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RESEARCH PAPER

STABILITY ANALYSIS FOR DISCRETE TIME ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider a discrete-time fractional model of abstract form involving the Riemann-Liouville-like difference operator. On account of the C_0-semigroups generated by a closed linear operator A and based on a distinguished class of sequences of operators, we show the existence of stable solutions for the nonlinear Cauchy problem by means of fixed point technique and the compact method. Moreover, we also establish the Ulam-Hyers-Rassias stability of the proposed problem. Two examples are presented to explain the main results.

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Key Words and Phrases: fractional difference operator; C_0-semigroups; stable solutions; Ulam-Hyers-Rassias stability

1. Introduction

Fractional calculus plays an increasingly important role in many fields due to its applications, such as physics, biology and engineering, etc. It provides an excellent tool for modeling the memory properties of viscoelasticity materials and the diffusion process of particles, see [11, 22, 23]. Meanwhile, the study of the fractional derivatives of differential equations is still a hot topic and there are many interesting results for the qualitative analysis, see the monographs [12, 17, 20] and the papers [8, 13] and reference therein. Compared with the continuous fractional differential models, some scholars

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found that the discrete time fractional differential equations (or called fractional difference equations) can still capture certain hidden aspects of real world phenomena with memory effects, further it will appear some new interesting properties which are different from the continuous case. Cermák, Gyori and Nechvátal [4] investigated the stability behaviors of discrete fractional systems. Wu and Baleanu [18, 19] considered the discrete fractional logistic map and its chaos whose point out there some new degrees of freedom in discrete fractional models. Abadias and Mianab [2] generalized the algebraic structure of Cesàro sums in the discrete fractional operators setting, several subjects of interest in harmonic and functional analysis are displayed. Goodrich and Lizama [9] showed the positivity, monotonicity, and convexity of functions under a different definition for fractional delta operator (see Definition 2.2) which can be derived by a transference principle from known fractional difference operators [3].

In this paper, we study the following nonlinear abstract fractional difference equations

$$\begin{cases} \Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{N}_0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where Δ^α is the Riemann-Liouville-like fractional difference operator of order $0 < \alpha \leq 1$, $f : \mathbb{N}_0 \to X$, A is the infinitesimal generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ with domain $D(A)$ defined on a Banach space X, $\mathbb{N}_0 = \{0, 1, 2, ...\}$. It is important to remark that such problem has been studied in [15] with A bounded. When $f = 0$ in (1.1), [14] considered the existence and stability for abstract difference equations with Caputo-like fractional difference operator by means of operator theory. Also in [10], the authors derived a structure of the solutions for the inhomogenous Cauchy problem of abstract fractional difference equations (1.1) and further investigated the existence results to the proposed problem.

To the best of our knowledge, there are few papers dealing with the analysis of existence for the discrete time abstract fractional differential equations, especially the Ulam-Hyers-Rassias stability study of problem (1.1) has not yet been investigated. However, there are some studies involving the fractional difference operators of Riemann-Liouville and/or Caputo-like, which focus on the stability results of Ulam-Hyers and Ulam-Hyers-Rassias in finite interval of discrete points, Chen and Zhou [7] considered the Ulam-Hyers stability of solutions for a discrete fractional boundary value problem, Chen, Bohner and Jia [5] studied the Ulam-Hyers stability of a initial value problem of Caputo-like fractional difference equations. Obviously, it's not a very captivating situation that it does not take the whole values on infinite interval of discrete points, for this purpose, we will consider the case of whole values on $\mathbb{N}$ in the current paper. It should be noted that the Ulam-Hyers stability may not exist on $\mathbb{N}$ of problem (1.1) due to the fact that the series $\sum_{j=0}^{\infty} k^{\alpha}(j) = \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}$ is divergent for $\alpha > 0$.

This paper is organized as follows: In Section 2, we introduce some important preliminary definitions and results. In Section 3, we consider a nonlinear discrete time abstract fractional differential equation modeled as (1.1), by using a different argument involving the compactness of the semigroup $T(t)$ associated with f satisfying growth type condition in the second variable, we obtain an existence criterion of stable solutions. In Section 4, the Ulam-Hyers-Rassias stability is also established.

2. **Preliminaries**

Let $\mathbb{N}_1 = \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X be a Banach space with norm $\|\cdot\|$, $\mathcal{B}(X)$ stands for the space of bounded linear operators from X into X with the norm $\|\cdot\|_{\mathcal{B}} := \|\cdot\|_{\mathcal{B}(X)}$. We also consider the essentially bounded vector-valued Banach space of sequences $l^{\infty}(\mathbb{N}_0; X)$, which is defined by

$$l^{\infty}(\mathbb{N}_0; X) := \Big\{u : \ \mathbb{N}_0 \to X, \ \sup_{n\in\mathbb{N}_0} \|u(n)\| < \infty\Big\},$$

endowed with the norm $\|u\|_{\infty} = \sup_{n\in\mathbb{N}_0} \|u(n)\|$. Additionally, for $\alpha > 0$, let $\Gamma(\cdot)$ denote the Euler Gamma function, we consider a scalar sequence $\{k^{\alpha}(n)\}_{n\in\mathbb{N}_0}$ defined by

$$k^{\alpha}(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0.$$

One can check that k^{α} possesses the semigroup property

$$(k^{\alpha} * k^{\beta})(n) = \sum_{j=0}^{n} k^{\alpha}(n-j)k^{\beta}(j) = k^{\alpha+\beta}(n), \quad n \in \mathbb{N}_0, \ \alpha > 0, \ \beta > 0,$$

where $*$ denotes the finite convolution. It is easy to see that for all $n \in \mathbb{N}_0$ and for any $\alpha \in (0,1]$, $k^{\alpha}(n) \in (0,1]$ and $k^{\alpha}(n)$ is a non-increasing sequence. By virtue of [21, pp.77 (1.18)] we also have

$$k^{\alpha}(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right), \ n \in \mathbb{N}, \ \alpha > 0. \tag{2.1}$$

Now, let us recall some definitions and properties of fractional difference/sum operators introduced by Lizama in [14]. These definitions were based on a transference principle which is applied to convert the definitions by Atici and Eloe [3], allowing the use of cleaner, simpler and transparent algebraic manipulations, [9].

Definition 2.1. Let $\alpha > 0$, the α-th order fractional sum operator is defined by

$$\Delta^{-\alpha}u(n) = \sum_{j=0}^{n} k^{\alpha}(n-j)u(j), \quad n \in \mathbb{N}_0.$$

DEFINITION 2.2. Let $\alpha \in (0,1]$, the α-th order fractional difference operator (in the sense of Riemann-Liouville-like) is defined by

$$\Delta^{\alpha}u(n) := \Delta \circ \Delta^{-(1-\alpha)}u(n), \quad n \in \mathbb{N}_0,$$

where Δ denotes the forward Euler operator by $\Delta u(n) := u(n+1) - u(n)$, $n \in \mathbb{N}_0$.

Our basic assumption is that the operator A in (1.1) is the infinitesimal generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$, which means that there exists a constant $M \geq 1$ such that $M = \sup_{t\in[0,\infty)} \|T(t)\|_{\mathcal{B}} < \infty$. It is well known from [16, pp.19, Theorem 5.2(i)] that A is closed and the domain $D(A)$ of operator A is dense in X. The norm of $D(A)$ is given by a graph norm defined as $\|x\|_A = \|x\| + \|Ax\|$ for any $x \in D(A)$. Next, we introduce the following notion of α-resolvent sequence that is an important tool to deal with abstract fractional difference equations.

DEFINITION 2.3. Let $\alpha > 0$ and let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. An operator-valued sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called an α-resolvent sequence generated by A if it satisfies the following conditions:

(i) $S_\alpha(n)x \in D(A)$ and $AS_\alpha(n)x = S_\alpha(n)Ax$, for all $n \in \mathbb{N}_0$ and $x \in D(A)$;
(ii) $S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

The main properties of α-resolvent sequences are contained in the following results.

LEMMA 2.1. ([1]) *Let $\rho(A)$ be the resolvent set of operator A and let $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ be an α-resolvent sequence generated by A. Then*

(i) $1 \in \rho(A)$, *and for all $x \in X$ we have that $S_\alpha(0)x = (I-A)^{-1}x$.*
(ii) *For all $x \in X$ we have that $S_\alpha(0)x \in D(A)$ and $S_\alpha(n)x \in D(A^2)$ for all $n \in \mathbb{N}$.*

We next get a strong relationship between C_0-semigroup in the setting of fractional version and α-resolvent sequence, one can find that the Poisson distribution also acts as a bridge between the discrete and continuous theories; for more details see [9, 14].

Lemma 2.2. ([10]) *Let $0 < \alpha \leq 1$ and let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X. Then, A generates an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ given by*

$$S_\alpha(n)x = \int_0^\infty \int_0^\infty p_n(t) f_{s,\alpha}(t) T(s)x ds dt, \quad n \in \mathbb{N}_0,\ x \in X. \tag{2.2}$$

where $p_n(t) := e^{-t}t^n/n!$ $(n \in \mathbb{N}_0,\ t \geq 0)$ is the Poisson distribution and function $f_{\alpha,s}(t)$ denotes the stable Lévy process given by

$$f_{s,\alpha}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt-sz^\alpha} dz, \quad \sigma > 0,\ s > 0,\ t \geq 0,\ 0 < \alpha < 1,$$

in which the branch of z^α is taken such that $Re(z^\alpha) > 0$ for $Re(z) > 0$.

It is mentioned that the stable Lévy process has the properties (i) $f_{s,\alpha}(\lambda) \geq 0,\ \lambda > 0,\ s > 0,\ \alpha \in (0,1)$; (ii) $\int_0^\infty f_{s,\alpha}(t)ds = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$; and (iii) for all $t > 0$, $\lambda \in \mathbb{C}$, we have

$$\int_0^\infty e^{-\lambda s} f_{s,\alpha}(t) ds = t^{\alpha-1} E_{\alpha,\alpha,}(-\lambda t^\alpha),$$

where $E_{\alpha,\alpha}(\cdot)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\alpha}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \alpha)}, \quad z \in \mathbb{C}.$$

Lemma 2.3. ([10]) *Let $0 < \alpha \leq 1$ and let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X. Then,*

$$\|S_\alpha(n)x\| \leq M k^\alpha(n)\|x\|, \quad \text{for } n \in \mathbb{N}_0,\ x \in X.$$

Furthermore, if A is the generator of a compact C_0-semigroup $\{T(t)\}_{t>0}$. Then, A generates a compact α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$.

Remark 2.1. Noting that if X is a finite dimension space, we see that the semigroup $T(t)$ can be rewritten by e^{At}, and if A is the generator of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ with respect to $\|A\|_{\mathcal{B}} \leq 1$. Then, A generates an α-resolvent sequence if and only if $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is given by

$$S_\alpha(n) = \sum_{j=0}^\infty A^j\, k^{\alpha j+\alpha}(n).$$

In fact, by the properties of stable Lévy process (iii) and Mittag-Leffler functions, we get for $n \in \mathbb{N}_0$

$$\int_0^\infty \int_0^\infty p_n(t) f_{s,\alpha}(t) e^{As} ds dt = \int_0^\infty p_n(t) t^{\alpha-1} E_{\alpha,\alpha,}(At^\alpha) dt$$
$$= \sum_{j=0}^\infty A^j k^{\alpha j+\alpha}(n).$$

Conversely, it is easy to check that $S_\alpha(n)$ satisfies the Definition 2.3 for every $n \in \mathbb{N}_0$.

3. Existence of stable solutions

In this section, we study the existence of stable solutions for the nonlinear discrete time abstract fractional differential equation (1.1). For this purpose, we introduce the next definition of solutions, which can be seen in [10].

Definition 3.1. Let $0 < \alpha < 1$ and A be the generator of an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$. We say that $u \in l^\infty(\mathbb{N}_0; D(A))$ is a solution of (1.1) if u satisfies $u(0) = u_0 \in D(A)$ and

$$u(n) = S_\alpha(n)(I - A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j) f(j, u(j)), \quad n \in \mathbb{N}.$$

According to Lemma 2.2, this definition is consistent with true solutions of (1.1). From Lemma 2.1 it follows that $S_\alpha(n)x \in D(A)$ for all $x \in X$, $n \in \mathbb{N}_0$ and $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$. In order to use the Schauder fixed point theorem, we need the next compactness result.

Lemma 3.1. *Let $U \subset l^\infty(\mathbb{N}; X)$ satisfy*

(a) *The set $H_n(U) = \{u(n) : \; u \in U\}$ is relatively compact in X, for all $n \in \mathbb{N}$.*

(b) $\lim_{n\to\infty} \sup_{u\in U} \|u(n)\| = 0$, *that is, for each $\varepsilon > 0$, there is a $N > 0$ such that $\|u(n)\| < \varepsilon$, for each $n \ge N$ and for all $u \in U$.*

Then U is relatively compact in $l^\infty(\mathbb{N}; X)$.

P r o o f. Let $\{u_m\}_{m=1}^\infty$ be a sequence in U, then by (a), for any given $n \in \mathbb{N}$, there exists a convergent subsequence $\{u_{m_k}\}_{k=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that $\lim_{k\to\infty} u_{m_k}(n) = u(n)$, i.e., for each $\varepsilon > 0$, there exists a constant $N^* = N^*(n, \varepsilon) > 0$, such that

$$\|u_{m_k}(n) - u(n)\| < \varepsilon, \quad \text{for } k > N^*.$$

From the assumption (b), for each $\varepsilon > 0$, there exists a constant $N' > 0$, such that for $n \geq N'$

$$\begin{aligned}\sup_{n \geq N'} \|u_{m_k}(n) - u_{m_j}(n)\| &\leq \sup_{n \geq N'} \|u_{m_k}(n)\| + \sup_{n \geq N'} \|u_{m_j}(n)\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Let N' be fixed. For each $1 \leq n < N'$, then for $j, k > N = N(N', \varepsilon)$, we have

$$\begin{aligned}&\sup_{1 \leq n < N'} \|u_{m_k}(n) - u_{m_j}(n)\| \\ \leq &\sup_{1 \leq n < N'} \|u_{m_k}(n) - u(n)\| + \sup_{1 \leq n < N'} \|u_{m_j}(n) - u(n)\| \\ < &\varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore, one has $\|u_{m_k} - u_{m_j}\|_\infty < \varepsilon$. This means that $\{u_{m_k}\}_{k=1}^\infty$ is a Cauchy subsequence in $l^\infty(\mathbb{N}; X)$. We thus derive that $\{u_{m_k}\}_{k=1}^\infty$ has a convergence element $u \in l^\infty(\mathbb{N}; X)$, which implies that U is relatively compact in $l^\infty(\mathbb{N}; X)$. □

For a given function $f : \mathbb{N}_0 \times X \to X$, the Nemytskii operator $N_f : l^\infty(\mathbb{N}; X) \to l^\infty(\mathbb{N}; X)$ (with f restricted to $\mathbb{N}$) is defined by

$$N_f(u)(n) := f(n, u(n)), \quad n \in \mathbb{N}.$$

In order to obtain our main result, we will need the following assumptions:

- (H1) A is the generator of a compact C_0-semigroup $\{T(t)\}_{t>0}$ and α-resolvent sequence defined in (2.2) for $0 < \alpha < 1$.
- (H2) There exist constant $L_f > 0$ and a positive sequence $a(\cdot) \in l^\infty(\mathbb{N}_0)$ such that $|a(n)| \leq L_f k^{1-\beta}(n) \quad (0 < \alpha < \beta < 1)$ and function $\|f(n, x)\| \leq a(n)\|x\|$, for all $n \in \mathbb{N}_0$ and $x \in X$.
- (H3) The Nemytskii operator N_f is continuous in $l^\infty(\mathbb{N}; X)$.

We mention that a vector-valued sequence $u \in l^\infty(\mathbb{N}_0, X)$ is said to be stable if $\|u(n)\| \to 0$, as $n \to \infty$. Obviously, f is stable for any $x \in X$ according to (H2) and the approximate behavior of $a(n) \to 0$ as $n \to \infty$ in view of (2.1). Now, we get the following main result.

Theorem 3.1. *Assume that operator A satisfies* (H1) *and f satisfies* (H2)-(H3). *Then, the problem (1.1) with $u_0 \in D(A)$ has at least one stable solution.*

P r o o f. Let us define the map $\mathcal{P} : l^\infty(\mathbb{N}_0; D(A)) \to l^\infty(\mathbb{N}_0; D(A))$ as follows

$$(\mathcal{P}u)(n) := S_\alpha(n)(I-A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j,u(j)),\ n \in \mathbb{N},$$

and $(\mathcal{P}u)(0) = u_0$. We first show that $\mathcal{P}$ is well defined. In fact, let $u \in l^\infty(\mathbb{N}_0; D(A))$ be given, it follows from (H2) that

$$\begin{aligned}\|(\mathcal{P}u)(n)\| \leq& \|S_\alpha(n)(I-A)u_0\| + \sum_{j=0}^{n-1} \|S_\alpha(n-1-j)f(j,u(j))\|\\ \leq& Mk^\alpha(n)\|(I-A)u_0\| + ML_f \sum_{j=0}^{n-1} k^\alpha(n-1-j)a(j)\|u(j)\|,\end{aligned}$$

additionally, in view of Lemma 2.3 and observing that $1+\alpha-\beta \in (0,1]$ for $0<\alpha<\beta<1$, using the semigroup relationship $(k^\alpha * k^{1-\beta})(n-1) = k^{1+\alpha-\beta}(n-1)$ for each $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\|(\mathcal{P}u)(n)\| \leq& Mk^\alpha(n)\|(I-A)u_0\| + ML_f \sum_{j=0}^{n-1} k^\alpha(n-1-j)k^{1-\beta}(j)\|u(j)\|\\ \leq& 2Mk^\alpha(n)\|u_0\|_A + ML_f\|u\|_\infty k^{1+\alpha-\beta}(n-1)\\ \leq& 2M\|u_0\|_A + ML_f\|u\|_\infty,\end{aligned}$$

where we also use the fact that $0 < k^s(n) \leq k^s(0) = 1$ owing to the nonincreasing property of $k^s(n)$ for each $0 < s \leq 1$ and $n \in \mathbb{N}$. This proves that $\mathcal{P}$ is well defined. Now, we show that $\mathcal{P}$ is continuous. Let $\{u_m\}_{m=1}^\infty \subset l^\infty(\mathbb{N}_0; D(A))$ be a sequence such that $u_m \to u$ as $m \to \infty$ in the norm topology of $l^\infty(\mathbb{N}_0; D(A))$. First, we have

$$\|f(j,u_m(j)) - f(j,u(j))\| \leq ML_f(\|u_m\|_\infty + \|u\|_\infty)k^{1-\beta}(j),$$

and using the semigroup property of $k^\alpha(n)$ for all $n \in \mathbb{N}_0$, we get

$$\begin{aligned}\|(\mathcal{P}u_m)(n) - (\mathcal{P}u)(n)\| \leq& M\sum_{j=0}^{n-1} k^\alpha(n-1-j)\|f(j,u_m(j)) - f(j,u(j))\|\\ \leq& ML_f(\|u_m\|_\infty + \|u\|_\infty)k^{1+\alpha-\beta}(n-1)\\ \leq& ML_f(\|u_m\|_\infty + \|u\|_\infty).\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, be virtue of the property of series, it is easy to check that

$$\|(\mathcal{P}u_m)(n) - (\mathcal{P}u)(n)\| \leq M\Delta^{-\alpha}\|(N_f(u_m) - N_f(u))(n-1)\| \to 0,$$

as $m \to \infty$, which implies that $\|\mathcal{P}u_m - \mathcal{P}u\|_\infty \to 0$ as $m \to \infty$. Therefore $\mathcal{P}$ is continuous.

Since $T(t)$ is compact for $t > 0$, then from Lemma 2.3, we know that the sequence of operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is compact. Let $r > 0$ be given. We define a set by

$$\mathcal{S}_r := \{\omega \in l^\infty(\mathbb{N}; D(A)) : \ \|\omega\|_\infty \le r\}.$$

Clearly, $\mathcal{S}_r$ is a bounded, closed and convex subset of $l^\infty(\mathbb{N}; D(A))$. In view of (H2), we can deduce that $\mathcal{P}$ maps $\mathcal{S}_r$ into itself. Thus, it remains to show that $\mathcal{P}$ is a compact operator.

In order to prove that $U := \mathcal{P}\mathcal{S}_r$ is relatively compact, we will use Lemma 3.1. We check that the conditions in this lemma are satisfied, and now we check that U satisfies all the assumptions:

(a) Let $v = \mathcal{P}u$ for any $u \in \mathcal{S}_r$. We have

$$v^\varepsilon(n) = (\mathcal{P}^\varepsilon u)(n) = \sum_{j=0}^{n-1} S_\alpha^\varepsilon(j) f(n-1-j, u(n-1-j)), \ n \in \mathbb{N},$$

where

$$\begin{aligned} S_\alpha^\varepsilon(j)x &:= \int_0^\infty \int_\varepsilon^\infty p_j(t) f_{s,\alpha}(t) T(s)x \ dsdt \\ &= T(\varepsilon) \int_0^\infty \int_\varepsilon^\infty p_j(t) f_{s,\alpha}(t) T(s-\varepsilon)x \, dsdt, \quad x \in X, \end{aligned}$$

where we use the semigroup property of $T(t)$. Hence, it remains to prove that $(\mathcal{Q} * f)(n-1)$ is bounded, where

$$\mathcal{Q}(j)x = \int_0^\infty \int_\varepsilon^\infty p_j(t) f_{s,\alpha}(t) T(s-\varepsilon)x \, dsdt.$$

In fact, noting that from the properties of stable Lévy process (ii), we have the following identity

$$\int_0^\infty \int_0^\infty p_j(t) f_{s,\alpha}(t) dsdt = k^\alpha(j), \quad j \in \mathbb{N}_0,$$

it is easy to check

$$\begin{aligned} &\Big\| \sum_{j=0}^{n-1} \mathcal{Q}(j) f(n-1-j, u(n-1-j)) \Big\| \\ \le &\sum_{j=0}^{n-1} \int_0^\infty \int_\varepsilon^\infty p_j(t) f_{s,\alpha}(t) \|T(s-\varepsilon) f(n-1-j, u(n-1-j))\| \, dsdt \\ \le &M L_f \sum_{j=0}^{n-1} k^\alpha(j) k^{1-\beta}(n-1-j) r \le M L_f r. \end{aligned}$$

For any fixed $\hat{n} \in \mathbb{N}$ and for all $n \geq \hat{n}+1$, since

$$\sum_{j=\hat{n}}^{n-1} k^{\alpha}(j)k^{1-\beta}(n-1-j) \leq k^{1+\alpha-\beta}(n-1),$$

and by (2.1) we have

$$k^{1+\alpha-\beta}(n-1) = \frac{1}{\Gamma(1+\alpha-\beta)}(n-1)^{\alpha-\beta}\left[1+O\left(\frac{1}{n-1}\right)\right],$$

for n large enough, it follows that for any $\delta > 0$ there is $n_* \in \mathbb{N}$ large enough such that $n_*+1 \leq n$ with n large enough and

$$\sum_{j=n_*}^{n-1} k^{\alpha}(j)k^{1-\beta}(n-1-j) < \frac{\delta}{2ML_f r}.$$

Therefore, one has

$$\begin{aligned}&\sum_{j=n_*}^{n-1} \|(S_\alpha(j)-S_\alpha^\varepsilon(j))f(n-1-j,u(n-1-j))\|\\ \leq&2ML_f\sum_{j=n_*}^{n-1} k^{\alpha}(j)k^{1-\beta}(n-1-j)\|u(n-1-j)\| < \delta.\end{aligned}$$

For all $n \in \mathbb{N}$, by the proof of [10, Corollary 3.2.], we know $\|(S_\alpha(n)-S_\alpha^\varepsilon(n))x\| \leq \varepsilon$ for all $n \in \mathbb{N}_0$ and for $x \in X$. Hence, we get

$$\sum_{j=0}^{n_*-1} \|(S_\alpha(j)-S_\alpha^\varepsilon(j))f(n-1-j,u(n-1-j))\| \leq ML_f r\varepsilon n_*.$$

Together the above arguments, we see that

$$\begin{aligned}\|v(n)-v^\varepsilon(n)\| \leq& \sum_{j=0}^{n_*-1} \|(S_\alpha(j)-S_\alpha^\varepsilon(j))f(n-1-j,u(n-1-j))\|\\ &+\sum_{j=n_*}^{n-1} \|(S_\alpha(j)-S_\alpha^\varepsilon(j))f(n-1-j,u(n-1-j))\|\\ \leq&ML_f r\varepsilon n_*+\delta.\end{aligned}$$

For the arbitrariness of δ, it yields that $\|v(n)-v^\varepsilon(n)\| \to 0$ as $\varepsilon \to 0$. We thus conclude that the set $H_n(U)$ is relatively compact in X for all $n \in \mathbb{N}$.

(b) Let $u \in \mathcal{S}_r$ and $v = \mathcal{P}u$. For each $n \in \mathbb{N}$ we have

$$\|v(n)\| \leq ML_f \sum_{j=0}^{n-1} k^{\alpha}(n-1-j)k^{1-\beta}(j)\|u(j)\|$$

$$\leq ML_f\|u\|_\infty k^{\alpha+1-\beta}(n-1),$$

which implies that $\lim_{n\to\infty}\|v(n)\| = 0$ independently of $u \in \mathcal{S}_r$. Therefore, $U = \mathcal{P}\mathcal{S}_r$ is relatively compact in $l^\infty(\mathbb{N}; D(A))$ from Lemma 3.1, and by applying the continuity of operator $\mathcal{P}$, we conclude that $\mathcal{P}$ is a completely continuous operator. Thus, the Schauder's fixed point theorem shows that $\mathcal{P}$ has at least one fixed point $u \in l^\infty(\mathbb{N}_0; D(A))$.

Additionally, let u^* be a solution of problem (1.1) in $l^\infty(\mathbb{N}_0; D(A))$, which means that there is a constant $C > 0$ such that $\|u^*\|_\infty \leq C$ and

$$u^*(n) = S_\alpha(n)(I-A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j,u^*(j)), \quad n \in \mathbb{N},$$

moreover, in view of (2.1) we have

$$\begin{aligned}
\|u^*(n)\| \leq& \|S_\alpha(n)(I-A)u_0\| + \sum_{j=0}^{n-1}\|S_\alpha(n-1-j)f(j,u^*(j))\| \\
\leq& Mk^\alpha(n)\|(I-A)u_0\| + ML_f\sum_{j=0}^{n-1}k^\alpha(n-1-j)k^{1-\beta}(j)\|u^*(j)\| \\
\leq& 2Mk^\alpha(n)\|u_0\|_A + ML_fCk^{1+\alpha-\beta}(n-1) \\
\to& 0, \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, u^* is a stable solution. The proof is completed. □

Remark 3.1. Theorem 3.1 shows that it is not necessary to use the Lipschitz condition to establish the existence for problem (1.1), and this is a general result of the paper [10].

Example 3.1. Let $\Omega = [0,\pi]$ and $X = L^2(\Omega)$. We consider the following discrete abstract Cauchy problem

$$\begin{cases} \Delta^\alpha u(n,z) = \dfrac{d^2}{dz^2}u(n+1,z) + k^{1-\beta}(n)u(n,z), & n\in\mathbb{N}_0,\ z\in\Omega, \\ u(n,0) = u(n,\pi) = 0, \quad n\in\mathbb{N}_0, \\ u(0,z) = 0,\ z\in\Omega, \end{cases} \tag{3.1}$$

where Δ^α is the Riemann-Liouville-like fractional difference operator of order $0<\alpha<\beta<1$.

Let us consider the operator $A : D(A) \subseteq X \to X$ defined by

$$D(A) = \left\{v\in X:\ v',\ v''\in X,\ v(0) = v(\pi) = 0\right\}, \quad Av = v''.$$

Clearly A is closed densely defined in X and it is well known that A generates a compact, uniformly bounded and analytic C_0-semigroup $\{T(t)\}_{t>0}$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-m^2$, $m \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $\phi_m(z) = \sqrt{2/\pi}\sin(mz)$. In addition, $\{\phi_m\}_{m\in\mathbb{N}}$ is an orthogonal basis for X, and

$$T(t)u = \sum_{m=1}^{\infty} e^{-m^2 t}(u,\phi_m)\phi_m, \quad u \in D(A).$$

Hence, by applying Lemmas 2.2-2.3, we get the discrete compact α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ as follows

$$S_\alpha(n)u = \sum_{m=1}^{\infty} \int_0^\infty p_n(t)t^{\alpha-1}E_{\alpha,\alpha}(-m^2t^\alpha)dt(u,\phi_m)\phi_m.$$

Let

$$\mathcal{S}_m(n) = \int_0^\infty p_n(t)t^{\alpha-1}E_{\alpha,\alpha}(-m^2t^\alpha)dt,$$

since the inequalities $|E_{\alpha,\alpha}(-m^2t^\alpha)| \le 1/\Gamma(\alpha)$ for all $m \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $|\mathcal{S}_m(n)| \le k^\alpha(n)$ for $n \in \mathbb{N}_0$, it follows that $\mathcal{S}_m(n)$ tend to zero as $n \to \infty$ for all $m \in \mathbb{N}$. Thus, we have

$$S_\alpha(n)u = \sum_{m=1}^{\infty} \mathcal{S}_m(n)(u,\phi_m)\phi_m, \quad n \in \mathbb{N}_0.$$

Therefore, let $f(n,u(n)) = k^{1-\beta}(n)u(n)$, the problem (3.1) possesses a stable solution by Theorem 3.1 and its expression form is given by

$$u(n) = \sum_{m=1}^{\infty}\sum_{j=0}^{n-1} \mathcal{S}_m(n-1-j)(f(j,u(j)),\phi_m)\phi_m, \ n \in \mathbb{N}.$$

4. **Ulam-Hyers-Rassias stability results**

In this section, we obtain the Ulam-Hyers-Rassias stability for problem (1.1). We now introduce the following adaptation definition of Ulam-Hyers-Rassias stability for the discrete form of fractional differential equation.

Definition 4.1. If $u(n)$ satisfies

$$\|\Delta^\alpha u(n) - Au(n+1) - f(n,u(n))\| \le \vartheta(n), \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $\vartheta(n) \ge 0$ for all $n \in \mathbb{N}_0$, and there exist a solution $v(n)$ of the problem (1.1) and a constant $C > 0$ independent of $u(n)$ and $v(n)$ with

$$\|u(n) - v(n)\| \le C\vartheta(n), \quad n \in \mathbb{N}_0,$$

for all $n \in \mathbb{N}_0$, then problem (1.1) is called the Ulam-Hyers-Rassias stability. In particular, if $\vartheta(n)$ is substituted for a constant in the above inequalities, then problem (1.1) is called the Ulam-Hyers stability.

REMARK 4.1. Obviously, v solves (4.1) if and only if there exists $g : \mathbb{N}_0 \to X$ satisfying

$$\|g(n)\| \leq \vartheta(n), \quad n \in \mathbb{N}_0.$$

such that

$$\Delta^{\alpha} v(n) = Av(n+1) + f(n, v(n)) + g(n), \quad n \in \mathbb{N}_0.$$

Furthermore, if $v \in l^{\infty}(\mathbb{N}_0, X)$ is a solution of inequality (4.1), there exists a constant $C > 0$ such that v is a solution of the following inequality

$$\begin{aligned}\Big\| v(n) - S_{\alpha}(n)(I-A)v(0) - \sum_{j=0}^{n-1} S_{\alpha}(n-1-j) f(j, v(j)) \Big\| \\ \leq C \sum_{j=0}^{n-1} \|S_{\alpha}(n-1-j)\|_{\mathcal{B}} \vartheta(j).\end{aligned}$$

REMARK 4.2. It is hard to get the Ulam-Hyers stability of problem (1.1), because if we substitute the sequence $\vartheta(n)$ for a constant, then from Remark 4.1, we see that

$$\Big\| v(n) - S_{\alpha}(n)(I-A)v(0) - \sum_{j=0}^{n-1} S_{\alpha}(n-1-j) f(j, v(j)) \Big\| \leq C \sum_{j=0}^{n-1} k^{\alpha}(j),$$

in which $\sum_{j=0}^{\infty} k^{\alpha}(j)$ is divergent according to the Raabe's discriminant, hence the above inequality does not make sense and we can not find a suitable stability in the sense of Ulam-Hyers.

(H4) there exists a nonnegative sequence $L(n)$, such that

$$\|f(n,x) - f(n,y)\| \leq L(n)\|x - y\|, \text{ for any } x, y \in X, \; n \in \mathbb{N}_0,$$

with respect to series $\sum_{j=0}^{\infty} L(j)$ convergence absolutely.

THEOREM 4.1. *Assume that (H4) holds. Let $\vartheta(n) : \mathbb{N}_0 \to \mathbb{R}_+$ be an increasing sequence such that $\sum_{j=0}^{n-1} \vartheta(j) \leq \vartheta(n)$ and let $u \in l^{\infty}(\mathbb{N}_0, D(A))$ be a solution of inequality (4.1), then problem (1.1) is Ulam-Hyers-Rassias stable.*

P r o o f. Let $v \in l^{\infty}(\mathbb{N}_0, D(A))$ be a solution of inequality (4.1). By Remark 4.1, from the property of $0 < k^{\alpha}(n) \leq 1$ for $\alpha \in (0,1)$, $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\left\| v(n) - S_{\alpha}(n)(I-A)v(0) - \sum_{j=0}^{n-1} S_{\alpha}(n-1-j)f(j,v(j)) \right\| \\
\leq &\sum_{j=0}^{n-1} \|S_{\alpha}(n-1-j)\|_{\mathcal{B}}\vartheta(j) \\
\leq &M\sum_{j=0}^{n-1} k^{\alpha}(n-1-j)\vartheta(j) \leq M\vartheta(n).
\end{aligned}$$

Let us denote by $u \in l^{\infty}(\mathbb{N}_0, D(A))$ the unique solution of the Cauchy problem

$$\begin{cases} \Delta^{\alpha}u(n) = Au(n+1) + f(n,u(n)), \quad n \in \mathbb{N}_0; \\ u(0) = v(0). \end{cases}$$

The solution u of above equation satisfies

$$u(n) = S_{\alpha}(n)(I-A)v(0) + (S_{\alpha} * f)(n-1, u(n-1)),$$

therefore, it follows that

$$\begin{aligned}
\|u(n) - v(n)\| \leq &M\vartheta(n) + \sum_{j=0}^{n-1} \|S_{\alpha}(n-1-j)(f(j,u(j)) - f(j,v(j)))\| \\
\leq &M\vartheta(n) + \sum_{j=0}^{n-1} k^{\alpha}(n-1-j)L(j)\|u(j)-v(j)\| \\
\leq &M\vartheta(n) + \sum_{j=0}^{n-1} L(j)\|u(j)-v(j)\|.
\end{aligned}$$

On the other hand, let $b(n) = \|u(n) - v(n)\|$, from $0 \leq b(n) \leq a(n) + \sum_{j=0}^{n-1} L(j)b(j)$ with respect to a increasing sequence $a(n)$ for all $n \in \mathbb{N}_0$, we get

$$b(n) \leq a(n)\prod_{j=1}^{n-1}(1+L(j)), \quad n \in \{2,3,\cdots\} =: \mathbb{N}_2.$$

In fact, in view of $b(0) = 0$, we have for $n = 1$ that $b(1) \leq a(1)$; for $n = 2$, we get that $b(2) \leq a(2) + a(1)L(1) \leq a(2)(1+L(1))$. Assume that it is true for some $n = k \in \mathbb{N}_2$. Let $n = k+1$, then the induction implies

$$b(k+1) \leq a(k) + \sum_{j=0}^{k} L(j)b(j)$$

$$
\begin{aligned}
&\le a(k) + \sum_{j=2}^{k} L(j)a(j) \prod_{i=1}^{j-1}(1+L(i)) + L(1)b(1) \\
&\le a(k)\left[1 + \sum_{j=2}^{k} L(j) \prod_{i=1}^{j-1}(1+L(i))\right] + L(1)a(1) \\
&\le a(k)\left[1 + L(1) + L(2)\prod_{j=1}^{1}(1+L(j)) + \cdots + L(k)\prod_{j=1}^{k-1}(1+L(j))\right],
\end{aligned}
$$

which implies the desired inequality. Since $\sum_{j=1}^{\infty} L(j)$ is convergent absolutely, it follows that $\prod_{j=1}^{\infty}(1+L(j))$ is convergent absolutely and then there exists a constant $M_* > 0$ such that

$$\prod_{j=1}^{\infty}(1+L(j)) \le M_*.$$

Thus, let $a(n) = M\vartheta(n)$, there exists a constant $C := MM_* > 0$ such that

$$\|u(n) - v(n)\| \le C\vartheta(n), \quad n \in \mathbb{N}_0.$$

Therefore, we conclude the desired result. The proof is completed. □

Example 4.1. For any $0 < \lambda < 1$ and $0 < \alpha < 1$, let us consider the following fractional difference equation

$$\Delta^{\alpha} u(n) = -\lambda u(n+1) + \nu g(n) \sin(u(n)), \quad n \in \mathbb{N}_0, \tag{4.2}$$

where $g(n)$ is a bounded sequence on $l^{\infty}(\mathbb{N}_0)$ with $n^2|g(n)| \le 1$, parameter $\nu > 0$. Clearly, $\lambda > 0$ is the generator of the exponentially bounded C_0-semigroup $T(t) = e^{-\lambda t}$ for $t \ge 0$. Hence, (H1) holds. Let $f(n,u) = \nu g(n)\sin(u)$, it is easy to check the condition (H4) and if $\vartheta(n) = 2^n$ for all $n \in \mathbb{N}_0$ and the inequality

$$\|\Delta^{\alpha} u(n) - Au(n+1) - f(n, u(n))\| \le \vartheta(n), \quad n \in \mathbb{N}_0,$$

holds, then (4.2) is the Ulam-Hyers-Rassias stable by Theorem 4.1.

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