



SHORT PAPER

EXISTENCE OF LOCAL SOLUTIONS FOR FRACTIONAL DIFFERENCE EQUATIONS WITH LEFT FOCAL BOUNDARY CONDITIONS

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Abstract

For $1<\nu\le 2$ a real number and $T\ge 3$ a natural number, conditions are given for the existence of solutions of the ν th order Atıcı-Eloe fractional difference equation, $\Delta^{\nu}y(t)+f(t+\nu-1,y(t+\nu-1))=0,\,t\in\{0,1,\ldots,T\},$ and satisfying the left focal boundary conditions $\Delta y(\nu-2)=y(\nu+T)=0.$

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1. Introduction

In this paper, for $1 < \nu \le 2$ a real number and $T \ge 3$ a natural number, we are concerned with local solutions of the nonlinear ν th order fractional difference equation,

$$\Delta^{\nu} y(t) + f(t+\nu-1, y(t+\nu-1)) = 0, \quad t \in \{0, 1, \dots, T\}, \tag{1.1}$$

satisfying the left focal boundary conditions,

$$\Delta y(\nu - 2) = 0, \quad y(\nu + T) = 0,$$
 (1.2)

where Δ is the forward difference operator with step-size 1, Δ^{ν} is the Atici-Eloe fractional difference, and $f(x,r): \{\nu-1,\nu,\dots,\nu+T-1\} \times \mathbb{R} \to \mathbb{R}$ is continuous. Our methods rely on application of the Leray-Schauder Nonlinear Alternative [17], which will involve imposing growth restrictions

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on f as well as restrictions on the natural number T. Since the paper involves restrictions on the size of T, classically such a result is called a local existence result as opposed to a global existence result. Primary motivation for this work is the recent paper by Henderson [18] devoted to local solutions of Dirichlet boundary value problems for fractional difference equations.

A good deal of the current interest in fractional difference equations, devoted to both their theoretical development and their applications, was spawned by the Atici and Eloe definitions [1, 2], in the context of discrete domains, of fractional sums and fractional differences. Soon afterwards, Goodrich further developed and extended those papers [9, 10, 11]. Atıcı and Eloe dealt in their first paper with initial value problems for fractional difference equations, and then followed in their second paper with applications of their definitions in obtaining positive solutions for Dirichlet boundary value problems for fractional difference equations. Their second work involved a Guo-Krasnosel'skii fixed point argument requiring the construction of a Green's function for their fractional problem. Some of Goodrich's subsequent research that required construction of appropriate Green's functions has focused on questions extending the Atici and Eloe results and can be found in [12, 13, 14, 15, 16]. For some other researchers' closely related works involving boundary value problems for fractional difference equations, we cite [4, 5, 8, 19, 20, 27, 28, 29, 30, 31].

It is natural that discrete fractional calculus and fractional difference equations often appear in modeling natural processes such as found in the paper by Atıcı and Şengül [3] and in the paper by Metzler et al. [22]. Especially prominent is the current use of boundary value problems for discrete fractional difference equations in its applications to discrete control processes; see, for example, this list of monographs devoted to discrete fractional control [6, 7, 21, 23, 24, 25].

2. Some preliminaries and the Green function

We begin this section with the Atıcı-Eloe definitions of fractional sum and fractional difference in the context of a discrete domain.

DEFINITION 2.1. Let $n \in \mathbb{N}$ and $n-1 < \kappa \le n$ be a real number, and let $a \in \mathbb{R}$. For $t \in \{a + \kappa, a + \kappa + 1, \ldots\}$, the κ th order Atıcı-Eloe fractional sum, $\Delta^{-\kappa}u$, of the function u is defined by

$$\Delta^{-\kappa}u(t) := \frac{1}{\Gamma(\kappa)} \sum_{s=a}^{t-\kappa} (t-s-1)^{(\kappa-1)} u(s),$$

where $t^{(\kappa)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\kappa)}$ is the falling function.

For $t \in \{a+n-\kappa, a+n+1-\kappa, \ldots\}$, the κ th order Atici-Eloe fractional difference, $\Delta^{\kappa}u$, of the function u is defined by

$$\Delta^{\kappa} u(t) := \Delta^{n - (n - \kappa)} u(t) := \Delta^{n} (\Delta^{-(n - \kappa)} u(t)),$$

where Δ is the forward difference defined by $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^{i}u(t) = \Delta(\Delta^{i-1}u(t)), i = 2, 3, \dots$

REMARK 2.1. We note that, for u defined on $\{a, a+1, \ldots\}$, then $\Delta^{-\kappa}u$ is defined on $\{a + \kappa, a + \kappa + 1, \ldots\}$.

In [11], for $1 < \nu \le 2$, Goodrich constructed the Green function, G(t, s), for $-\Delta^{\nu}y(t)=0, t\in\{0,1,\ldots,T\}$, satisfying the left focal boundary conditions (1.2). In particular, by direct computation, Goodrich obtained

$$G(t,s) = \frac{c(\nu,T)}{\Gamma(\nu)} \begin{cases} \Gamma(\nu)(\nu+T-s-1)^{(\nu-1)}, & t=\nu-2,\nu-1, \\ [(2-\nu)t^{(\nu-1)}+(\nu-1)^{(\nu-2)}] \times \\ (\nu+T-s-1)^{(\nu-1)}, & 0 \le t-\nu \le s \le T, \\ [(2-\nu)t^{(\nu-1)}+(\nu-1)^{(\nu-2)}] \times \\ (\nu+T-s-1)^{(\nu-1)}, & 0 \le t-\nu \le T, \end{cases}$$

with

$$c(\nu, T) = \frac{1}{(\nu + T)^{(\nu - 2)} [(\nu - 1) + (2 - \nu)(T + 2)]}.$$

Goodrich also obtained the following properties of G(t,s) which will be of importance to us:

(a) For each $s \in \{0, ..., T\}$.

$$\Delta_t G(\nu - 2, s) = 0$$
 and $G(\nu + T, s) = 0$.

(b)
$$G(t,s) > 0$$
, for $(t,s) \in \{\nu - 2, \dots, \nu + T - 1\} \times \{0, \dots, T\}$.

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(c) $\max_{t \in \{\nu - 2, \dots, \nu + T\}} G(t,s) = G(s + \nu - 1, s)$, for $s \in \{0, \dots, T\}$.

We remark that y is a solution of the linear discrete fractional difference equation, $\Delta^{\nu}y(t) + h(t+\nu-1) = 0$, $t \in \{0,\ldots,T\}$, and satisfying (1.2), if and only if $y: \{\nu - 2, \dots, \nu + T\} \to \mathbb{R}$ has the form

$$y(t) = \sum_{s=0}^{T} G(t, s)h(s + \nu - 1), \quad t \in {\{\nu - 2, \dots, \nu + T\}}.$$

In the next section, G(t, s) will play the role of the kernel of a completely continuous summation operator.

3. Local existence of solutions

In this section, we make application of the Leray-Schauder Nonlinear Alternative [17] in obtaining local solutions of (1.1)-(1.2). In doing so, we impose restrictions on f and restrictions on the natural number T.

THEOREM 3.1. [Leray-Schauder Nonlinear Alternative]. Let $(E, \| \cdot \|)$ be a Banach space, K be a closed and convex subset of E, U be a relatively open subset of K such that $0 \in U$, and $N : \overline{U} \to K$ be completely continuous. Then, either

(i) u = Nu has a solution in \overline{U} ,

or

(ii) There exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Nu$.

For $1 < \nu \le 2$ a real number and $T \ge 3$ a natural number, assume $f(x,r): \{\nu-1,\nu,\ldots,\nu+T-1\} \times \mathbb{R} \to \mathbb{R}$ is continuous, and let

$$m := \max_{s \in \{0, \dots, T\}} G(s + \nu - 1, s).$$

We now present the result of this paper.

THEOREM 3.2. Assume

(A) There exist $\sigma: \{\nu-1, \dots, T+\nu-1\} \to [0, \infty)$ and a nondecreasing function $\psi: [0, \infty) \to [0, \infty)$ such that

$$|f(x,r)| \le \sigma(x)\psi(|r|), \quad (x,r) \in \{\nu - 1, \dots, T + \nu - 1\} \times \mathbb{R},$$

and

(B) There exists $\ell > 0$ such that

$$\frac{\ell}{m\psi(\ell)(T+1)\max_{x\in\{\nu-1,...,T+\nu-1\}}\sigma(x)} > 1.$$

Then, (1.1)-(1.2) has a solution defined on $\{\nu-2,\ldots,T+\nu\}$.

Proof. Let the Banach space $E:=\{h:\{\nu-2,\ldots,T+\nu\}\to\mathbb{R}\mid \Delta h(\nu-2)=h(T+\nu)=0\}$ be equipped with the norm

$$||h|| := \max_{x \in \{\nu - 2, \dots, T + \nu\}} |h(x)|.$$

We seek fixed points of the mapping $N: E \to E$ defined by

$$(Nh)(t) := \sum_{s=0}^{T} G(t,s) f(s+\nu-1,h(s+\nu-1)), \ h \in E, \ t \in \{\nu-2,\dots,T+\nu\},$$

where G(t, s) is the Green function of Section 2. We note that $u \in E$ is a solution of (1.1)-(1.2) if and only if u is a fixed point of N.

We first show that N maps bounded sets into bounded sets. In that direction, for r > 0, let

$$B_r := \{ h \in E \mid ||h|| \le r \}$$

be a bounded subset of E. Then, by (A), for $t \in \{\nu - 2, \dots, T + \nu\}$ and $h \in B_r$,

$$|(Nh)(t)| \leq \sum_{s=0}^{T} G(t,s)|f(s+\nu-1,h(s+\nu-1))|$$

$$\leq \sum_{s=0}^{T} G(s+\nu-1,s)|f(s+\nu-1,h(s+\nu-1))|$$

$$\leq \sum_{s=0}^{T} m\sigma(s+\nu-1)\psi(|h(s+\nu-1)|)$$

$$\leq m\sum_{s=0}^{T} \max_{x\in\{\nu-1,\dots,T+\nu-1\}} \sigma(x)\psi(r)$$

$$= m\psi(r) \max_{x\in\{\nu-1,\dots,T+\nu-1\}} \sigma(x)(T+1).$$

Hence, $||Nh|| \le m\psi(r) \max_{x \in \{\nu-1,\dots,T+\nu-1\}} \sigma(x)(T+1)$, and so N maps B_r into a bounded set.

Since $\{\nu-2,\ldots,T+\nu\}$ is a discrete set, it follows immediately that N maps B_r into an equicontinuous set. Therefore, by the Arzelà-Ascoli Theorem, N is completely continuous.

Next, we suppose $h \in E$ and that for some $0 < \lambda < 1$, $h = \lambda Nh$. Then, for $t \in \{\nu - 2, \dots, T + \nu\}$, and again by (A),

$$\begin{split} |h(t)| &= |\lambda(Nh)(t)| \\ &\leq \sum_{s=0}^T G(t,s) |f(s+\nu-1,h(s+\nu-1))| \\ &\leq \sum_{s=0}^T G(s+\nu-1,s) |f(s+\nu-1,h(s+\nu-1))| \\ &\leq m \sum_{s=0}^T \sigma(s+\nu-1) \psi(\|h\|) \\ &\leq m \max_{x \in \{\nu-1,\dots,T+\nu-1\}} \sigma(x) \psi(\|h\|) (T+1), \end{split}$$

which yields

$$\frac{\|h\|}{m \max_{x \in \{\nu-1, \dots, T+\nu-1\}} \sigma(x) \psi(\|h\|)(T+1)} \le 1.$$

It follows from (B) that $||h|| \neq \ell$. If we set

$$U := \{ h \in E \mid ||h|| < \ell \},\$$

then the operator $N:\overline{U}\to E$ is completely continuous. From the choice of U, then there is no $h\in\partial U$ such that $h=\lambda Nh$, for some $0<\lambda<1$. It follows from Theorem 3.1 that N has a fixed point $y\in\overline{U}$, which is a desired solution of (1.1)-(1.2).

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