, $d(a, b) = \mathrm{dist}([a], [b])$ for all $a, b \in A_0 \cup A_1$.

We show how to construct A_2 intersecting every element of B_2 so that $A = A_0 \cup A_1 \cup A_2$ intersects every element of B, and $d(a, b) = \mathrm{dist}([a], [b])$ for all $a, b \in A$.

Clearly, for every ball in B_2, point (i) of Lemma 2 holds. Let $C_0, C_1, \dots$ be an enumeration of all balls in B_2. By Lemma 3, there is $x_0 \in C_0$ such that $d(x_0, y) = \mathrm{dist}([x_0], [y])$ for every $D \in B$ with $x_0 \notin D$, and for every $y \in D$. Put $D_0 = C_0$. Suppose that we have constructed $x_0, \dots, x_n$, $D_0, \dots, D_n$ so that $x_i \in C_i$, and x_i is chosen as in Lemma 3 for the ball D_i, $i \leq n$. If $x_{i_0} \in C_{n+1}$ for some $i_0 \leq n$, then put $x_{n+1} = x_{i_0}$, $D_{n+1} = D_{i_0}$. Otherwise, fix x_{n+1} for C_{n+1} as in Lemma 3, and put $D_{n+1} = C_{n+1}$.

By the construction, we have that $\{x_n\}$ intersects every element of B_2, and $d(x_n, x_m) = \mathrm{dist}([x_n], [x_m])$ for every n, m. By Lemma 1, $A_2 = \{x_n\}$ is as required.

Now, by [1, Lemma 4.10], there exists a d-transversal Y such that $A \subseteq Y$. Obviously, A intersects every element of B. □

References

[1] M. Malicki, Separable ultrametric spaces and their isometry groups, *Forum Math.* **26** (2014), 1663–1683.