



RESEARCH PAPER

WELL-POSEDNESS FOR WEAK AND STRONG SOLUTIONS OF NON-HOMOGENEOUS INITIAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFUSION EQUATIONS

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Abstract

We study the well-posedness for initial boundary value problems associated with time fractional diffusion equations with non-homogenous boundary and initial values. We consider both weak and strong solutions for the problems. For weak solutions, we introduce a definition of solutions which allows to prove the existence of solution to the initial boundary value problems with non-zero initial and boundary values and non-homogeneous source terms lying in some negative-order Sobolev spaces. For strong solutions, we introduce an optimal compatibility condition and prove the existence of the solutions. We introduce also some sharp conditions guaranteeing the existence of solutions with more regularity in time and space.

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Key Words and Phrases: fractional diffusion equation; initial boundary value problem; well-posedness; weak and strong solutions

1. Introduction

1.1. **Statement of the problem.** Let Ω be a bounded and connected open subset of \mathbb{R}^d , $d \geq 2$, with \mathcal{C}^2 boundary $\partial \Omega$. Let $a := (a_{i,j})_{1 \leq i,j \leq d} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^{d^2})$ be symmetric, that is

$$a_{i,j}(x) = a_{j,i}(x), \ x \in \Omega, \ i, j = 1, \dots, d,$$

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and fulfill the ellipticity condition: there exists a constant c > 0 such that

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geqslant c|\xi|^2, \quad \text{for each } x \in \overline{\Omega}, \ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$
 (1.1)

Assume that $q \in L^{\infty}(\Omega)$ and

there exists a constant $q_0 > 0$ such that $q(x) \ge q_0$ for $x \in \Omega$, (1.2) and define the operator \mathcal{A} by

$$\mathcal{A}u(x) := -\sum_{i,j=1}^{d} \partial_{x_i} \left(a_{i,j}(x) \partial_{x_j} u(x) \right) + q(x)u(x), \ x \in \Omega.$$

Throughout the article, we set

$$Q := (0, T) \times \Omega, \quad \Sigma := (0, T) \times \partial \Omega.$$

Next, for $\alpha \in (0,1) \cup (1,2)$, $T \in (0,+\infty)$ and $\rho \in L^{\infty}(\Omega)$ obeying

$$0 < \rho_0 \leqslant \rho(x) \leqslant \rho_M < +\infty, \ x \in \Omega, \tag{1.3}$$

we consider the following initial boundary value problem (IBVP):

$$\begin{cases}
(\rho(x)\partial_t^{\alpha} + \mathcal{A})u(t, x) &= F(t, x), & (t, x) \in Q, \\
\tau_{\chi}u(t, x) &= f(t, x), & (t, x) \in \Sigma, \\
\partial_t^k u(0, x) &= u_k, & x \in \Omega, \ k = 0, \dots, \lceil \alpha \rceil - 1,
\end{cases} (1.4)$$

where $\chi = 0, 1, [\cdot]$ denotes the ceiling function:

$$\lceil \alpha \rceil = \begin{cases} 1 & \text{if } 0 < \alpha < 1, \\ 2 & \text{if } 1 < \alpha < 2, \end{cases}$$

and ∂_t^{α} denotes the fractional Caputo derivative of order α with respect to t, defined by

$$\partial_t^{\alpha} u(t,x) := \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^t (t-s)^{\lceil \alpha \rceil - 1 - \alpha} \partial_s^{\lceil \alpha \rceil} u(s,x) ds, \ (t,x) \in Q. \ (1.5)$$

Here the boundary operators τ_{χ} , $\chi = 0, 1$, are defined by:

- 1) $\tau_0 u := u$
- 2) $\tau_1 u := \partial_{\nu_A} u$, where ∂_{ν_A} stands for the normal derivative with respect to $a = (a_{i,j})_{1 \leq i,j \leq d}$, and is given by

$$\partial_{\nu_{\mathcal{A}}} h(x) := \sum_{i,j=1}^{d} a_{i,j}(x) \partial_{x_j} h(x) \nu_i(x), \ x \in \partial\Omega,$$

and $\nu = (\nu_1, \dots, \nu_d)$ is the outward unit normal vector to $\partial \Omega$.

REMARK 1. We can omit the condition (1.2) but for simplicity, we assume it. All the results of this article can be easily extended to Robin boundary conditions. Moreover, applying the fixed point argument, one can extend our results to a more general equation:

$$\partial_t^{\alpha} u + \mathcal{A}u + B(t, x) \cdot \nabla_x u + V(t, x)u = 0,$$

where some suitable assumptions are imposed on the coefficients $B \in L^{\infty}(Q)^d$, $V \in L^{\infty}(Q)$. However, in this article, in order to keep descriptions concise, we do not consider such extensions of our results.

In the present article, we study the well-posedness for problem (1.4) in a strong and a weak senses. In the weak sense we will prove the well-posedness of (1.4) when data f and u_0 , $u_{\lceil \alpha \rceil - 1}$ are lying in some negative-order Sobolev spaces. The strong solutions of (1.4) correspond to smooth solutions of this problem in time and space.

1.2. Motivations and a short bibliographical review. Recall that the initial boundary value problem (1.4) is often used for describing anomalous diffusion for several physical phenomenon such as diffusion of substances in heterogeneous media, diffusion of fluid flow in inhomogeneous anisotropic porous media, diffusion of carriers in amorphous photoconductors, diffusion in a turbulent flow (see e.g., [1], [21]). The case $\alpha \in (0,1)$ corresponds to a subdiffusive model, while the case $\alpha \in (1,2)$ corresponds to a super diffusive case.

The well-posedness for the problem (1.4) has been intensively studied these last decades. Many authors considered problem (1.4) for $\alpha \in (0,1)$ with f=0 or $\Omega=\mathbb{R}^d$. For $\Omega=\mathbb{R}^d$, one can refer to [2] where the existence of classical solutions of (1.4) is proved by mean of a representation formula involving the Green functions for (1.4). This formulation of the problem has been extended by [27], who proposed a variational formulation of (1.4) in some abstract framework allowing to consider elliptic operators \mathcal{A} depending on $t \in (0,T)$. We refer also to [18] as for (1.4) with a time dependent elliptic operators A, where the authors applied the approach of [27] to establish the existence of strong solutions under suitable assumptions. In [14, 24], the authors proved the existence of solutions to (1.4) in a bounded domain by mean of an eigenfunction representation involving the Mittag-Leffter functions. The definition of solutions of [14] can be formulated in terms of Laplace transform in time of the solutions. On the basis of this last definition, the works [13, 19] proved the existence of solutions of more complex equations than (1.4), including a fractional diffusion equation with distributed and variable order. Some similar approach has been considered by [22] for time and space fractional diffusion equations. Moreover, we can refer to the monograph [17].

All the above mentioned results considered (1.4) with the homogenous boundary conditions or $\Omega = \mathbb{R}^d$. Several authors considered also the wellposedness for problem (1.4) with non-homogeneous boundary conditions. These results can be classified into two categories: the existence of solutions of (1.4) in a weak sense and a strong sense, according to the cases where data belong to some negative-order Sobolev spaces and to smoother spaces respectively. We can refer to the series of works [8, 10] where the authors proved the existence of solutions to (1.4) for $u_0 = u_{\lceil \alpha \rceil - 1} = 0$ and f lying in some negative-order Sobolev space in time and space, when $\mathcal{A} = -\Delta$ and $\chi = 0$ (i.e., the Dirichlet boundary condition). The proof is based on a single layer representation of solutions and some results (e.g., [2, 25]) concerning the Green functions for fractional diffusion equations with constant coefficients. In [9], the author extended this approach to (1.4) for $\alpha \in (0,1)$ with non-homogeneous Robin boundary condition and non-vanishing initial condition lying in some Hölder spaces. The article [26] proved the unique existence of weak solution to (1.4) by the transposition (e.g., Lions and Magenes [20]) when $u_0 = 0, F = 0, \alpha \in (0,1)$ and $f \in L^2(0,T;H^{-\theta}(\partial\Omega))$ with $\theta > 0$. As for strong solutions, we refer to the work [6] where the author proved the existence of strong solution u to (1.4) such that u is continuous in time, belongs to a class C^2 in space and $\partial_t^{\alpha} u$ is Hölder continuous in time and space.

We remark that the well-posdness for problem (1.4) with non-homogenous boundary conditions is meaningful also for other mathematical problems such as optimal control problems (see e.g., [26]) or inverse problems (see e.g., [3, 7, 11, 12, 15]).

2. Statement of the main results

This section is devoted to the statement of our main results. For this purpose, we first introduce a definition of solutions of (1.4) for $f \in L^1(0,T;H^{-\theta}(\partial\Omega))$ with $\theta \in \left[\frac{1}{2},+\infty\right)$ and $(u_0,u_{\lceil\alpha\rceil-1},F)$ in some negative-order Sobolev spaces. To give a suitable definition of such solutions, we define the operators A_{χ} , $\chi=0,1$, in $L^2(\Omega;\rho dx)$ by

$$A_{\chi}u = \rho^{-1}\mathcal{A}u, \quad D(A_{\chi}) = \{g \in H^1(\Omega): \ \rho^{-1}\mathcal{A}g \in L^2(\Omega), \ \tau_{\chi}g_{|\partial\Omega} = 0\}.$$

In view of our assumptions, we have $D(A_{\chi}) = \{g \in H^2(\Omega) : \tau_{\chi} g_{|\partial\Omega} = 0\}$. Recall that the operators A_{χ} , $\chi = 0, 1$ are strictly positive self-adjoint operators with compact resolvent. Therefore, for $\chi = 0, 1$, the spectrum of A_{χ} consists of a non-decreasing sequence of strictly positive eigenvalues $(\lambda_{\chi,n})_{n\geqslant 1}$. Here and henceforth we number $\lambda_{\chi,n}$ with the multiplicities for $\chi = 0, 1$. In the Hilbert space $L^2(\Omega; \rho dx)$, we introduce an orthonormal basis of eigenfunctions $(\varphi_{\chi,n})_{n\geqslant 1}$ of A_{χ} associated with the eigenvalues

 $(\lambda_{\chi,n})_{n\geqslant 1}$. From now on, by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $L^2(\Omega; \rho dx)$ and we set $\mathbb{N} = \{1, 2, \ldots\}$. Let us observe that according to the condition imposed on ρ , we have $L^2(\Omega; \rho dx) = L^2(\Omega)$ with the equivalent norms. For all $s \geqslant 0$, we denote by A_{χ}^s the operator defined by

$$A_{\chi}^{s}g = \sum_{n=1}^{+\infty} \langle g, \varphi_{n} \rangle \lambda_{\chi,n}^{s} \varphi_{\chi,n},$$

$$g \in D(A_{\chi}^{s}) := \left\{ h \in L^{2}(\Omega) : \sum_{n=1}^{+\infty} |\langle g, \varphi_{\chi,n} \rangle|^{2} \lambda_{\chi,n}^{2s} < \infty \right\}$$

and in $D(A_{\gamma}^{s})$ we introduce the norm

$$||g||_{D(A_{\chi}^s)} = \left(\sum_{n=1}^{+\infty} |\langle g, \varphi_{\chi,n} \rangle|^2 \lambda_{\chi,n}^{2s}\right)^{\frac{1}{2}}, \quad g \in D(A_{\chi}^s).$$

We define $D(A_{\chi}^{-s}) = D(A_{\chi}^{s})'$ by the dual space to $D(A_{\chi}^{s})$, which is a Hilbert space with the norm

$$\|g\|_{D(A_{\chi}^{-s})} = \left(\sum_{n=1}^{\infty} \left| \langle g, \varphi_{\chi,n} \rangle_{-s} \right|^2 \lambda_{\chi,n}^{-2s} \right)^{\frac{1}{2}}.$$

Here $\langle \cdot, \cdot \rangle_{-s}$ denotes the duality bracket between $D(A_{\chi}^{-s})$ and $D(A_{\chi}^{s})$. By the duality, we see that $D(A_{\chi}^{-\frac{1}{2}})$ is embedded continuously into $H^{-1}(\Omega)$, because $H_0^1(\Omega)$ is embedded continuously into $D(A_{\chi}^{\frac{1}{2}})$.

For any $k \in \mathbb{N}$, we consider the following condition (H_k) :

$$\rho, \ a_{i,j} \in \mathcal{C}^{2(k-1)+1}(\overline{\Omega}), \ i,j=1,\ldots,d, \quad q \in W^{2(k-1),\infty}(\Omega), \quad \partial\Omega \text{ is of } \mathcal{C}^{2k}.$$

$$(H_k)$$

In view of [5, Theorem 2.5.1.1] (see also [4, Theorem 8.13]), for any $k \in \mathbb{N}$, condition (H_k) implies that the space $D(A_{\chi}^{\ell})$ is embedded continuously into $H^{2\ell}(\Omega)$ for any $\ell = 0, \ldots, k$ and $\chi = 0, 1$. Therefore, by the interpolation, we deduce that condition (H_k) implies that the space $D(A_{\chi}^s)$ is embedded continuously into $H^{2s}(\Omega)$ for any $s \in [0, k]$.

Let us fix $\theta \in \left[\frac{1}{2}, +\infty\right)$, $\kappa = \frac{\theta}{2} - \frac{1}{4}$ and $k = 1 + \lceil \kappa \rceil$. Using the above properties and assuming that (H_k) is fulfilled, for $\mu \geq 0$, $h \in H^{-\theta-\chi}(\partial\Omega)$ and $\Phi \in D(A_{\chi}^{-\kappa-1})$, we define the solution $y \in D(A_{\chi}^{-\kappa})$ to the following boundary value problem:

$$\begin{cases} \rho(x)^{-1} \mathcal{A} y(x) + \mu y(x) &= \Phi(x), \quad x \in \Omega, \\ \tau_{\chi} y(x) &= h(x), \quad x \in \partial \Omega, \end{cases}$$
 (2.1)

in the transposition sense corresponding to the unique element of $D(A_{\chi}^{-\kappa})$ satisfying

$$\langle y, G \rangle_{-\kappa} = -(-1)^{\chi} \left\langle h, \tau_{\chi}^* (A_{\chi} + \mu)^{-1} G \right\rangle_{H^{-\theta - \chi}(\partial\Omega), H^{\theta + \chi}(\partial\Omega)}$$
$$+ \left\langle \Phi, (A_{\chi} + \mu)^{-1} G \right\rangle_{-1 - \kappa}$$
 (2.2)

for all $G \in D(A_{\chi}^{\kappa})$. Here τ_{χ}^{*} denotes the formal adjoint operator to τ_{χ} , and we have $\tau_{0}^{*} = \tau_{1}$ and $\tau_{1}^{*} = \tau_{0}$. Then, we define the solution to (1.4) in the following way.

DEFINITION 2.1. Fix $\theta \in \left[\frac{1}{2}, +\infty\right)$, $\kappa = \frac{2\theta-1}{4}$, $k = 1 + \lceil \kappa \rceil$ and assume that condition (H_k) is fulfilled. Let $f \in L^1(0,T;H^{-\theta-\chi}(\partial\Omega))$, $F \in L^1(0,T;\rho D(A_\chi^{-\kappa-1}))$, $u_0,u_{\lceil \alpha \rceil-1} \in D(A_\chi^{-\kappa-1})$. We say that u is a weak solution to (1.4) if there exist $\varepsilon > 0$ and $v \in L^1_{loc}(0,+\infty;D(A_\chi^{-\varepsilon-\kappa}))$ satisfying $u = v_{|Q}$ and the following properties:

- (i) $\inf\{\lambda>0:\ t\mapsto e^{-\lambda t}v(t,\cdot)\in L^1(0,+\infty;D(A_\chi^{-\varepsilon-\kappa}))\}=0,$
- (ii) for all p > 0, the Laplace transform

$$\mathcal{L}v(p,\cdot) := \int_0^{+\infty} e^{-pt} v(t,\cdot) dt$$

of v is lying in $D(A_{\chi}^{-\kappa})$ and it solves the boundary value problem

$$\begin{cases}
\rho(x)^{-1}\mathcal{A}\mathcal{L}v(p,x) + p^{\alpha}\mathcal{L}v(p,x) &= G(p,x), & x \in \Omega, \\
\tau_{\chi}\mathcal{L}v(p,x)(x) &= \int_{0}^{T} e^{-pt} f(t,x)dt, & x \in \partial\Omega,
\end{cases} (2.3)$$

with
$$G(p,\cdot) = \int_0^T e^{-pt} \rho^{-1} F(t,\cdot) dt + \sum_{m=0}^{\lceil \alpha \rceil - 1} p^{\alpha - 1 - m} u_m$$
.

We give also the following definition of strong solutions to (1.4).

DEFINITION 2.2. We say that (1.4) admits a strong solution if there exists a weak solution u to (1.4) lying in $W^{\lceil \alpha \rceil, 1}(0, T; H^{-1}(\Omega)) \cap L^1(0, T; H^2(\Omega))$ such that

$$\rho(x)\partial_t^\alpha u + \mathcal{A}u = F \tag{2.4}$$

holds true in $L^1(0,T;L^2(\Omega))$ and

$$\begin{cases}
\tau_{\chi} u(t,x) &= f(t,x), & (t,x) \in \Sigma, \\
\partial_t^k u(0,x) &= u_k(x), & x \in \Omega, k = 0, \dots, \lceil \alpha \rceil - 1.
\end{cases}$$
(2.5)

REMARK 2. Let us observe that for any function $v \in W^{\lceil \alpha \rceil, 1}_{loc}(0, +\infty; H^{-1}(\Omega)) \cap L^1_{loc}(0, +\infty; H^2(\Omega))$ satisfying

$$e^{-pt}v(t,\cdot)\in W^{\lceil\alpha\rceil,1}(0,+\infty;H^{-1}(\Omega))\cap L^1(0,+\infty;H^2(\Omega)),\quad p>0,\quad (2.6)$$
 we have

$$\begin{split} &\mathcal{L}(\rho\partial_t^\alpha v + \mathcal{A}v)(p,\cdot) \\ &= \mathcal{A}\mathcal{L}v(p,\cdot) + \rho \left(p^\alpha \mathcal{L}v(p,\cdot) - \sum_{m=0}^{\lceil \alpha \rceil - 1} p^{\alpha - 1 - m} \partial_t^m v(0,\cdot)\right). \end{split}$$

Our choice for the definition of weak solutions of (1.4) is based on the above identity. Moreover, from this identity and the property of weak solutions stated in Remark 3 (see below), one can verify that any $u \in W^{\lceil \alpha \rceil, 1}(0, T; H^{-1}(\Omega)) \cap L^1(0, T; H^2(\Omega))$ which satisfies (2.4)-(2.5) and can be extended to a function $v \in W^{\lceil \alpha \rceil, 1}_{loc}(0, +\infty; H^{-1}(\Omega)) \cap L^1_{loc}(0, +\infty; H^2(\Omega))$ satisfying (2.6), is a weak solution in the sense of Definition 2.1.

2.1. Well-posedness for weak solutions. In this subsection, applying the definition of weak solutions given by Definition 2.1, we state our results of well-posedness of (1.4) when $f \in L^1(0,T;H^{-\theta-\chi}(\partial\Omega)),\ u_0,u_{\lceil\alpha\rceil-1} \in \rho D(A_\chi^{-\kappa-1})$ for some $\theta \in \left[\frac{1}{2},+\infty\right)$ and $\kappa = \frac{2\theta-1}{4}$. Our first main result can be stated as follows.

THEOREM 2.3. Let $\alpha \in (0,1) \cup (0,2)$, $\chi = 0,1$, $\theta \in \left[\frac{1}{2},+\infty\right)$, $\kappa = \frac{\theta}{2} - \frac{1}{4}$, $k = 1 + \lceil \kappa \rceil$ and let (H_k) be fulfilled. Let $r \in [1,+\infty)$, β be given by

$$\beta = \begin{cases} 1 & \text{if } r < \alpha^{-1} \\ \alpha^{-1} r^{-1} & \text{if } r \geqslant \alpha^{-1} \end{cases}$$

and let $f \in L^r(0,T;H^{-\theta-\chi}(\partial\Omega)), F \in L^r(0,T;\rho D(A_\chi^{-\frac{\theta}{2}-\frac{3}{2}}))$. Consider also

$$\begin{cases} u_0 \in D(A_{\chi}^{\frac{1}{4} - \beta - \frac{\theta}{2}}) & \text{for } \alpha \in (0, 1), \\ u_0 \in D(A_{\chi}^{-\frac{1}{\alpha r} + \frac{1}{4} - \frac{\theta}{2}}), & u_1 \in D(A_{\chi}^{-\alpha^{-1}(1 + r^{-1}) + \frac{1}{4} - \frac{\theta}{2}}) & \text{for } \alpha \in (1, 2). \end{cases}$$

Then problem (1.4) admits a unique solution u lying in

$$\bigcap_{\epsilon>0} L^r(0,T;D(A_\chi^{-\varepsilon+\frac{1}{4}-\frac{\theta}{2}})).$$

Moreover, for any $\varepsilon > 0$, we have estimates:

Case $\alpha \in (0,1)$:

$$||u||_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon+\frac{1}{4}-\frac{\theta}{2}}))} \leq C_{\varepsilon} \Big(||f||_{L^{r}(0,T;H^{-\theta-\chi}(\partial\Omega))} + ||\rho^{-1}F||_{L^{r}(0,T;D(A_{\chi}^{-\frac{\theta}{2}-\frac{3}{4}}))} + ||u_{0}||_{D(A_{\chi}^{-\beta+\frac{1}{4}-\frac{\theta}{2}})} \Big).$$

$$(2.7)$$

Case $\alpha \in (1,2)$:

$$\|u\|_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon+\frac{1}{4}-\frac{\theta}{2}}))} \leq C_{\varepsilon} \left(\|f\|_{L^{r}(0,T;H^{-\theta-\chi}(\partial\Omega))} + \|\rho^{-1}F\|_{L^{r}(0,T;D(A_{\chi}^{-1-\kappa}))} \right) + C_{\varepsilon} \left(\|u_{0}\|_{D(A_{\chi}^{-\frac{1}{\alpha r}-\kappa})} + \|u_{1}\|_{D(A_{\chi}^{-\frac{(1+r-1)}{\alpha}-\kappa})} \right).$$
(2.8)

In both cases, the constant C_{ε} depends on ε , r, ρ , α , θ , A, Ω and T.

REMARK 3. Note that even if Definition 2.1 of weak solutions depends on the final time T, the solution that we obtain in Theorem 2.3 is independent of T. Namely, fix $T_1, T_2 \in (0 + \infty)$ with $T_1 < T_2$ and consider $f \in L^r(0, T_2; H^{-\theta-\chi}(\partial\Omega)), \ \rho^{-1}F \in L^r(0, T_2; D(A_\chi^{-\frac{3}{4}-\frac{\theta}{2}}))$. Consider the unique weak solutions u_ℓ , $\ell=1,2$ to (1.4) for $T=T_\ell$, which are given by Theorem 2.3. According to the expression of the weak solution given in the proof of Theorem 2.3 in terms of Fourier series, we can verify that the restriction of u_2 to $(0,T_1)\times\Omega$ coincides with u_1 .

In a special case of r=2 and zero initial values, we can improve Theorem 2.3, as follows.

THEOREM 2.4. Let the condition of Theorem 2.3 be fulfilled with r=2, $\rho=1$ and $u_0=u_{\lceil\alpha\rceil-1}=0$, and $\theta\geq\frac{1}{2}$. Then the unique weak solution u of (1.4) is lying in $L^2(0,T;D(A_\chi^{\frac{1}{4}-\frac{\theta}{2}}))$ with $\partial_t^\alpha u\in L^2(0,T;D(A_\chi^{-\frac{3}{4}-\frac{\theta}{2}}))$. Moreover, we have

$$||u||_{L^{2}(0,T;D(A_{\chi}^{\frac{1}{4}-\frac{\theta}{2}}))} + ||\partial_{t}^{\alpha}u||_{L^{2}(0,T;D(A_{\chi}^{-\frac{3}{4}-\frac{\theta}{2}}))} \leq C \left(||f||_{L^{2}(0,T;H^{-\theta}(\partial\Omega))} + ||F||_{L^{2}(0,T;D(A_{\chi}^{-\frac{3}{4}-\frac{\theta}{2}}))}\right).$$
(2.9)

Under the assumption $F \equiv 0$, Theorem 4.1 in [26] established the same conclusion in the case of $0 < \alpha < 1$ and arbitrary $\theta > 0$. On the other hand, this theorem holds for $\alpha \in (0,1) \cup (1,2)$ and non-zero F but requires that $\theta \geq \frac{1}{2}$.

2.2. Well-posedness for strong solutions. In this subsection, we state our results related to the well-posedness of strong solutions of (1.4). We treat separately the cases $\alpha \in (0,1)$ and $\alpha \in (1,2)$. Let us start with $\alpha \in (0,1)$. For this purpose, we introduce a compatibility condition on data $f \in W^{1,1}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega)), F \in W^{1,1}(0,T;\rho D(A_{\chi}^{-1})),$ and $u_0 \in (0,1)$

 $L^2(\Omega)$ which requires that u_0 solves the boundary value problem in the transposition sense:

$$\begin{cases}
Au_0(x) = F(0,x), & x \in \Omega, \\
\tau_{\chi}u_0(x) = f(0,x), & x \in \partial\Omega, \quad \chi = 0, 1.
\end{cases}$$
(2.10)

We can now state our result for $\alpha \in (0,1)$.

THEOREM 2.5. Let $\alpha \in (0,1)$, $\chi = 0,1$, $\rho \in \mathcal{C}^2(\overline{\Omega})$ and $m \in \mathbb{N}$ be fixed. Assume also that condition (H_m) is fulfilled. Let $u_0 \in L^2(\Omega)$, $r \in [1, +\infty)$ and

$$f \in \bigcap_{k=0}^{m} W^{m-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega)),$$

$$F \in \bigcap_{k=1}^{m} W^{m-k,r}(0,T;H^{2(k-1)}(\Omega)) \cap W^{m,r}(0,T;\rho D(A_{\chi}^{-1})).$$

For $m \ge 2$, we assume that

$$\partial_t^k f(0,\cdot) = 0, \quad \partial_t^k F(0,\cdot) = 0, \quad k = 1,\dots, m-1.$$
 (2.11)

If u_0 satisfies the compatibility condition (2.10), then the unique weak solution u to (1.4) is a strong solution lying in

$$\bigcap_{k=1}^{m} W^{m-k,r}(0,T;H^{2k}(\Omega)) \cap W^{m,r}(0,T;H^{-\varepsilon}(\Omega)),$$

where $\varepsilon > 0$ is arbitrary. Moreover, for any $\varepsilon > 0$, we have

$$\|u\|_{W^{m,r}(0,T;H^{-\varepsilon}(\Omega))} + \sum_{k=1}^{m} \|u\|_{W^{m-k,r}(0,T;H^{2k}(\Omega))}$$

$$\leqslant C_{\varepsilon} \sum_{k=1}^{m} \left(\|f\|_{W^{m-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega))} + \|F\|_{W^{m-k,r}(0,T;H^{2(k-1)}(\Omega))} \right)$$

$$+ C_{\varepsilon} \left(\|f\|_{W^{m,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + \|\rho^{-1}F\|_{W^{m,r}(0,T;D(A_{\chi}^{-1}))} \right),$$
where the constant C_{ε} depends on ε , r , ρ , α , A , Ω and T .

Remark 4. We recall that the compatibility condition (2.10) yields an a priori estimate of u_0 :

$$||u_0||_{H^k(\Omega)} \le C \left(||f(0,\cdot)||_{H^{k-\frac{1}{2}-\chi}(\partial\Omega)} + ||\rho^{-1}F(0,\cdot)||_{D(A_{\chi}^{\frac{k-2}{2}})} \right), \quad k = 0, 2.$$

Therefore the right-hand side of (2.12) needs not be involved directly with the norm of u_0 , which is the same as for (2.14) and (2.17) which are stated below.

For $\alpha \in (0,1)$, the compatibility condition (2.10) corresponds to an optimal condition guaranteeing the existence of smooth solutions as described in Theorem 2.5. Similarly the additional condition (2.11) is an optimal condition guaranteeing the existence of solutions with higher regularity. Indeed we can prove the following

PROPOSITION 2.6. For $f \in \mathcal{C}^{\infty}(\overline{\Sigma})$, $F \in \mathcal{C}^{\infty}(\overline{Q})$ and $u_0 \in \mathcal{C}^{\infty}(\overline{\Omega})$, if the compatibility condition (2.10) is not fulfilled, then the solution u of (1.4) does not belong to $W^{1,(1-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Moreover, if there exists $m \geq 2$ such that (2.11) fails, then $u \notin W^{m,(1-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$.

For $\alpha \in (1,2)$, we recall first two results whose proof can be find in [16]. We consider two different situations. In the first case, we state a result with a weaker regularity assumption and the compatibility condition (2.10). In the second result, we consider smoother data under more compatibility conditions.

The first result that we will recall, will be stated only with the compatibility condition (2.10).

THEOREM 2.7. (Theorem 1.7, [16]) Let $\alpha \in (1,2)$, $\chi = 0,1$ and let $\delta \in (0,1/4)$, $r \in [1,+\infty)$ satisfy

$$r < \frac{1}{1 - \alpha \delta}.\tag{2.13}$$

Assume $u_0 \in H^1(\Omega)$, $u_1 \in H^{2\delta}(\Omega)$ and

$$f \in W^{2,r}(0,T;H^{\frac{1}{2}-\chi}(\partial\Omega)) \cap L^r(0,T;H^{\frac{3}{2}-\chi}(\partial\Omega)),$$

$$F \in W^{2,r}(0,T;D(A_{\chi}^{-\frac{1}{2}})) \cap L^r(0,T;L^2(\Omega)).$$

If u_0 satisfies (2.10), then the unique weak solution u to (1.4) is a strong solution lying in

$$L^r(0,T;H^2(\Omega))\cap W^{1,r}(0,T;H^1(\Omega))\cap W^{2,r}(0,T;L^2(\Omega)).$$

Moreover, we have

$$||u||_{W^{2,r}(0,T;L^{2}(\Omega))} + ||u||_{W^{1,r}(0,T;H^{1}(\Omega))} + ||u||_{L^{r}(0,T;H^{2}(\Omega))}$$

$$\leq C \left(||f||_{W^{2,r}(0,T;H^{\frac{1}{2}-\chi}(\partial\Omega))} + ||f||_{L^{r}(0,T;H^{\frac{3}{2}-\chi}(\partial\Omega))} + ||u_{1}||_{H^{2\delta}(\Omega)} \right)$$

$$+ C \left(||F||_{W^{2,r}(0,T;D(A_{\chi}^{-\frac{1}{2}}))} + ||f||_{L^{r}(0,T;L^{2}(\Omega))} \right),$$
(2.14)

where the constant C > 0 depends on $r, \rho, \alpha, A, \Omega$ and T.

Next we discuss a solutions with higher regularity requiring the two compatibility conditions (2.10) and (2.15): we assume that u_1 satisfies

$$\begin{cases}
Au_1(x) = \partial_t F(0, x), & x \in \Omega, \\
\tau_{\chi} u_1(x) = \partial_t f(0, x), & x \in \partial \Omega.
\end{cases}$$
(2.15)

THEOREM 2.8. (Theorem 1.8, [16]) Let $\alpha \in (1,2)$, $\chi = 0, 1$ and $m_1 \in \mathbb{N}$ be fixed. Assume also that condition (H_{m_1+1}) is fulfilled and fix $3 \leq m \leq 2m_1 + 2$. Let $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$, $r \in [1, +\infty)$ and let

$$f \in \bigcap_{k=2}^{m} W^{m-k,r}(0,T;H^{k-\frac{1}{2}-\chi}(\partial\Omega)) \cap W^{m,r}(0,T;H^{\frac{1}{2}-\chi}(\partial\Omega)),$$

$$F \in \bigcap_{k=2}^{m} W^{m-k,r}(0,T;H^{k-2}(\Omega)) \cap W^{m,r}(0,T;D(A_{\chi}^{-\frac{1}{2}})).$$

For $m \ge 4$, we assume also that

$$\partial_t^k f(0,\cdot) = 0, \quad \partial_t^k F(0,\cdot) = 0, \quad k = 3,\dots, m-1.$$
 (2.16)

If u_0 and u_1 satisfy the compatibility conditions (2.10) and (2.15), then the unique weak solution u of (1.4) is a strong solution lying in

$$\bigcap_{k=0}^{m} W^{m-k,r}(0,T;H^k(\Omega)).$$

Moreover, we have

$$\sum_{k=0}^{m} \|u\|_{W^{m-k,r}(0,T;H^{k}(\Omega))}$$

$$\leq C \left(\sum_{k=2}^{m} \|f\|_{W^{m-k,r}(0,T;H^{k-\frac{1}{2}-\chi}(\partial\Omega))} + \|F\|_{W^{m-k,r}(0,T;H^{k-2}(\Omega))} \right) (2.17)$$

$$+ C \left(\|f\|_{W^{m,r}(0,T;H^{\frac{1}{2}-\chi}(\partial\Omega))} + \|F\|_{W^{m,r}(0,T;D(A_{\chi}^{-\frac{1}{2}}))} \right),$$

where the constant C > 0 depends on ρ , r, α , \mathcal{A} , Ω and T.

For $\alpha \in (1,2)$, the compatibility condition (2.10) corresponds to an optimal condition guaranteeing the existence of solutions with the smoothness of Theorem 2.7, while both conditions (2.10) and (2.15) are required for the existence of smooth solution as stated in Theorem 2.8. In the same way, the additional condition (2.16) is an optimal condition guaranteeing existence of solutions with higher regularity. Indeed we can prove

PROPOSITION 2.9. For $f \in \mathcal{C}^{\infty}(\overline{\Sigma})$ and $u_0, u_1 \in \mathcal{C}^{\infty}(\overline{\Omega})$, if compatibility condition (2.10) is not fulfilled, then the solution u to (1.4) does not belong to $W^{2,(2-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Moreover if (2.10) is fulfilled but not (2.15), then $u \notin W^{3,(2-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Finally, if there exists $m \geqslant 4$ such that (2.16) is not fulfilled, then $u \notin W^{m,(2-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$.

2.3. Comments about our results. To the best of our knowledge, Theorem 2.3 is the first result of well-posedness of (1.4) with such weak assumptions imposed on the boundary value f, the initial values $u_0, u_{\lceil \alpha \rceil - 1}$ and the source term F, as long as the elliptic part \mathcal{A} is t-independent and symmetric. Indeed, all other comparable results have been stated with less general equations and smoother data (see for instance [8, 9, 10]) or with zero initial value (see [26]). In addition, we state these well-posedness for solutions lying in L^r in time with $r \in [1, +\infty]$, while other comparable results are restricted to solutions lying in L^2 in time.

Theorems 2.5, 2.7 and 2.8 are concerned with the well-posedness in the strong sense of (1.4). Our aim is to obtain an optimal condition guaranteeing the existence of smooth solutions to (1.4) in time and space. It is known that there exist even smooth data satisfying a usual compatibility conditions, but that the regularity in time of the solution to (1.4) can not exceed $W^{1,(\lceil\alpha\rceil-\alpha)^{-1}}$ (see Propositions 2.6 and 2.9 for more details). Moreover, most of results concerning the existence of smooth solutions to (1.4)have mainly established some Hölder continuity in time of the solutions (e.g. [6]) or the regularity which is weaker than H^{α} in time (e.g. [18]). In Theorems 2.5 and 2.8, we prove the existence of solutions lying $W^{m,r}$ in time with arbitrary $m \in \mathbb{N}$ and $r \in [1, +\infty]$ by assuming some compatibility conditions (2.10) and (2.11) (respectively (2.10), (2.15), and (2.16)) for $\alpha \in (0, 1)$ (respectively $\alpha \in (1, 2)$).

We have obtained Theorem 2.3 by means of the representation of the solutions which explicitly involves the boundary value f. In the same way, we derive and prove an optimality of the compatibility conditions (2.10), (2.11), (2.15) and (2.16) by using the Fourier series representation of the solution to (1.4). The representation of solutions to (1.4) which we used for the proof of Theorem 2.3, is the key ingredient for the well-posedness of (1.4).

Indeed, in order to reach the regularity stated in Theorems 2.5, 2.7 and 2.8, we do not know whether we can use a classical lifting arguments with results on the existence of the solutions with homogeneous boundary conditions like [14, 18, 24].

2.4. **Outline.** This paper is organized as follows. Section 3 is devoted to the proof of Theorems 2.3 and 2.4. In Section 4 we prove Theorem 2.5. Section 5 is devoted to the proofs of Propositions 2.6 and 2.9 emphasizing the optimality of the compatibility conditions (2.10), (2.11), (2.15) and (2.16).

3. Proof of Theorems 2.3 and 2.4

We start with a lemma. Then we prove Theorem 2.3 for $\alpha \in (0,1)$, and then for the case $\alpha \in (1,2)$. Finally, we prove Theorem 2.4.

3.1. **Preliminary lemma.** Henceforth C>0 denotes generic constants which depend on \mathcal{A} , ρ , θ and r, α , T and Ω , and C may change from line to line.

We prove the following lemma.

Lemma 3.1. Let $\chi=0,1,\,\theta\in\left[\frac{1}{2},+\infty\right),\,\kappa=\frac{2\theta-1}{4},\,k=1+\lceil\kappa\rceil$ and let condition (H_k) be fulfilled. Then, for any $h\in H^{-\theta-\chi}(\partial\Omega)$, we have

$$\sum_{n=1}^{\infty} \left| \lambda_{\chi,n}^{-1-\kappa} \left\langle h, \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} \right|^2 \leqslant C^2 \left\| h \right\|_{H^{-\theta-\chi}(\partial\Omega)}^2. \tag{3.1}$$

Proof. Let $y\in D(A_\chi^{-\kappa})$ be the solution in the transposition sense to (2.1) with $\mu=0$ and $\Phi=0$. Since $A_\chi\varphi_{\chi,n}=\lambda_{\chi,n}\varphi_{\chi,n}$, we have

$$\langle y, \lambda_{\chi,n} \varphi_{\chi,n} \rangle_{-\kappa} = -(-1)^{\chi} \langle h, \tau_{\chi}^* \varphi_{\chi,n} \rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)}, \quad n \in \mathbb{N}.$$

Therefore

$$\langle y, \varphi_{\chi, n} \rangle_{-\kappa} = -(-1)^{\chi} \lambda_{\chi, n}^{-1} \langle h, \tau_{\chi}^* \varphi_{\chi, n} \rangle_{H^{-\theta - \chi}(\partial \Omega), H^{\theta + \chi}(\partial \Omega)}$$

and

$$\sum_{n=1}^{\infty} \left| \lambda_{\chi,n}^{-1-\kappa} \left\langle h, \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} \right|^2 = \|y\|_{D(A_{\chi}^{-\kappa})}^2. \tag{3.2}$$

Furthermore, for any $G \in D(A_{\chi}^{\kappa})$, we have

$$\begin{split} \left| \left\langle y, G \right\rangle_{-\kappa} \right| &= \left| \left\langle h, \tau_{\chi}^* A_{\chi}^{-1} G \right\rangle_{H^{-\theta - \chi}(\partial\Omega), H^{\theta + \chi}(\partial\Omega)} \right| \\ &\leqslant \left\| h \right\|_{H^{-\theta - \chi}(\partial\Omega)} \left\| \tau_{\chi}^* A_{\chi}^{-1} G \right\|_{H^{\theta + \chi}(\partial\Omega)} \\ &\leqslant C \left\| h \right\|_{H^{-\theta - \chi}(\partial\Omega)} \left\| A_{\chi}^{-1} G \right\|_{H^{2(1+\kappa)}(\Omega)} \\ &\leqslant C \left\| h \right\|_{H^{-\theta - \chi}(\partial\Omega)} \left\| A_{\chi}^{-1} G \right\|_{D(A_{\chi}^{1+\kappa})} \\ &\leqslant C \left\| h \right\|_{H^{-\theta - \chi}(\partial\Omega)} \left\| G \right\|_{D(A_{\chi}^{\kappa})}. \end{split}$$

Therefore, we obtain

$$||y||_{D(A_{\chi}^{-\kappa})} \leqslant C ||h||_{H^{-\theta-\chi}(\partial\Omega)}$$

and combining this with (3.2), we deduce (3.1).

By this lemma, we are now in position to complete the proof of Theorem 2.3.

3.2. **Proof of Theorem 2.3 for** $\alpha \in (0,1)$. For all $n \in \mathbb{N}$, we set

$$u_{1,n}(t) := -(-1)^{\chi} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha})$$

$$\times \mathbb{I}_{(0,T)}(s) \left\langle f(s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega),H^{\theta+\chi}(\partial\Omega)} ds,$$

$$u_{2,n}(t) := E_{\alpha,1}(-\lambda_{\chi,n}t^{\alpha}) \left\langle u_0, \varphi_{\chi,n} \right\rangle_{-1-\kappa},$$

$$u_{3,n}(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \mathbb{I}_{(0,T)}(s) \left\langle \rho^{-1} F(s,\cdot), \varphi_{\chi,n} \right\rangle_{-\kappa-1} ds,$$

and

$$u_n(t) := u_{1,n}(t) + u_{2,n}(t) + u_{3,n}(t),$$

where $\mathbb{I}_{(0,T)}$ denotes the characteristic function of (0,T). Here, for $\beta_1, \beta_2 > 0$, E_{β_1,β_2} denotes the Mittag-Leffler function given by

$$E_{\beta_1,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 k + \beta_2)}, \quad z \in \mathbb{C}.$$

We will divide the proof of Theorem 2.3 into three steps: In the first step, we will prove that for any $\varepsilon > 0$, the sequence

$$\sum_{n=1}^{N} u_n(t)\varphi_{\chi,n}, \quad N \in \mathbb{N},$$

converges to u in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$ as $N\to\infty$. In the second step, we prove that the same sequence converges in $L^r_{loc}(0,+\infty;D(A_\chi^{-\varepsilon-\kappa}))$ to a function v satisfying the conditions (i) and (ii) of Definition 2.1. In the third step, we complete the proof of the theorem.

In addition to the generic constant C > 0, for all $\varepsilon > 0$ by $C_{\varepsilon} > 0$ we denote generic constants depending also on ε .

Step 1. Fix $\varepsilon > 0$. Let us show that the sequence

$$\sum_{n=1}^{N} u_n(t)\varphi_{\chi,n}, \quad N \in \mathbb{N}$$

converges in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$. For this purpose, it suffices to prove that the sequences

$$\sum_{n=1}^{N} u_{i,n}(t)\varphi_{\chi,n}, \quad i = 1, 2, 3, \ N \in \mathbb{N},$$

converge in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$. First we prove this result for the case i=1. For m< n and almost all $t\in (0,T)$, we have

$$\left\| \sum_{\ell=m}^{n} u_{1,\ell}(t) \varphi_{\chi,\ell} \right\|_{D(A_{\chi}^{-\varepsilon-\kappa})} = \left\| \sum_{\ell=m}^{n} \int_{0}^{t} H_{\ell}(t,s) ds \varphi_{\chi,\ell} \right\|_{D(A_{\chi}^{-\varepsilon-\kappa})}$$

$$\leq \int_{0}^{t} \left\| \sum_{\ell=m}^{n} H_{\ell}(t,s) \varphi_{\chi,\ell} \right\|_{D(A_{\chi}^{-\varepsilon-\kappa})} ds,$$

where, for a.e. 0 < s < t < T, we consider

$$H_{\ell}(t,s) := (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,\ell}(t-s)^{\alpha}) \left\langle f(s,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi}(\partial\Omega),H^{\theta+\chi}(\partial\Omega)}.$$

In view of formula (1.148) of [23, Theorem 1.6], for almost all $s \in (0, t)$, we find

$$\left\| \sum_{\ell=m}^{n} H_{\ell}(t,s) \varphi_{\chi,\ell} \right\|_{D(A_{\chi}^{-\varepsilon-\kappa})}^{2}$$

$$\leq \sum_{\ell=m}^{n} \left| \lambda_{\chi,\ell}^{-\varepsilon-\kappa} (t-s)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_{\chi,\ell} (t-s)^{\alpha}) \left\langle f(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi},H^{\theta+\chi}} \right|^{2}$$

$$\leq C(t-s)^{2(\varepsilon\alpha-1)} \sum_{\ell=m}^{n} \left| \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi},H^{\theta+\chi}} \right|^{2}.$$

Hence

$$\left\| \sum_{\ell=m}^{n} u_{1,\ell}(t) \varphi_{\chi,\ell} \right\|_{D(A_{\chi}^{-\varepsilon-\kappa})} \le C \int_{0}^{t} (t-s)^{\varepsilon\alpha-1} \left(\sum_{\ell=m}^{n} \left| \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi},H^{\theta+\chi}} \right|^{2} \right)^{\frac{1}{2}} ds.$$

Applying the Young inequality for convolution, we obtain

$$\left\| \sum_{\ell=m}^{n} u_{1,\ell}(t) \varphi_{\chi,\ell} \right\|_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon-\kappa}))} \\ \leq C_{\varepsilon} \left\| \sum_{\ell=m}^{n} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle \varphi_{\chi,\ell} \right\|_{L^{r}(0,T;L^{2}(\Omega))}$$

Using Lemma 3.1, for almost all $t \in (0,T)$ and all $N \in \mathbb{N}$, we obtain

$$\begin{split} & \left\| \sum_{\ell=1}^{N} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle \varphi_{\chi,\ell} \right\|_{L^{2}(\Omega)} \\ & \leq \left(\sum_{\ell=1}^{+\infty} \left| \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} \right|^{2} \right)^{\frac{1}{2}} \\ & \leq C \left\| f(t,\cdot) \right\|_{H^{-\theta-\chi}(\partial\Omega)} \end{split}$$

and the limit

$$\lim_{N \to \infty} \sum_{\ell=1}^{N} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} \varphi_{\chi,\ell},$$

exists in $L^2(\Omega)$ for almost all $t \in (0,T)$. In the same way, by $f \in L^r(0,T; H^{-\theta-\chi}(\partial\Omega))$, applying the Lebesgue dominate convergence theorem for functions taking values in $L^2(\Omega)$, we deduce that the limit

$$\lim_{N \to \infty} \sum_{\ell=1}^{N} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} \varphi_{\chi,\ell},$$

exists in $L^r(0,T;L^2(\Omega))$. In particular, it is a Cauchy sequence and so we have

$$\lim_{m,n\to\infty} \left\| \sum_{\ell=1}^{n} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle \varphi_{\chi,\ell} \right.$$
$$- \sum_{\ell=1}^{m} \lambda_{\chi,\ell}^{-1-\kappa} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle \varphi_{\chi,\ell} \right\|_{L^{r}(0,T;L^{2}(\Omega))} = 0,$$

which implies that

$$\lim_{m,n\to\infty} \left\| \sum_{\ell=m}^n u_{1,\ell}(t) \varphi_{\chi,\ell} \right\|_{L^r(0,T;D(A_{\gamma}^{-\varepsilon-\kappa}))} = 0.$$

Thus the sequence

$$\sum_{n=1}^{N} u_{1,n}(t)\varphi_{\chi,n}, \quad N \in \mathbb{N}$$

is a Cauchy sequence in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$, which yields the convergence in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$. Similarly, we can show that the sequence

$$\sum_{n=1}^{N} u_{3,n}(t)\varphi_{\chi,n}, \quad N \in \mathbb{N},$$

converges in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$. Moreover formula (1.148) of [23, Theorem 1.6] yields

$$\sum_{n=1}^{\infty} \left| \lambda_{\chi,n}^{-\varepsilon - \kappa} u_{2,n}(t) \right|^{2} \leqslant C t^{2(\varepsilon - \beta)\alpha} \sum_{n=1}^{\infty} \left| \lambda_{\chi,n}^{-\beta - \kappa} \left\langle u_{0}, \varphi_{\chi,n} \right\rangle \right|^{2}$$

$$\leqslant C t^{2(\varepsilon - \beta)\alpha} \left\| u_{0} \right\|_{D(A_{\gamma}^{-\kappa - \beta})}^{2}$$

and, combining this with the fact that $r\beta\alpha=1$, we can conclude that the limit

$$\lim_{N \to \infty} \sum_{n=1}^{N} u_{2,n}(t) \varphi_{\chi,n},$$

exists in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$. These results prove that the sequence

$$\sum_{n=1}^{N} u_n(t)\varphi_{\chi,n}, \quad N \in \mathbb{N},$$

converges in $L^r(0,T;D(A_{\chi}^{-\varepsilon-\kappa}))$.

Step 2. Fix $\varepsilon > 0$. Using the arguments of Step 1, we can define $v \in L^r_{loc}(0, +\infty; D(A^{-\varepsilon-\kappa}_{\chi}))$ by

$$v(t,\cdot) := \sum_{n=1}^{\infty} u_n(t)\varphi_{\chi,n}.$$

In this step, we will show that v fulfills conditions (i) and (ii) of Definition 2.1. Fixing p > 0, repeating the arguments of Step 1, using (3.1) and applying Young's inequality, we deduce that, for almost all t > 0, we have

$$\left\| e^{-pt} \sum_{n=1}^{\infty} u_{1,n}(t) \varphi_{\chi,n} \right\|_{L^{1}(0,+\infty;D(A_{\chi}^{-\varepsilon-\kappa}))}$$

$$\leq C \left\| \left(e^{-pt} t^{\varepsilon\alpha-1} \mathbb{I}_{(0,+\infty)}(t) \right) * \left(\mathbb{I}_{(0,T)}(t) \left\| f(t,\cdot) \right\|_{H^{-\theta-\chi}(\partial\Omega)} \right) \right\|_{L^{1}(0,+\infty)}$$

$$\leq C \left(\int_{0}^{+\infty} e^{-pt} t^{\varepsilon\alpha-1} dt \right) \|f\|_{L^{r}(0,T;H^{-\theta-\chi}(\partial\Omega))} < \infty.$$

Therefore, we have

$$e^{-pt}\sum_{n=1}^{\infty}u_{1,n}(t)\varphi_{\chi,n}\in L^1(0,+\infty;D(A_{\chi}^{-\varepsilon-\kappa})),\quad p>0.$$

In the same way, one can verify that

$$e^{-pt} \sum_{n=1}^{\infty} u_{i,n}(t) \varphi_{\chi,n} \in L^1(0, +\infty; D(A_{\chi}^{-\varepsilon - \kappa})), \quad p > 0, \ i = 2, 3$$

and conclude that condition (i) of Definition 2.1 is fulfilled.

Now let us consider condition (ii) of Definition 2.1. Applying the properties of Mittag-Leffler functions (see for instance [23]), we see that the Laplace transform $\mathcal{L}v(p,\cdot)$ of v satisfies

$$\mathcal{L}v(p,\cdot) = \sum_{n=1}^{\infty} \mathcal{L}u_n(p)\varphi_{\chi,n} \in D(A_{\chi}^{-\varepsilon-\kappa}),$$

where

$$\mathcal{L}u_n(p) = \int_0^{+\infty} e^{-pt} u_n(t) dt$$

$$= \frac{\left\langle p^{\alpha - 1} u_0 + \int_0^T e^{-pt} \rho^{-1} F(t, \cdot) dt, \varphi_{\chi, n} \right\rangle_{-\kappa - 1}}{p^{\alpha} + \lambda_{\chi, n}}$$

$$- \frac{(-1)^{\chi} \left\langle \int_0^T e^{-pt} f(t, \cdot) dt, \tau_{\chi}^* \varphi_{\chi, n} \right\rangle_{H^{-\theta - \chi}, H^{\theta + \chi}}}{p^{\alpha} + \lambda_{\chi, n}}$$

Now let $V(p,x) := (\mathcal{L}v)(p,x)$ be the solution in the transposition sense of the boundary value problem

$$\begin{cases} \rho(x)^{-1} \mathcal{A} V(p,\cdot) + p^{\alpha} V(p,\cdot) &= \hat{F}(p,\cdot) + p^{\alpha-1} u_0, & \text{in } \Omega, \\ \tau_{\chi} V(p,x)(x) &= \int_0^T e^{-pt} f(t,x) dt, & x \in \partial \Omega, \end{cases}$$

where

$$\hat{F}(p,\cdot) = \int_0^T e^{-pt} \rho^{-1} F(t,\cdot) dt.$$

By the definition of the transposition, setting $G = \varphi_{\chi,n}$ in (2.2) and applying $(\mathcal{A} + p^{\alpha})\varphi_{\chi,n} = (\lambda_{\chi,n} + p^{\alpha})\varphi_{\chi,n}$, we obtain

$$\begin{split} &(\lambda_{\chi,n}+p^{\alpha}) \left\langle V(p,\cdot),\varphi_{\chi,n}\right\rangle_{-\kappa} \\ &= -(-1)^{\chi} \left\langle \int_{0}^{T} e^{-pt} f(t,\cdot) dt, \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega),H^{\theta+\chi}(\partial\Omega)} \\ &+ \left\langle p^{\alpha-1} u_{0} + \int_{0}^{T} e^{-pt} \rho^{-1} F(t,\cdot) dt, \varphi_{\chi,n} \right\rangle_{-\kappa-1} \\ &= (\lambda_{\chi,n}+p^{\alpha}) \mathcal{L} u_{n}(p). \end{split}$$

Therefore $(\mathcal{L}v)(p,x) = V(p,x)$ for p > 0 and $x \in \Omega$, and so we can verify condition (ii) of Definition 2.1.

Step 3. In this step we complete the proof of Theorem 2.3. The above argument shows that u is a weak solution to (1.4) in the sense of Definition 2.1. Moreover, the uniqueness of this solution is guaranteed by the uniqueness of the Laplace transform of a function and the uniqueness of the solutions of the boundary value problem (2.3). Therefore, the proof

of the theorem will be completed if we show that for all $\varepsilon > 0$, estimate (2.7) holds true. For this purpose, let us first consider

$$\left\| \sum_{n=1}^{\infty} u_{1,n} \varphi_{\chi,n} \right\|_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon-\kappa}))}.$$

Repeating the arguments of Step 1 and Lemma 3.1, we deduce that

$$\left\| \sum_{n=1}^{\infty} u_{1,n} \varphi_{\chi,n} \right\|_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon-\kappa}))} \le C \left\| \int_{0}^{t} (t-s)^{\varepsilon\alpha-1} \|f(s,\cdot)\|_{H^{-\theta-\chi}(\partial\Omega)} ds \right\|_{L^{r}(0,T)}.$$

Therefore Young's inequality for convolution yields

$$\left\| \sum_{n=1}^{\infty} u_{1,n} \varphi_{\chi,n} \right\|_{L^{r}(0,T;D(A_{\gamma}^{-\varepsilon-\kappa}))} \leqslant C_{\varepsilon} \|f\|_{L^{r}(0,T;H^{-\theta-\chi}(\partial\Omega))}.$$

In the same way, we obtain

$$\left\| \sum_{n=1}^{\infty} u_{i,n} \varphi_{\chi,n} \right\|_{L^{r}(0,T;D(A_{\chi}^{-\varepsilon-\kappa}))} \le C_{\varepsilon} \left(\|u_{0}\|_{D(A_{\chi}^{-\kappa-\beta})} + \|\rho^{-1}F\|_{L^{r}(0,T;D(A_{\chi}^{-\kappa-1}))} \right), \ i = 2, 3.$$

Combining these estimates, we deduce (2.7). This completes the proof of Theorem 2.3 for $\alpha \in (0,1)$.

3.3. **Proof of Theorem 2.3 for** $\alpha \in (1,2)$. We fix $\varepsilon \in (0,\alpha^{-1})$. For all $n \in \mathbb{N}$, we set

$$\begin{split} u_{1,n}(t) &:= - (-1)^\chi \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^\alpha) \\ & \times \mathbb{I}_{(0,T)}(s) \left\langle f(s,\cdot), \tau_\chi^* \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega),H^{\theta+\chi}(\partial\Omega)} ds, \\ u_{2,n}(t) &:= E_{\alpha,1}(-\lambda_{\chi,n}t^\alpha) \left\langle u_0, \varphi_{\chi,n} \right\rangle_{-\alpha^{-1}-\kappa}, \\ u_{3,n}(t) &:= t E_{\alpha,2}(-\lambda_{\chi,n}t^\alpha) \left\langle u_1, \varphi_{\chi,n} \right\rangle_{-\alpha^{-1}(1+r^{-1})-\kappa}, \\ u_{4,n}(t) &:= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^\alpha) \mathbb{I}_{(0,T)}(s) \left\langle \rho^{-1} F(s,\cdot), \varphi_{\chi,n} \right\rangle_{-\kappa-1} ds \\ \text{and} \end{split}$$

$$u_n := u_{1,n} + u_{2,n} + u_{3,n} + u_{4,n}.$$

We will show that the sequence

$$\sum_{n=1}^{N} u_n(t)\varphi_{\chi,n}, \quad N \in \mathbb{N},$$

converges in $L^r(0,T;D(A_\chi^{-\varepsilon-\kappa}))$ to the unique solution u to (1.4). For this purpose, we remark that formula (1.148) of [23, Theorem 1.6] implies that

$$|u_{2,n}(t)| \leqslant Ct^{\varepsilon\alpha-r^{-1}}\lambda_{\chi,n}^{\varepsilon-\alpha^{-1}r^{-1}} \left| \langle u_0, \varphi_{\chi,n} \rangle_{-\alpha^{-1}-\kappa} \right|$$

and

$$|u_{3,n}(t)| \leqslant Ct^{\varepsilon\alpha-r^{-1}} \lambda_{\chi,n}^{\varepsilon-\alpha^{-1}(1-r^{-1})} \left| \langle u_1, \varphi_{\chi,n} \rangle_{-\alpha^{-1}(1+r^{-1})-\kappa} \right|.$$

Therefore, repeating the arguments for the case $\alpha \in (0,1)$, we can complete the proof of Theorem 2.3 for $\alpha \in (1,2)$.

3.4. **Proof of Theorem 2.4.** Since $u_0 = u_{\lceil \alpha \rceil - 1} = 0$ and $\rho = 1$, the unique weak solution u to (1.4) is given by

$$u(t,\cdot) = \sum_{n=1}^{\infty} u_n(t)\varphi_{\chi,n}, \quad t \in (0,T),$$

where

$$u_n(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \lambda_{\chi,n}^{\kappa+1} G_n(s) ds,$$

and

$$G_n(t) := \frac{(-1)^{\chi+1} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\theta-\chi}(\partial\Omega), H^{\theta+\chi}(\partial\Omega)} + \left\langle F(t,\cdot), \varphi_{\chi,n} \right\rangle_{-\kappa-1}}{\lambda_{\chi,n}^{\kappa+1}}$$

for all $n \in \mathbb{N}$ and $t \in (0, T)$.

Here we notice that

$$\langle A_{\chi}^{-1-\kappa} u(t,\cdot), \varphi_{\chi,n} \rangle = \lambda_{\chi,n}^{-\kappa-1} u_n(t)$$

$$= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) G_n(s) ds.$$

Applying Lemma 3.1, we see

$$G(t,\cdot) = \sum_{n=1}^{\infty} G_n(t)\varphi_{\chi,n} \in L^2((0,T) \times \Omega),$$

and

$$||G||_{L^{2}((0,T)\times\Omega)} \leq C\left(||f||_{L^{2}(0,T;H^{-\theta}(\partial\Omega))} + ||F||_{L^{2}(0,T;D(A_{\chi}^{-\kappa-1}))}\right).$$
(3.3)

On the other hand, in view of [14, Lemma A.2] (see also [24, Theorem 2.2]), we have $A_{\chi}^{-1-\kappa}u \in L^2(0,T;D(A_{\chi})), A_{\chi}^{-1-\kappa}\partial_t^{\alpha}u = \partial_t^{\alpha}A_{\chi}^{-1-\kappa}u \in L^2((0,T)\times\Omega)$ and

$$||A_{\chi}^{-1-\kappa}u||_{L^{2}(0,T;D(A_{\chi}))} + ||\partial_{t}^{\alpha}A_{\chi}^{-1-\kappa}u||_{L^{2}((0,T)\times\Omega)} \leqslant C ||G||_{L^{2}((0,T)\times\Omega)}.$$
(3.4)

From these results, one can easily verify that $u \in L^2(0,T;D(A_{\chi}^{-\kappa})), \partial_t^{\alpha} u \in L^2(0,T;D(A_{\chi}^{-1-\kappa}))$ and estimates (3.3)-(3.4) imply (2.9). This completes the proof of the theorem.

4. Proof of Theorem 2.5

4.1. For m=1. Let us start with the case m=1. For this purpose, we fix $\varepsilon \in (0,1/4)$. In view of Theorem 2.3, the solution $u \in L^r(0,T;D(A_\chi^{-\varepsilon})$ to (1.4) is given by

$$u(t,\cdot) = \sum_{n=1}^{\infty} u_n(t)\varphi_{\chi,n},$$

where we set

$$u_{1,n}(t) := -(-1)^{\chi} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \left\langle f(s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{L^2(\partial\Omega)} ds,$$

$$u_{2,n}(t) := E_{\alpha,1}(-\lambda_{\chi,n}t^{\alpha}) \langle u_0, \varphi_{\chi,n} \rangle,$$

$$u_{3,n}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \mathbb{I}_{(0,T)}(s) \left\langle \rho^{-1} F(s,\cdot), \varphi_{\chi,n} \right\rangle ds$$

and

$$u_n(t) = u_{1,n}(t) + u_{2,n}(t) + u_{3,n}(t), \quad n \in \mathbb{N}.$$

First we see that $u_{1,n} \in W^{1,1}(0,T)$ and

$$u'_{1,n}(t) := \frac{du_{1,n}}{dt}(t)$$

$$= -(-1)^{\chi} \partial_t \left(\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{L^2(\partial\Omega)} ds \right)$$

$$= -(-1)^{\chi} \left\langle f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}(\partial\Omega), H^{\frac{1}{2}+\chi}(\partial\Omega)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} t^{\alpha})$$

$$-(-1)^{\chi} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}} ds.$$

$$(4.1)$$

Similarly, using [24, Lemma 3.2], we deduce that $u_{i,n} \in W^{1,1}(0,T)$, i = 2, 3, where

$$u'_{2,n}(t) = -\lambda_{\chi,n} \langle u_0, \varphi_{\chi,n} \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} t^{\alpha}),$$

$$u'_{3,n}(t) = \left\langle \rho^{-1} F(0,\cdot), \varphi_{\chi,n} \right\rangle_{-1} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} t^{\alpha})$$
$$+ \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle \rho^{-1} \partial_t F(t-s,\cdot), \varphi_{\chi,n} \right\rangle_{-1} ds.$$

On the other hand, the compatibility condition (2.10) and the representation (2.2) of the solution in the transposition sense to the elliptic problem (2.1), imply that

$$\lambda_{\chi,n} \langle u_0, \varphi_{\chi,n} \rangle = -(-1)^{\chi} \langle f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \rangle_{H^{-\frac{1}{2}-\chi}(\partial\Omega), H^{\frac{1}{2}+\chi}(\partial\Omega)} + \langle \rho^{-1} F(0,\cdot), \varphi_{\chi,n} \rangle_{-1}.$$

Therefore, we have

$$u'_{2,n}(t) = K_n t^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_{\chi,n} t^{\alpha}),$$

where

$$K_n := (-1)^{\chi} \left\langle f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}(\partial\Omega), H^{\frac{1}{2}+\chi}(\partial\Omega)} - \left\langle \rho^{-1} F(0,\cdot), \varphi_{\chi,n} \right\rangle_{-1}.$$

Combining this with (4.1), we obtain

$$u'_{n}(t) = -(-1)^{\chi} \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha})$$

$$\times \left\langle \partial_{s} f(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}(\partial\Omega), H^{\frac{1}{2}+\chi}(\partial\Omega)} ds$$

$$+ \int_{0}^{t} s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle \rho^{-1} \partial_{t} F(t-s,\cdot), \varphi_{\chi,n} \right\rangle_{-1} ds.$$

Thus, repeating the arguments in the proof of Theorem 2.3, we deduce that

$$\sum_{k=1}^{n} u_k'(t)\varphi_{\chi,k}, \quad n \in \mathbb{N}$$

converges in $L^r(0,T;D(A_{_Y}^{-\varepsilon}))$ and

$$\left\| \sum_{k=1}^{\infty} u_k' \varphi_{\chi,k} \right\|_{L^r(0,T;D(A_{\chi}^{-\varepsilon}))} \leq C_{\varepsilon} (\left\| \partial_t f \right\|_{L^r(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + \left\| \rho^{-1} \partial_t F \right\|_{L^r(0,T;D(A_{\chi}^{-1}))}).$$

Therefore $u \in W^{1,r}(0,T;D(A_\chi^{-\varepsilon}))$ and

$$||u||_{W^{1,r}(0,T;D(A_{\chi}^{-\varepsilon}))} \le C_{\varepsilon} \left(||f||_{W^{1,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + ||\rho^{-1}F||_{W^{1,r}(0,T;D(A_{\chi}^{-1}))} \right).$$

Moreover, in view of [20, Theorem 11.1, Chapter 1], since $D(A_{\chi}^{\varepsilon}) = H_0^{2\varepsilon}(\Omega)$ for $\varepsilon \in (0,1/4)$ with the equivalent norms, the duality yields $H^{-2\varepsilon}(\Omega) = D(A_{\chi}^{-\varepsilon})$. Thus, the last inequality can be rewritten as

$$||u||_{W^{1,r}(0,T;H^{-2\varepsilon}(\Omega))} \leq C_{\varepsilon} \left(||f||_{W^{1,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + ||\rho^{-1}F||_{W^{1,r}(0,T;D(A_{\chi}^{-1}))} \right).$$

$$(4.2)$$

In order to complete the proof for m=1, we need to prove that $u \in L^r(0,T;H^2(\Omega))$ satisfies

$$||u||_{L^r(0,T;H^2(\Omega))}$$

$$\leqslant C \left(\sum_{k=0}^{1} \|f\|_{W^{1-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega))} + \|\rho^{-1}F\|_{W^{1-k,r}(0,T;D(A_{\chi}^{k-1}))} \right). \tag{4.3}$$

Using [24, Lemma 3.2] and integrating by parts, for almost all $t \in (0, T)$, one can verify that

$$\begin{split} u_{1,n}(t) &= -(-1)^{\chi} \int_{0}^{t} s^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{L^{2}(\partial \Omega)} ds \\ &= (-1)^{\chi} \int_{0}^{t} \partial_{s} \left(\frac{E_{\alpha,1}(-\lambda_{\chi,n} s^{\alpha})}{\lambda_{\chi,n}} \right) \left\langle f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{L^{2}(\partial \Omega)} ds \\ &= (-1)^{\chi} \int_{0}^{t} E_{\alpha,1}(-\lambda_{\chi,n} s^{\alpha}) \frac{\left\langle \partial_{t} f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,n}} ds \\ &+ (-1)^{\chi} E_{\alpha,1}(-\lambda_{\chi,n} t^{\alpha}) \frac{\left\langle f(0,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}(\partial \Omega)}}{\lambda_{\chi,n}} \\ &- (-1)^{\chi} \frac{\left\langle f(t,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{L^{2}(\partial \Omega)}}{\lambda_{\chi,n}}. \end{split}$$

Similarly, we find

$$u_{3,n}(t) = \int_0^t E_{\alpha,1}(-\lambda_{\chi,n}s^{\alpha}) \frac{\langle \rho^{-1}\partial_t F(t-s,\cdot), \varphi_{\chi,n} \rangle_{-1}}{\lambda_{\chi,n}} ds$$
$$-E_{\alpha,1}(-\lambda_{\chi,n}t^{\alpha}) \frac{\langle \rho^{-1}F(0,\cdot), \varphi_{\chi,n} \rangle_{-1}}{\lambda_{\chi,n}} + \frac{\langle \rho^{-1}F(t,\cdot), \varphi_{\chi,n} \rangle}{\lambda_{\chi,n}}.$$

Thus, applying again the compatibility condition (2.10), we find

$$u_{n}(t) = (-1)^{\chi} \int_{0}^{t} E_{\alpha,1}(-\lambda_{\chi,n}s^{\alpha}) \frac{\langle \partial_{t} f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,n}} ds$$

$$+ \int_{0}^{t} E_{\alpha,1}(-\lambda_{\chi,n}s^{\alpha}) \frac{\langle \rho^{-1} \partial_{t} F(t-s,\cdot), \varphi_{\chi,n} \rangle_{-1}}{\lambda_{\chi,n}} ds$$

$$+ \frac{(-1)^{\chi+1} \langle f(t,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \rangle_{L^{2}(\partial\Omega)} + \langle \rho^{-1} F(t,\cdot), \varphi_{\chi,n} \rangle}{\lambda_{\chi,n}}.$$

$$(4.4)$$

We set

$$w_{1,n}(t) := (-1)^{\chi} \int_0^t E_{\alpha,1}(-\lambda_{\chi,n} s^{\alpha}) \frac{\left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,n}} ds,$$

$$w_{2,n}(t) := \int_0^t E_{\alpha,1}(-\lambda_{\chi,n} s^{\alpha}) \frac{\left\langle \rho^{-1} \partial_t F(t-s,\cdot), \varphi_{\chi,n} \right\rangle_{-1}}{\lambda_{\chi,n}} ds,$$

and

$$w_{3,n}(t) := \frac{(-1)^{\chi+1} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}} + \left\langle \rho^{-1} F(t,\cdot), \varphi_{\chi,n} \right\rangle}{\lambda_{\chi,n}}.$$

In view of (4.4), the condition

$$\sum_{n=1}^{\infty} w_{i,n} \varphi_{\chi,n} \in L^r(0,T; H^2(\Omega)), \quad i = 1, 2, 3,$$
(4.5)

implies that $u \in L^r(0,T;H^2(\Omega))$. Moreover, the fact that, for i = 1,2,3, we have

$$\left\| \sum_{n=1}^{\infty} w_{i,n} \varphi_{\chi,n} \right\|_{L^{r}(0,T;H^{2}(\Omega))} \leq C \left(\sum_{k=0}^{1} \|f\|_{W^{1-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega))} + \|\rho^{-1}F\|_{W^{1-k,r}(0,T;D(A_{\chi}^{k-1}))} \right), \tag{4.6}$$

implies (4.3). In order to complete the proof of our result for m=1, we have to prove (4.5) and (4.6). Applying formula (1.148) of [23, Theorem 1.6], similarly to the proof of Theorem 2.3, for m < n we have

$$\begin{split} & \left\| \sum_{\ell=m}^{n} w_{1,\ell} \varphi_{\chi,\ell} \right\|_{L^{r}(0,T;D(A_{\chi}))} \\ & \leqslant C \left\| \int_{0}^{t} s^{-\alpha} \left\| \sum_{\ell=m}^{n} \frac{\left\langle \partial_{t} f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\frac{1}{2}-\chi},H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,\ell}} \varphi_{\chi,\ell} \right\|_{L^{2}(\Omega)} ds \right\|_{L^{r}(0,T)} \\ & \leqslant C \left\| \sum_{\ell=m}^{n} \frac{\left\langle \partial_{t} f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\frac{1}{2}-\chi},H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,\ell}} \varphi_{\chi,\ell} \right\|_{L^{r}(0,T;L^{2}(\Omega))} . \end{split}$$

On the other hand, in view of Lemma 3.1 and $\partial_t f \in L^r(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))$, the sequence

$$\sum_{\ell=1}^{N} \frac{\left\langle \partial_{t} f(t-s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,\ell} \right\rangle_{H^{-\frac{1}{2}-\chi}(\partial\Omega), H^{\frac{1}{2}+\chi}(\partial\Omega)}}{\lambda_{\chi,\ell}}, \quad N \in \mathbb{N}$$

converges in $L^r(0,T;L^2(\Omega))$. Therefore we have

$$\lim_{m,n\to\infty} \left\| \sum_{\ell=m}^{n} \frac{\left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,\ell} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}}}{\lambda_{\chi,\ell}} \varphi_{\chi,\ell} \right\|_{L^r(0,T;L^2(\Omega))} = 0,$$

which implies that the sequence (4.5) with i = 1 converges in $L^r(0, T; D(A_{\chi}))$. Moreover, since $D(A_{\chi})$ is embedded continuously into $H^2(\Omega)$, it follows that (4.5) with i = 1 holds true. The proof for (4.5) with i = 2 is similar and so omitted.

Using similar arguments again, we obtain estimate (4.6) for i=1,2. Now let y satisfy $y(t,\cdot) \in H^1(\Omega)$ for almost all $t \in (0,T)$ and satisfy

$$\begin{cases} \mathcal{A}y(t,x) &= F(t,x), \quad x \in \Omega, \\ \tau_{\chi}y(t,x) &= f(t,\cdot), \quad x \in \partial\Omega. \end{cases}$$
 (4.7)

Combining $f \in L^r(0,T;H^{\frac{3}{2}-\chi}(\partial\Omega))$ and $F \in L^r(0,T;L^2(\Omega))$ with the elliptic regularity of the operator \mathcal{A} , we have that $y \in L^r(0,T;H^2(\Omega))$ and

$$||y||_{L^r(0,T;H^2(\Omega))} \le C(||f||_{L^r(0,T;H^{\frac{3}{2}-\chi}(\partial\Omega))} + ||F||_{L^r(0,T;L^2(\Omega))}).$$
 (4.8)

Moreover, following the proof of Lemma 3.1, one can verify that

$$\langle y(t,\cdot), \varphi_{\chi,n} \rangle$$

$$= \frac{(-1)^{\chi+1} \left\langle f(t,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2}-\chi}, H^{\frac{1}{2}+\chi}} + \left\langle \rho^{-1} F(t,\cdot), \varphi_{\chi,n} \right\rangle}{\lambda_{\chi,n}}$$

$$= w_{3,n}(t). \tag{4.9}$$

Thus, we have

$$\sum_{n=1}^{\infty} w_{3,n}(t)\varphi_{\chi,n} = y(t,\cdot)$$

and by $y \in L^r(0,T;H^2(\Omega))$, we deduce (4.5) with i=3. In addition, we obtain (4.6) for i=3 from estimate (4.8). This proves that $u \in L^r(0,T;H^2(\Omega))$. Therefore we prove that $u \in W^{1,r}(0,T;H^{-\varepsilon}(\Omega)) \cap L^r(0,T;H^2(\Omega))$ for all $\varepsilon \in (0,1/2)$ and u satisfies (2.12). In order to complete the proof of Theorem 2.5, it suffices to verify that u is a strong solution to (1.4) in the sense of Definition 2.2. For this purpose, let

$$\widetilde{f} \in W^{1,r}(0,+\infty;H^{-\frac{1}{2}-\chi}(\partial\Omega)) \cap L^r(0,+\infty;H^{\frac{3}{2}-\chi}(\partial\Omega)),$$

$$\widetilde{F} \in W^{1,r}(0,+\infty;\rho D(A_\chi^{-1})) \cap L^r(0,+\infty;L^2(\Omega)),$$

satisfy supp $\widetilde{f} \subset [0, T+1) \times \partial \Omega$, supp $\widetilde{F} \subset [0, T+1) \times \overline{\Omega}$ and

$$\widetilde{f}|_{\Sigma} = f, \quad \widetilde{F}|_{Q} = F.$$
 (4.10)

Now we set

$$\widetilde{v}(t,\cdot) := \sum_{n=1}^{\infty} \widetilde{v}_n(t) \varphi_{\chi,n}$$

with

$$\widetilde{v}_{\infty}(t)$$

$$:= -(-1)^{\chi} \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \left\langle \widetilde{f}(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{L^{2}(\partial\Omega)} ds$$

$$+ \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \left\langle \widetilde{F}(s,\cdot), \varphi_{\chi,n} \right\rangle_{L^{2}(\Omega)} ds$$

$$+ E_{\alpha,1}(-\lambda_{\chi,n}t^{\alpha}) \left\langle u_{0}, \varphi_{\chi,n} \right\rangle.$$

From the above arguments, one can verify that $\widetilde{v} \in W^{1,r}_{loc}(0,+\infty;H^{-\varepsilon}(\Omega)) \cap L^r_{loc}(0,+\infty;H^2(\Omega))$ and

$$e^{-pt}\widetilde{v}(t,\cdot) \in W^{1,1}(0,+\infty;H^{-\varepsilon}(\Omega)) \cap L^1(0,+\infty;H^2(\Omega))$$
(4.11)

for p > 0. Moreover, following the proof of Theorem 2.5, we see that, for all $(p, x) \in (0, +\infty) \times \Omega$, the Laplace transform $\mathcal{L}\widetilde{v}(p, x)$ of \widetilde{v} satisfies

$$p^{\alpha}(\mathcal{L}\widetilde{v}(p,x)-p^{-1}u_0(x))+\rho^{-1}\mathcal{AL}\widetilde{v}(p,x)=\int_0^{T+1}e^{-pt}\rho^{-1}\widetilde{F}(t,x)dt.$$

Henceforth we write $D'(\Omega) = \mathcal{C}_0^{\infty}(\Omega)'$. In view of (4.11), this identity implies that, for all p > 0 and all $\psi \in \mathcal{C}_0^{\infty}(\Omega)$, we have

$$\begin{split} &\mathcal{L}\left[\langle (\rho \partial_t^\alpha \widetilde{v} + \mathcal{A} \widetilde{v})(t, \cdot), \psi \rangle_{D'(\Omega), \mathcal{C}_0^\infty(\Omega)} \right](p) \\ &= \langle \mathcal{L}[(\rho \partial_t^\alpha \widetilde{v} + \mathcal{A} \widetilde{v})](p, \cdot), \psi \rangle_{D'(\Omega), \mathcal{C}_0^\infty(\Omega)} \\ &= \langle \rho p^\alpha (\mathcal{L} \widetilde{v}(p, \cdot) - u_0) + \mathcal{A} \mathcal{L} \widetilde{v}(p, \cdot), \psi \rangle_{D'(\Omega), \mathcal{C}_0^\infty(\Omega)} \\ &= \left\langle \int_0^{T+1} e^{-pt} \widetilde{F}(t, \cdot) dt, \psi \right\rangle_{D'(\Omega), \mathcal{C}_0^\infty(\Omega)} \\ &= \mathcal{L}\left[\left\langle \widetilde{F}(t, \cdot), \psi \right\rangle_{D'(\Omega), \mathcal{C}_0^\infty(\Omega)} \right](p). \end{split}$$

Therefore, for almost all t > 0, we have

$$\left\langle (\rho \partial_t^{\alpha} \widetilde{v} + \mathcal{A} \widetilde{v})(t, \cdot) - \widetilde{F}(t, \cdot), \psi \right\rangle_{D'(\Omega), \mathcal{C}_0^{\infty}(\Omega)} = 0.$$

On the other hand, in view of (4.10) one can easily verify that $\tilde{v} = u$ in $(0,T) \times \Omega$. Hence for almost all $t \in (0,T)$ we have

$$\langle (\rho \partial_t^{\alpha} u + \mathcal{A} u)(t, \cdot) - F(t, \cdot), \psi \rangle_{D'(\Omega), \mathcal{C}_0^{\infty}(\Omega)} = 0, \quad \psi \in \mathcal{C}_0^{\infty}(\Omega).$$

Furthermore, by $u \in L^r(0,T;H^2(\Omega))$, we have $\partial_t^{\alpha} u = -\rho^{-1} \mathcal{A} u + F \in L^r(0,T;L^2(\Omega))$. Therefore (2.4) holds true in $L^r(0,T;L^2(\Omega))$. In the same way, applying (4.11), one can verify that condition (2.5) is also fulfilled and u is a strong solution to (1.4). This completes the proof of Theorem 2.5 for m=1.

4.2. For $m \ge 2$. We will consider only the case m = 2. The case $m \ge 3$ can be deduced in a similar way by an iteration argument. Applying the result for m = 1, for all $\varepsilon \in (0, 1/2)$, we obtain that $u \in W^{1,r}(0, T; H^{-\varepsilon}(\Omega)) \cap L^r(0, T; H^2(\Omega))$ and

$$\partial_t u(t,\cdot) = \sum_{n=1}^{\infty} u'_n(t)\varphi_{\chi,n},$$

where

$$u'_{n}(t)$$

$$= -(-1)^{\chi} \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \left\langle \partial_{s} f(s,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n} \right\rangle_{L^{2}(\partial\Omega)} ds$$

$$+ \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n}(t-s)^{\alpha}) \left\langle \rho^{-1} \partial_{s} F(s,\cdot), \varphi_{\chi,n} \right\rangle ds.$$
(4.12)

Applying (2.11), we obtain

$$u_n''(t)$$

$$= -(-1)^{\chi} \int_0^t s^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_{\chi,n} s^{\alpha}) \left\langle \partial_s^2 f(s,\cdot), \tau_{\chi}^* \varphi_{\chi,n} \right\rangle_{H^{-\frac{1}{2} - \chi}, H^{\frac{1}{2} + \chi}} ds$$

$$+ \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_{\chi,n} (t - s)^{\alpha}) \left\langle \rho^{-1} \partial_s^2 F(s,\cdot), \varphi_{\chi,n} \right\rangle_{-1} ds.$$

and, following the proof for m=1, we reach $u\in W^{2,r}(0,T;H^{-\varepsilon}(\Omega))$ and

$$||u||_{W^{2,r}(0,T;H^{-\varepsilon}(\Omega))} \leq C_{\varepsilon} \left(||f||_{W^{2,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + ||F||_{W^{2,r}(0,T;D(A_{\chi}^{-\frac{1}{2}}))} \right). \tag{4.13}$$

Now let us prove that $\partial_t u \in L^r(0,T;H^2(\Omega))$. For this purpose, in view of (4.12), applying the arguments for the proof of Theorem 2.3, we see that $v = \partial_t u$ is the weak solution to

$$\begin{cases} (\rho(x)\partial_t^{\alpha} + \mathcal{A})v(t,x) &=& \partial_t F(t,x), & (t,x) \in Q, \\ \tau_{\chi}v(t,x) &=& \partial_t f(t,x), & (t,x) \in \Sigma, \\ v(0,x) &=& 0, & x \in \Omega. \end{cases}$$

Since

$$\partial_t f \in L^r(0, T; H^{\frac{3}{2} - \chi}(\partial \Omega)) \cap W^{1,r}(0, T; H^{-\frac{1}{2} - \chi}(\partial \Omega)),$$
$$\partial_t F \in L^r(0, T; L^2(\Omega)) \cap W^{1,r}(0, T; \rho D(A_{\nu}^{-1}))$$

and (2.11) is fulfilled, applying the theorem for m=1, we obtain $\partial_t u \in L^r(0,T;H^2(\Omega)) \cap W^{1,r}(0,T;H^{-\varepsilon}(\Omega))$ and

$$||u||_{W^{1,r}(0,T;H^2(\Omega))} + ||u||_{W^{2,r}(0,T;H^{-\varepsilon}(\Omega))}$$

$$\leqslant C_{\varepsilon} \left(\sum_{k=1}^{2} \|f\|_{W^{2-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega))} + \|F\|_{W^{2-k,r}(0,T;H^{2(k-1)}(\Omega))} \right) (4.14)
+ C_{\varepsilon} \left(\|f\|_{W^{2,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + \|\rho^{-1}F\|_{W^{2,r}(0,T;D(A_{\chi}^{-1}))} \right).$$

Recalling that u is a strong solution to (1.4), we see that $u(t,\cdot)$ solves the boundary value problem

for almost all $t \in (0,T)$. On the other hand, since $u \in W^{1,r}(0,T;H^2(\Omega))$, Young's inequality for convolution yields $\partial_t^{\alpha}u \in L^r(0,T;H^2(\Omega))$. Therefore, by $f \in L^r(0,T;H^{2+\frac{3}{2}-\chi}(\partial\Omega))$ and $F \in L^r(0,T;H^2(\Omega))$, applying the elliptic regularity of the operator $\mathcal A$ guaranteed by condition (H_2) , we see that $u \in L^r(0,T;H^4(\Omega))$ and

$$\begin{aligned} &\|u\|_{L^{r}(0,T;H^{4}(\Omega))} \\ &\leqslant C(\|u\|_{W^{1,r}(0,T;H^{2}(\Omega))} + \|f\|_{L^{r}(0,T;H^{2+\frac{3}{2}-\chi}(\partial\Omega))} + \|F\|_{L^{r}(0,T;H^{2}(\Omega))}). \end{aligned}$$

Combining this with (4.14), we find

$$||u||_{W^{1,r}(0,T;H^{2}(\Omega))} + ||u||_{W^{2,r}(0,T;H^{-\varepsilon}(\Omega))} + ||u||_{L^{r}(0,T;H^{4}(\Omega))}$$

$$\leq C_{\varepsilon} \sum_{k=1}^{2} \left(||f||_{W^{2-k,r}(0,T;H^{2k-\frac{1}{2}-\chi}(\partial\Omega))} + ||F||_{W^{2-k,r}(0,T;H^{2(k-1)}(\Omega))} \right)$$

$$+ C_{\varepsilon} \left(||f||_{W^{2,r}(0,T;H^{-\frac{1}{2}-\chi}(\partial\Omega))} + ||\rho^{-1}F||_{W^{2,r}(0,T;D(A_{\chi}^{-1}))} \right).$$

This completes the proof of Theorem 2.5.

5. Proof of Propositions 2.6 and 2.9

Proof of Proposition 2.6. We start with the first statement of Proposition 2.6. For this purpose, we fix $f \in \mathcal{C}^{\infty}([0,T] \times \partial\Omega)$, $F \in \mathcal{C}^{\infty}([0,T] \times \overline{\Omega})$ and $u_0 \in \mathcal{C}^{\infty}(\overline{\Omega})$ which do not satisfy (2.10). We assume also that (1.4) admits a unique solution $u \in W^{1,1}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. In view of (4.9), there exists $n_0 \in \mathbb{N}$ such that

$$b_0 := -(-1)^{\chi} \left\langle f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2(\partial\Omega)} + \left\langle \rho^{-1} F(0,\cdot), \varphi_{\chi,n_0} \right\rangle - \lambda_{\chi,n_0} \left\langle u_0, \varphi_{\chi,n_0} \right\rangle \neq 0.$$
(5.1)

We will prove that $u \notin W^{1,(1-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Following the argumentation of Theorem 2.5, we can show that

$$\langle \partial_t u(t,\cdot), \varphi_{\chi,n_0} \rangle_{-\frac{1}{2}}$$

$$= -(-1)^{\chi} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2(\partial\Omega)} ds$$

$$+ \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \rho^{-1} \partial_t F(t-s,\cdot), \varphi_{\chi,n_0} \right\rangle ds$$

$$+ b_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} t^{\alpha}).$$

On the other hand, since $E_{\alpha,\alpha}(-\lambda_{\chi,n_0}t^{\alpha}) \in \mathcal{C}[0,T]$ and $E_{\alpha,\alpha}(0) > 0$, we see that there exists $t_0 \in (0,T)$ and $c_0 > 0$ such that

$$\inf_{t \in [0, t_0]} |E_{\alpha, \alpha}(-\lambda_{\chi, n} t^{\alpha})| = c_0.$$

Therefore, for almost all $t \in (0, t_0)$, we have

$$|\langle \partial_t u(t,\cdot), \varphi_{\chi,n_0} \rangle_{-\frac{1}{2}}|$$

$$\geqslant c_0 |b_0| t^{\alpha-1}$$

$$- \left| -(-1)^{\chi} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2(\partial\Omega)} ds \right|$$

$$- \left| \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \rho^{-1} \partial_t F(t-s,\cdot), \varphi_{\chi,n_0} \right\rangle ds \right|.$$

Moreover, since $f \in \mathcal{C}^{\infty}([0,T] \times \partial \Omega)$, $F \in \mathcal{C}^{\infty}([0,T] \times \overline{\Omega})$, it is clear that

 c_1

$$:= \left\| (-1)^{\chi+1} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \partial_t f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2} ds \right\|_{L^{\infty}(0,T)}$$

$$+ \left\| \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \rho^{-1} \partial_t F(t-s,\cdot), \varphi_{\chi,n_0} \right\rangle ds \right\|_{L^{\infty}(0,T)} < \infty.$$

Thus, for almost all $t \in (0, t_0)$, we have

$$|\langle \partial_t u(t,\cdot), \varphi_{\chi,n_0} \rangle_{-\frac{1}{2}}| \geqslant c_0 |b_0| t^{\alpha-1} - c_1$$

and condition (5.1) clearly implies that

$$|\langle \partial_t u(t,\cdot), \varphi_{\chi,n_0} \rangle| \notin L^{(1-\alpha)^{-1}}(0,T).$$

Thus, we have $u \notin W^{1,(1-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$, which completes the proof of the first statement of Proposition 2.6.

For the second statement of Proposition 2.6, let us consider $f \in \mathcal{C}^{\infty}([0,T] \times \partial \Omega)$, $F \in \mathcal{C}^{\infty}([0,T] \times \overline{\Omega})$ and $u_0 \in \mathcal{C}^{\infty}(\overline{\Omega})$ satisfying (2.10) and let (1.4) admit a unique solution $u \in W^{2,1}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Assume also that (2.11) is not fulfilled for m=2. Then we first show that there exists $n_1 \in \mathbb{N}$ such that

$$-(-1)^{\chi} \left\langle \partial_t f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n_1} \right\rangle_{L^2(\partial\Omega)} + \left\langle \rho^{-1} \partial_t F(0,\cdot), \varphi_{\chi,n_1} \right\rangle \neq 0.$$
 (5.2)

Indeed, assuming the contrary, we deduce that

$$\lambda_{\chi,k} \left\langle A_{\chi}^{-1} \rho^{-1} \partial_t F(0,\cdot), \varphi_{\chi,k} \right\rangle = -(-1)^{\chi} \left\langle -\partial_t f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,k} \right\rangle_{L^2(\partial\Omega)}, \quad k \in \mathbb{N}.$$

Then, following formula (4.9), we deduce that $G = A_{\chi}^{-1} \rho^{-1} \partial_t F(0, \cdot)$ solves the boundary value problem

$$\begin{cases} AG(x) = 0, & x \in \Omega, \\ \tau_{\chi}G(x) = -\partial_t f(0, x), & x \in \partial\Omega. \end{cases}$$

On the other hand, since $\rho^{-1}\partial_t F(0,\cdot) \in L^2(\Omega)$ and $G = A_{\chi}^{-1}\rho^{-1}\partial_t F(0,\cdot)$, we obtain $\partial_t F(0,\cdot) = \rho A_{\chi}G = \mathcal{A}G = 0$ and $\partial_t f(0,\cdot) = -\tau_{\chi}G = 0$. This contradicts that (2.11) is not fulfilled for m=2. Thus, there exists $n_1 \in \mathbb{N}$ such that (5.2) holds true. Repeating the arguments for Theorem 2.5, in view of (2.10), one can verify that

$$\begin{split} &\left\langle \partial_t^2 u(t,\cdot), \varphi_{\chi,n_1} \right\rangle_{-\frac{1}{2}} \\ &= -(-1)^{\chi} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_1} s^{\alpha}) \left\langle \partial_t^2 f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n_1} \right\rangle_{L^2(\partial\Omega)} ds \\ &+ \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_1} s^{\alpha}) \left\langle \rho^{-1} \partial_t^2 F(t-s,\cdot), \varphi_{\chi,n_1} \right\rangle ds \\ &- (-1)^{\chi} \left\langle \partial_t f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n_1} \right\rangle_{L^2(\partial\Omega)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_1} t^{\alpha}) \\ &+ \left\langle \rho^{-1} \partial_t F(0,\cdot), \varphi_{\chi,n_1} \right\rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_1} t^{\alpha}). \end{split}$$

Therefore, similarly to the proof of the first statement of Proposition 2.6, by (5.2) we deduce that $\langle \partial_t^2 u(t,\cdot), \varphi_{\chi,n_1} \rangle_{-\frac{1}{2}} \notin L^{(\alpha-1)^{-1}}(0,T)$, which implies that $\partial_t^2 u \notin L^{(\alpha-1)^{-1}}(0,T;D(A_\chi^{-\frac{1}{2}}))$. Thus the proof of the proposition is completed.

Proof of Proposition 2.9. We start with the proof of the first statement. For this purpose, we fix $f \in \mathcal{C}^{\infty}([0,T] \times \partial\Omega)$, $F \in \mathcal{C}^{\infty}([0,T] \times \overline{\Omega})$ and $u_0, u_1 \in \mathcal{C}^{\infty}(\overline{\Omega})$ such that (2.10) is not fulfilled. Then, there exists $n_0 \in \mathbb{N}$ such that (5.1) is not fulfilled.

Let us assume that the solution u to (1.4) is lying in $W^{2,1}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. We will prove that $u \notin W^{2,(2-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. Combining [24, Lemma 3.2] with Proposition 2.6, we deduce that

$$\begin{split} \left\langle \partial_t^2 u(t,\cdot), \varphi_{\chi,n_0} \right\rangle_{-\frac{1}{2}} \\ &= -(-1)^{\chi} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \partial_t^2 f(t-s,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2(\partial\Omega)} ds \\ &+ \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} s^{\alpha}) \left\langle \rho^{-1} \partial_t^2 F(t-s,\cdot), \varphi_{\chi,n_0} \right\rangle ds \\ &+ d_0 t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_{\chi,n_0} t^{\alpha}) + e_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\chi,n_0} t^{\alpha}), \end{split}$$

where

$$d_0 = -(-1)^{\chi} \left\langle f(0,\cdot), \tau_{\chi}^* \varphi_{\chi,n_0} \right\rangle_{L^2(\partial\Omega)} + \left\langle \rho^{-1} F(0,\cdot), \varphi_{\chi,n_0} \right\rangle$$
$$-\lambda_{\chi,n_0} \left\langle u_0, \varphi_{\chi,n_0} \right\rangle$$

and

$$e_{0} = -(-1)^{\chi} \left\langle \partial_{t} f(0,\cdot), \tau_{\chi}^{*} \varphi_{\chi,n_{0}} \right\rangle_{L^{2}(\partial\Omega)} + \left\langle \rho^{-1} \partial_{t} F(0,\cdot), \varphi_{\chi,n_{0}} \right\rangle$$
$$-\lambda_{\chi,n_{0}} \left\langle u_{1}, \varphi_{\chi,n_{0}} \right\rangle.$$

Therefore, in a similar way to Proposition 2.6, one can show that there exists $t_1 \in (0,T)$, $c_2, c_3 > 0$ such that

$$\left| \left\langle \partial_t^2 u(t,\cdot), \varphi_{\chi,n_0} \right\rangle_{-\frac{1}{2}} \right| \geqslant c_3 \left| d_0 \right| t^{\alpha-2} - c_2, \quad t \in (0,t_1),$$

which proves that $u \notin W^{2,(2-\alpha)^{-1}}(0,T;D(A_{\chi}^{-\frac{1}{2}}))$. The rest part of the proposition can be proved similarly and is omitted.

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