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RESEARCH PAPER

A FRACTIONAL GENERALIZATION OF THE DIRICHLET DISTRIBUTION AND RELATED DISTRIBUTIONS

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Abstract

This paper is devoted to a fractional generalization of the Dirichlet distribution. The form of the multivariate distribution is derived assuming that the n partitions of the interval $[0, W_n]$ are independent and identically distributed random variables following the generalized Mittag-Leffler distribution. The expected value and variance of the one-dimensional marginal are derived as well as the form of its probability density function. A related generalized Dirichlet distribution is studied that provides a reasonable approximation for some values of the parameters. The relation between this distribution and other generalizations of the Dirichlet distribution is discussed. Monte Carlo simulations of the one-dimensional marginals for both distributions are presented.

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Key Words and Phrases: fractional Dirichlet distribution; generalized Dirichlet distribution; three-parameter Mittag-Leffler functions; fractional Poisson process; wealth distribution; power-law tails

1. Introduction

Let us consider a finite sequence of n positive random variables $Z_1, ..., Z_n$. For instance, these variables can represent the wealth of n economic agents if indebtedness is not allowed. Let us denote the sum as $W_n = Z_1 + \ldots + Z_n$. In the wealth interpretation this is the total wealth. If we define the

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wealth fraction of the i-th agent as $Q_i = Z_i/W_n$, we get a partition of the interval $[0,1]$ represented by the sequence $\mathbf{Q} = (Q_1, \ldots, Q_n)$ such that $Q_1 + \ldots + Q_n = 1$ almost surely. We are particularly interested in multivariate distributions for the sequence $\mathbf{Q}$ whose one-dimensional marginals have heavy tails. If we further assume that the random variables $Z_1, \ldots, Z_n$ are independent and identically distributed, there is a nice and immediate relationship with point processes of renewal type. In this case, the variables Z_i can be interpreted as inter-event intervals and the partial sums $W_k = \sum_{i=1}^{k} Z_i$, with $k \leq n$, are the epochs of the events.

In order to clarify the relationship, we start by recalling some basic facts on the time-fractional Poisson process as we are going to use and generalize it in the next section. From [1, 14] we know that the time-fractional Poisson process $N^\nu = (N^\nu(t))_{t\geq 0}$, $\nu \in (0,1]$, can be defined as a renewal process with independent and identically distributed inter-event waiting times $\mathcal{T}_j$, $j \in \mathbb{N}^* = \{1, 2, \ldots\}$, with probability density function (pdf)

$$\mathbb{P}(\mathcal{T}_j \in \mathrm{d}t) = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu)\, \mathrm{d}t, \qquad \lambda > 0,\ t > 0, \tag{1.1}$$

where

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)}, \qquad z, \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0, \tag{1.2}$$

is the two-parameter Mittag–Leffler function. Note that for $\nu = 1$, the waiting times $\mathcal{T}_j$ are exponentially distributed and N^1 is the homogeneous Poisson process. Moreover, the Laplace transform of the pdf (1.1) takes a very compact form. Indeed, we have

$$\int_0^\infty e^{-zt} \mathbb{P}(\mathcal{T}_j \in \mathrm{d}t) = \frac{\lambda}{\lambda + z^\nu}, \qquad z > 0. \tag{1.3}$$

Let us now indicate with T_k, $k \in \mathbb{N}^*$, the random occurrence time of the k-th event of the stream of events defining N^ν. From the renewal structure of N^ν we readily obtain that the Laplace transform of T_k reads

$$\int_0^\infty e^{-zt} \mathbb{P}(T_k \in \mathrm{d}t) = \left(\frac{\lambda}{\lambda + z^\nu}\right)^k, \qquad z > 0, \tag{1.4}$$

which in turn corresponds to the Laplace transform of a function involving the three-parameter Mittag–Leffler function (also known as the Prabhakar function – see [7]). In particular, the three-parameter Mittag–Leffler function is defined as

$$E^\delta_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)} \frac{\Gamma(\delta + r)}{r!\,\Gamma(\delta)}, \qquad z, \alpha, \beta, \delta \in \mathbb{C},\ \Re(\alpha) > 0, \tag{1.5}$$

and we know by direct calculation that (see e.g. [15], formula (2.3.24))

$$\int_0^\infty t^{\beta-1}e^{-zt}E^{\delta}_{\alpha,\beta}(\zeta t^\alpha)\,\mathrm{d}t = z^{-\beta}\left(1-\zeta z^{-\beta}\right)^{-\delta}, \tag{1.6}$$

where $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(z)>0$, $z>|\zeta|^{1/\Re(\alpha)}$. Using (1.6), we obtain

$$\mathbb{P}(T_k\in\mathrm{d}t) = \lambda^k t^{\nu k-1}E^k_{\nu,\nu k}(-\lambda t^\nu)\,\mathrm{d}t, \qquad \lambda>0,\ t>0,\ \nu\in(0,1],\ k\in\mathbb{N}^*. \tag{1.7}$$

Remark 1.1. Note that, for $\nu=1$, the above density reduces to that of an Erlang(λ,k) distributed random variable. This can be seen by simply noticing that

$$\mathbb{P}(T_k\in\mathrm{d}t) = \mathrm{d}t\,\lambda^k t^{k-1}\sum_{r=0}^{\infty}\frac{(-\lambda t)^r\Gamma(r+k)}{\Gamma(r+k)\Gamma(k)r!} = \mathrm{d}t\,\frac{\lambda^k t^{k-1}e^{-\lambda t}}{(k-1)!}, \qquad k\in\mathbb{N}^*. \tag{1.8}$$

Remark 1.2. The Erlang(λ,k) distribution is a special case of the Gamma(a,c) distribution. Consider a sequence $Z_1,\ldots,Z_n$ of independent random variables each following a Gamma distribution of parameter $(a_1,c),\ldots,(a_n,c)$. It is well known that their sum W_n is still a Gamma of parameter $(a_1+\ldots+a_n,c)$. Then the sequence of fractions $Q_1,\ldots,Q_n$ has a joint $(N-1)$-dimensional Dirichlet distribution of parameters $a_1,\ldots,a_n$ with density

$$f_{\mathbf{Q}}(q_1,\ldots,q_{n-1}) = \frac{\Gamma(a_1+\ldots+a_n)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_n)}q_1^{a_1-1}\cdots\left(1-\sum_{i=1}^{n-1}q_i\right)^{a_n-1} \tag{1.9}$$

with $q_1+\ldots+q_n=1$ and is independent of W_n.

The proof of the results in Remark 1.2 can be found in several textbooks and lecture notes (see e.g. [2], Lemma 1.5).

Remark 1.3. The random variables $\mathcal{T}_j$ have the following asymptotic behaviour for $t\to\infty$ [8]:

$$\mathbb{P}(\mathcal{T}_1>t)\sim\frac{\sin(\nu\pi)}{\pi}\frac{\Gamma(\nu)}{t^\nu}, \quad t\gg 1; \tag{1.10}$$

therefore, their sums T_k belong to the basin of attraction of the ν-stable subordinator.

Remark 1.4. The distributions considered in the present paper belong to the class of distributions on the simplex discussed in [5] (see (2.2) below and [3]).

This paper contains the following material. Section 2 concerns the definition and properties of the fractional Dirichlet distribution. Section 3 mirrors Section 2 and is devoted to the generalized Dirichlet distribution. Section 4 explains how to simulate the fractional Dirichlet distribution and presents the results of Monte Carlo simulations in order to illustrate the relation between the fractional Dirichlet distribution and the generalized Dirichlet distribution.

2. **Construction of the fractional Dirichlet distribution**

Based on Remark 1.1 and Remark 1.2, we now define a generalization of the Gamma distribution and we immediately present a fractional generalization of the Dirichlet distribution.

Definition 2.1 (Fractional Gamma distribution). Let X be a positive real valued random variable with distribution

$$\mu(\mathrm{d}x) = \mathbb{P}(X \in \mathrm{d}x) = \lambda^{\beta} x^{\nu\beta-1} E^{\beta}_{\nu,\nu\beta}(-\lambda x^{\nu})\,\mathrm{d}x, \tag{2.1}$$

where $\lambda > 0$, $x > 0$, $\beta > 0$, $\nu \in (0,1]$. Then X is said to be distributed as a fractional Gamma of parameters λ, β, ν (we write $X \sim FG(\lambda, \beta, \nu)$) (see [19]; for applications to renewal processes see [4, 16, 17, 18]).

Remark 2.1. The Laplace transform of μ reads

$$\int_0^{\infty} e^{-zx}\mu(\mathrm{d}x) = \left(\frac{\lambda}{\lambda + z^{\nu}}\right)^{\beta}, \qquad z > 0. \tag{2.2}$$

By means of (2.1), we will construct a generalization of the Dirichlet distribution. We consider n independent random variables Z_i, $i = 1, \ldots, n$, distributed as fractional Gamma random variables of parameters $(1, \beta_i, \nu)$, $\nu \in (0,1]$, $\beta_i > 0$, $i = 1, \ldots, n$, respectively. Furthermore, define the sum $W = Z_1 + \ldots + Z_n$, set $Q_i = Z_i/W$, $i = 1, \ldots, n$, and consider the transformation

$$(Z_1, \ldots, Z_n) \longrightarrow \left(WQ_1, \ldots, WQ_{n-1}, W\Big(1 - \sum_{i=1}^{n-1} Q_i\Big)\right). \tag{2.3}$$

vskip-2pt Note that, from (2.2), the distribution of W is fractional Gamma as well, i.e. $W \sim FG(1, \bar{\beta}, \nu)$, where $\bar{\beta} = \sum_{i=1}^{n} \beta_i$. The joint pdf of the vector $(W, \boldsymbol{Q}) = (W, Q_1, \ldots, Q_{n-1})$ reads

$$f_{(W,\boldsymbol{Q})}(y, q_1, \ldots, q_{n-1}) \tag{2.4}$$

$$
\begin{aligned}
&= \left[\prod_{i=1}^{n-1}(yq_i)^{\nu\beta_i-1}E^{\beta_i}_{\nu,\nu\beta_i}(-(yq_i)^\nu)\right]\left[y(1-\sum_{i=1}^{n-1}q_i)\right]^{\nu\beta_n-1}\\
&\quad\times E^{\beta_n}_{\nu,\nu\beta_n}\left(-\Big(y-y\sum_{i=1}^{n-1}q_i\Big)^\nu\right)y^{n-1}\\
&= \left(\prod_{i=1}^{n-1}y^{\nu\beta_i-1}\right)y^{\nu\beta_n-1}y^{n-1}\left[\prod_{i=1}^{n-1}q_i^{\nu\beta_i-1}E^{\beta_i}_{\nu,\nu\beta_i}(-(yq_i)^\nu)\right]\\
&\quad\times\left(1-\sum_{i=1}^{n-1}q_i\right)^{\nu\beta_n-1}E^{\beta_n}_{\nu,\nu\beta_n}\left(-\Big(y-y\sum_{i=1}^{n-1}q_i\Big)^\nu\right)\\
&= y^{\nu\bar\beta-1}\left[\prod_{i=1}^{n-1}q_i^{\nu\beta_i-1}E^{\beta_i}_{\nu,\nu\beta_i}(-(yq_i)^\nu)\right]\left(1-\sum_{i=1}^{n-1}q_i\right)^{\nu\beta_n-1}\\
&\quad\times E^{\beta_n}_{\nu,\nu\beta_n}\left(-\Big(y-y\sum_{i=1}^{n-1}q_i\Big)^\nu\right).
\end{aligned}
$$

The joint pdf of $\boldsymbol{Q}=(Q_1,\dots,Q_{n-1})$ is then obtained by marginalization. Hence,

$$
\begin{aligned}
f_{\boldsymbol{Q}}(q_1,\dots,q_{n-1}) &= \left(\prod_{i=1}^{n-1}q_i^{\nu\beta_i-1}\right)\left(1-\sum_{i=1}^{n-1}q_i\right)^{\nu\beta_n-1}\\
&\quad\times\int_0^\infty y^{\nu\bar\beta-1}\prod_{i=1}^{n-1}E^{\beta_i}_{\nu,\nu\beta_i}(-(yq_i)^\nu)E^{\beta_n}_{\nu,\nu\beta_n}\left(-\Big(y-y\sum_{i=1}^{n-1}q_i\Big)^\nu\right)\mathrm{d}y.
\end{aligned}
\tag{2.5}
$$

Remark 2.2. On the n-dimensional simplex Δ_n the probability density of the random vector $(Q_1,\dots,Q_n)$, where $\sum_n Q_n=1$ a.s., writes

$$
\begin{aligned}
&\mathbb{P}\left((Q_1,\dots,Q_n)\in\mathrm{d}(q_1,\dots,q_n)\right)\\
&=\prod_{i=1}^{n}q_i^{\nu\beta_i-1}\int_0^\infty y^{\nu\bar\beta-1}\prod_{i=1}^{n}E^{\beta_i}_{\nu,\nu\beta_i}\left(-(yq_i)^\nu\right)\,\mathrm{d}y.
\end{aligned}
\tag{2.6}
$$

Notice that for $\nu=1$ the integral in the rhs of (2.6) can be easily solved and the Dirichlet($\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)$) is obtained. In this case $(Q_1,\dots,Q_n)$ is uniformly distributed on Δ_n for $\beta_i=1$, $i\in\mathbb{N}^*$.

If $\nu\in(0,1)$ with $\beta_i=1$, we have

$$\mathbb{P}\left((Q_1,\dots,Q_n)\in \mathrm{d}(q_1,\dots,q_n)\right)=\prod_{i=1}^{n} q_i^{\nu-1}\int_0^\infty y^{\nu n-1}\prod_{i=1}^{n} E_{\nu,\nu}(-(yq_i)^\nu)\,\mathrm{d}y, \tag{2.7}$$

which is symmetric but not uniform.

If we let instead $\beta_i = 1/\nu$ (again symmetric), we obtain

$$\mathbb{P}\left((Q_1,\dots,Q_n)\in \mathrm{d}(q_1,\dots,q_n)\right)=\int_0^\infty y^{n-1}\prod_{i=1}^{n} E^{1/\nu}_{\nu,1}(-(yq_i)^\nu)\,\mathrm{d}y. \tag{2.8}$$

2.1. **Properties.** The derivation of the marginal moments can be done explicitly using the formulas in Section 2.2 of [5].

Proposition 2.1. *Let $\boldsymbol{Q}=(Q_1,\dots,Q_{n-1})$ be a random vector distributed with pdf* (2.5). *For each $j=1,\dots,n-1$, we have,*

$$\mathbb{E}Q_j=\frac{\beta_j}{\bar\beta}, \tag{2.9}$$

$$\mathbb{V}\mathrm{ar}\,Q_j=\frac{\beta_j(\bar\beta-\beta_j)}{\bar\beta^2(\bar\beta+1)}\left(1+\bar\beta(1-\nu)\right). \tag{2.10}$$

P r o o f. By Proposition 2 of [5] we have

$$\begin{aligned}\mathbb{E}Q_j &= -\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1+z^\nu}\right)^{\beta_j}\right)\left(\frac{1}{1+z^\nu}\right)^{\bar\beta-\beta_j}\mathrm{d}z \qquad (2.11)\\ &=\int_0^\infty \frac{\beta_j\nu z^{\nu-1}}{1+z^\nu}(1+z^\nu)^{-\bar\beta}\mathrm{d}z\\ &=\beta_j\int_0^\infty\frac{\mathrm{d}w}{(1+w)^{\bar\beta+1}}=\frac{\beta_j}{\bar\beta}.\end{aligned}$$

Similarly, the second moment writes

$$\begin{aligned}\mathbb{E}Q_j^2 &= \int_0^\infty z\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{1+z^\nu}\right)^{\beta_j}\right)\left(\frac{1}{1+z^\nu}\right)^{\bar\beta-\beta_j}\mathrm{d}z \qquad (2.12)\\ &=\int_0^\infty\left[\frac{\nu^2z^{2\nu-2}\beta_j(\beta_j+1)}{(1+z^\nu)^{\beta_j+2}}-\frac{\beta_j\nu(\nu-1)z^{\nu-2}}{(1+z^\nu)^{\beta_j+1}}\right](1+z^\nu)^{-\bar\beta+\beta_j}\mathrm{d}z\\ &\overset{z^\nu=w}{=}\nu\beta_j(\beta_+1)\int_0^\infty\frac{w}{(1+w)^{\bar\beta+2}}\mathrm{d}w+(1-\nu)\beta_j\int_0^\infty\frac{\mathrm{d}w}{(1+w)^{\bar\beta+1}}\\ &=\nu\frac{\beta_j(\beta_j+1)}{\bar\beta(\bar\beta+1)}+(1-\nu)\frac{\beta_j}{\bar\beta},\end{aligned}$$

and hence after some computation

$$\mathbb{V}\mathrm{ar}\, Q_j = \frac{\beta_j(\bar{\beta}-\beta_j)}{\bar{\beta}^2(\bar{\beta}+1)}\left(1+\bar{\beta}(1-\nu)\right). \tag{2.13}$$

□

Remark 2.3. Notice that the first factor of the variance (2.10) is in fact the variance of a one-dimensional marginal of a Dirichlet($\boldsymbol{\beta}$) distribution. It follows that the marginals are overdispersed with respect to those of a Dirichlet($\boldsymbol{\beta}$) distribution.

We now proceed by analyzing the aggregation property and therefore the marginal distributions.

Proposition 2.2 (Aggregation property). *Consider the pdf defined in equation* (2.5) *and the random variable* $\mathcal{Q} = \sum_{j=1}^{k} Q_{i_j}$ *where* $1 < k < n$ *and* i_j *denotes any permutation of the indices. Then the random variable* $\mathcal{Z} = W\mathcal{Q}$ *has pdf* (2.1) *with* $\beta = \sum_{j=1}^{k} \beta_{i_j}$.

P r o o f. The proof is immediate considering that $\mathcal{Q}$ comes from the sum of i.i.d. positive random variables each one with Laplace transform given by (2.2) and then divided by W. Therefore $\mathcal{Z} = W\mathcal{Q}$ has pdf given by (2.1) with $\beta = \sum_{j=1}^{k} \beta_{i_j}$. □

An immediate corollary of this result is

Corollary 2.1 (Marginal pdf). *Consider the pdf in equation* (2.5). *Then its marginal on* Q_i *is given by*

$$f_{Q_i}(q_i) = q_i^{\nu\beta_i-1}(1-q_i)^{\nu(\bar{\beta}-\beta_i)-1} \times \int_0^\infty y^{\nu\bar{\beta}-1} E^{\beta_i}_{\nu,\nu\beta_i}(-(yq_i)^\nu) E^{\bar{\beta}-\beta_i}_{\nu,\nu(\bar{\beta}-\beta_i)}((-y(1-q_i)^\nu))\,\mathrm{d}y. \tag{2.14}$$

As the three-parameter Mittag-Leffler function has a representation as an H-function [15],

$$H^{m,n}_{p,r}(z) = H^{m,n}_{p,r}\left[z \,\middle|\, \begin{matrix}(a_i, A_i)_{1,p}\\ (b_i, B_i)_{1,r}\end{matrix}\right], \tag{2.15}$$

for suitable choices of (a_i, A_i) and (b_i, B_i), the marginal pdf (2.14) can be expressed in terms of an H-function too.

PROPOSITION 2.3. *If Q_i is the random variable with pdf* (2.14), *then*

$$f_{Q_i}(q_i) = \frac{1}{\nu\Gamma(\beta_i)\Gamma(\bar\beta-\beta_i)} q_i^{\nu\beta_i-1}(1-q_i)^{-(\nu\beta_i+1)} \tag{2.16}$$

$$\times \lim_{\varepsilon\downarrow 0} H^{2,2}_{3,3}\left[\left(\frac{q_i}{1-q_i}\right)^{\nu}\left|\begin{matrix}(1-\beta_i,1) & (1-\bar\beta+\varepsilon,1) & (\nu(\varepsilon-\beta_i),\nu)\\ (0,1) & (\varepsilon-\beta_i,1) & (1-\nu\beta_i,\nu)\end{matrix}\right.\right].$$

P r o o f. In the integral (2.14) set

$$\delta_1=\beta_i,\ \delta_2=\bar\beta-\beta_i,\ \bar q_1=q_i^{\nu},\ \text{ and }\ \bar q_2=(1-q_i)^{\nu}. \tag{2.17}$$

Denote with $I^{\nu\bar\beta}_{\delta_1,\delta_2}$ the resulting integral and observe that

$$I^{\nu\bar\beta}_{\delta_1,\delta_2} = \frac{1}{\nu}\int_0^{\infty} y^{\delta_1+\delta_2-1}E^{\delta_1}_{\nu,\nu\delta_1}(-y\bar q_1)E^{\delta_2}_{\nu,\nu\delta_2}(-y\bar q_2)\,\mathrm{d}y. \tag{2.18}$$

For $\delta>0$ and $\nu\in(0,1]$ we have

$$E^{\delta}_{\nu,\nu\delta}(z) = \frac{1}{\Gamma(\delta)}H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(1-\delta,1)\\ (0,1)\ \ (1-\nu\delta,\nu)\end{matrix}\right.\right]. \tag{2.19}$$

For $\eta\in(0,\bar\beta)$, by using (2.19) and Theorem 2.9 in [12], we have

$$\begin{aligned} I^{\eta}_{\delta_1,\delta_2} &= \int_0^{\infty} y^{\eta-1}E^{\delta_1}_{\nu,\nu\delta_1}(-y\bar q_1)E^{\delta_2}_{\nu,\nu\delta_2}(-y\bar q_2)\,\mathrm{d}y \qquad (2.20)\\ &= \frac{\bar q_2^{-\eta}}{\Gamma(\delta_1)\Gamma(\delta_2)}H^{2,2}_{3,3}\left[\frac{\bar q_1}{\bar q_2}\left|\begin{matrix}(1-\delta_1,1) & (1-\eta,1) & (\nu(\delta_2-\eta),\nu)\\ (0,1) & (\delta_2-\eta,1) & (1-\nu\delta_1,\nu)\end{matrix}\right.\right]. \end{aligned}$$

Set $\eta=\bar\beta-\varepsilon$ in (2.20) with $\varepsilon\in(0,\bar\beta)$, and use (2.17) to recover $I^{\bar\beta-\varepsilon}_{\beta_i,\bar\beta-\beta_i}$. If ε is sufficiently small, the poles $-l,\beta_i-\varepsilon-l,(\nu\beta_i-1-l)/\nu,l=0,1,2,\ldots$, do not coincide with the poles $\beta_i+k,\bar\beta-\varepsilon+k,(\nu(\varepsilon-\beta_i)+k)/\nu,k=0,1,2,\ldots$. Then, according to Theorem 1.1 in [12], the H-function in (2.16) makes sense for all $q_i\in(0,1)$ as $A_1+A_2-A_3+B_1+B_2-B_3=4-2\nu>0$. The claim follows by taking the limit as $\varepsilon\downarrow 0$ of $I^{\bar\beta-\varepsilon}_{\beta_i,\bar\beta-\beta_i}$. □

REMARK 2.4. By using Properties 2.1, 2.3 and 2.5 of [12], the H-function in (2.16) can be rewritten interchanging β_i with $\bar\beta-\beta_i$ and q_i with $1-q_i$, which corresponds to commuting the two Mittag-Leffler functions in (2.14).

According to Theorems 1.3 and 1.4 in [12], since

$$\sum_{i=1}^{3}(B_i - A_i) = 0, \qquad \prod_{i=1}^{3}\frac{B_i^{B_i}}{A_i^{A_i}} = 1, \tag{2.21}$$
$$\sum_{i=1}^{3}(b_i - a_i) = \bar{\beta}(1-\nu) + \nu(\bar{\beta} - \varepsilon) > 0,$$

the H-function in (2.16) has a power series expansion. The following propositions rely on this property.

PROPOSITION 2.4. *For $q_i < 1/2$ and β_i not a positive integer*

$$f_{Q_i}(q_i) = \frac{q_i^{\nu\beta_i - 1}(1-q_i)^{-(\nu\beta_i+1)}}{\nu\Gamma(\beta_i)\Gamma(\bar{\beta}-\beta_i)}$$
$$\times \left[\Gamma(\bar{\beta})\frac{\Gamma(-\beta_i)\Gamma(\beta_i)}{\Gamma(-\nu\beta_i)\Gamma(\nu\beta_i)} + \sum_{k=1}^{\infty}(-1)^k D_k \left(\frac{q_i}{1-q_i}\right)^{\nu k}\right], \tag{2.22}$$

where

$$D_k = \frac{\Gamma(\bar{\beta}+k)\Gamma(-\beta_i - k)\,\Gamma(\beta_i + k)}{k!\,\Gamma(-\nu(\beta_i+k))\,\Gamma(\nu(\beta_i+k))} + \frac{(1-q_i)^{\nu\beta_i}\Gamma(\beta_i - k)\,\Gamma(\bar{\beta}-\beta_i+k)}{q_i^{\nu\beta_i}\Gamma(-\nu k)\,\Gamma(\nu k)}. \tag{2.23}$$

P r o o f. Consider the H-function $H_{3,3}^{2,2}$ in (2.16). If β_i is not a positive integer, we have $B_1(b_2+l) \neq B_2(b_1+k)$ for $l, k = 0, 1, 2, \ldots$. Thus, thanks to (2.21), from Theorem 1.3 of [12], $H_{3,3}^{2,2}$ is an analytical function in $q_i^{\nu}/(1-q_i)^{\nu}$ and has the following power series expansion for $q_i < 1/2$:

$$\sum_{k=0}^{\infty}\frac{\Gamma(b_2-k)\,\Gamma(1-a_1+k)\,\Gamma(1-a_2+k)}{\Gamma(1-b_3+\nu k)\,\Gamma(a_3-\nu k)}\frac{(-1)^k}{k!}\left(\frac{q_i}{1-q_i}\right)^{\nu k} + \left(\frac{q_i}{1-q_i}\right)^{\nu b_2}$$
$$+\sum_{k=0}^{\infty}\frac{\Gamma(-b_2-k)\,\Gamma(1-a_1+b_2+k)\,\Gamma(1-a_2+b_2+k)}{\Gamma(1-b_3+\nu(b_2+k))\,\Gamma(a_3-\nu(b_2+k))}\frac{(-1)^k}{k!}\left(\frac{q_i}{1-q_i}\right)^{\nu k}. \tag{2.24}$$

The claim follows replacing $a_1 = 1-\beta_i, a_2 = 1-\bar{\beta}+\varepsilon, a_3 = \nu(\varepsilon-\beta_i), b_2 = \varepsilon - \beta_i$ and $b_3 = 1-\nu\beta_i$ in (2.24) and taking the limit as $\varepsilon \downarrow 0$. □

REMARK 2.5. If $\nu(\beta_i + k)$ is not a positive integer for $k = 0, 1, 2, \ldots$ thanks to the reflection formula for the gamma function [12], we might simplify the expansion in (2.22) using

$$\frac{\Gamma(-\beta_i-k)\,\Gamma(\beta_i+k)}{\Gamma(-\nu(\beta_i+k))\,\Gamma(\nu(\beta_i+k))}=\nu\frac{\sin(\pi\nu(\beta_i+k))}{\sin(\pi(\beta_i+k))}. \tag{2.25}$$

Similarly we get $\Gamma(-\nu k)\,\Gamma(\nu k)=-\pi/(k\nu\sin(\pi\nu k))$ if νk is not a positive integer.

PROPOSITION 2.5. *For $q_i>1/2$ and $\bar\beta-\beta_i$ not a positive integer*

$$f_{Q_i}(q_i)=\frac{q_i^{-(\nu(\bar\beta-\beta_i)+1)}(1-q_i)^{\nu(\bar\beta-\beta_i)-1}}{\nu\Gamma(\beta_i)\Gamma(\bar\beta-\beta_i)}$$

$$\times\left[\Gamma(\bar\beta)\frac{\Gamma(-(\bar\beta-\beta_i))\Gamma(\bar\beta-\beta_i)}{\Gamma(-\nu(\bar\beta-\beta_i))\Gamma(\nu(\bar\beta-\beta_i))}+\sum_{k=1}^{\infty}(-1)^kD_k\left(\frac{1-q_i}{q_i}\right)^{\nu k}\right], \tag{2.26}$$

where

$$\begin{aligned}D_k=&\frac{\Gamma(\bar\beta+k)}{k!}\frac{\Gamma(-(\bar\beta-\beta_i+k))\Gamma(\bar\beta-\beta_i+k)}{\Gamma(-\nu(\bar\beta-\beta_i+k))\Gamma(\nu(\bar\beta-\beta_i+k))}\\&+\left(\frac{q_i}{1-q_i}\right)^{\nu(\bar\beta-\beta_i)}\frac{\Gamma(\beta_i+k)\,\Gamma(\bar\beta-\beta_i-k)}{\Gamma(-\nu k)\,\Gamma(\nu k)}.\end{aligned} \tag{2.27}$$

P r o o f. Consider again the H-function $H_{3,3}^{2,2}$ in (2.16). If $\bar\beta-\beta_i$ is not a positive integer, we have $A_1(1-a_2+l)\neq A_2(1-a_1+k)$ for $l,k=0,1,2,\dots$ From (2.21) and Theorem 1.4 of [12], $H_{3,3}^{2,2}$ is an analytical function in $q_i^\nu/(1-q_i)^\nu$ and has the following power series expansion for $q_i>1/2$:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\Gamma(1-a_1+k)\,\Gamma(b_2+1-a_1+k)\,\Gamma(a_1-a_2-k)}{\Gamma(a_3+\nu(1-a_1+k))\,\Gamma(1-b_3-\nu(1-a_1+k))}\frac{(-1)^k}{k!}\left[\frac{q_i}{1-q_i}\right]^{\nu(a_1-1-k)}\\&+\sum_{k=0}^{\infty}\frac{\Gamma(1-a_2+k)\,\Gamma(b_2+1-a_2+k)\,\Gamma(a_2-a_1-k)}{\Gamma(a_3+\nu(1-a_2+k))\,\Gamma(1-b_3-\nu(1-a_2+k))}\frac{(-1)^k}{k!}\left[\frac{q_i}{1-q_i}\right]^{\nu(a_2-1-k)}\end{aligned} \tag{2.28}$$

The claim follows replacing $a_1=1-\beta_i, a_2=1-\bar\beta+\varepsilon, a_3=\nu(\varepsilon-\beta_i), b_2=\varepsilon-\beta_i$ and $b_3=1-\nu\beta_i$ in (2.28) and taking the limit as $\varepsilon\downarrow 0$. □

By using the reflection formula for the gamma function, also the expansion (2.26) might be simplified similarly to what has been addressed in Remark 2.5.

3. **An alternative generalization**

We now give an alternative generalization with desirable properties which in addition approximates the fractional Dirichlet distribution with density (2.5) for appropriate values of the parameters.

Let us thus consider a random vector $\boldsymbol{Q} = (Q_1, \ldots, Q_{n-1})$, $n \geq 2$, with the following probability density function:

$$f_{\boldsymbol{Q}}(q_1, \ldots, q_{n-1}) = \left(\prod_{i=1}^{n-1} q_i^{\nu\beta_i - 1} \right) \left(1 - \sum_{i=1}^{n-1} q_i \right)^{\nu\beta_n - 1} \times \frac{\nu^{n-1}\Gamma(\bar{\beta})}{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_n)} \left(q_1^{\nu} + \ldots + \left(1 - \sum_{i=1}^{n-1} q_i \right)^{\nu} \right)^{-\bar{\beta}}, \tag{3.1}$$

for $q_1, \ldots, q_{n-1} \in (0,1)$, $q_1 + \ldots + q_{n-1} < 1$, $\nu > 0$, $\beta_i > 0$, $i = 1, \ldots, n$, $\bar{\beta} = \beta_1 + \cdots + \beta_n$.

For the sake of clarity we check that $f_{\boldsymbol{Q}}(q_1, \ldots, q_{n-1})$ as given in (3.1) is a genuine probability density function. This will follow by proving that

$$\frac{\nu^{n-1}\,\Gamma(\bar{\beta})}{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_n)} \tag{3.2}$$

in the rhs of (3.1) plays the role of a normalization coefficient.

THEOREM 3.1. *We have*

$$\int_0^1 \mathrm{d}q_1 \cdots \int_0^{1-q_1-\cdots-q_{n-2}} \mathrm{d}q_{n-1} \left(\prod_{i=1}^{n-1} q_i^{\nu\beta_i - 1} \right) \left(1 - \sum_{i=1}^{n-1} q_i \right)^{\nu\beta_n - 1} \times \left(q_1^{\nu} + \ldots + \left(1 - \sum_{i=1}^{n-1} q_i \right)^{\nu} \right)^{-\bar{\beta}} = \frac{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_n)}{\nu^{n-1}\,\Gamma(\bar{\beta})}. \tag{3.3}$$

P r o o f. Observe that the lhs of (3.3) can be rewritten as

$$I = \int_0^1 \mathrm{d}q_1 \cdots \int_0^{1-q_1-\cdots-q_{n-2}} \mathrm{d}q_{n-1} \left(\prod_{i=1}^{n-1} q_i^{-1} \right) \times (1-\bar{q})^{-1} \prod_{i=1}^{n-1} \left(\frac{q_i}{1-\bar{q}} \right)^{\nu\beta_i} \left(1 + \sum_{i=1}^{n-1} \left(\frac{q_i}{1-\bar{q}} \right)^{\nu} \right)^{-\bar{\beta}}, \tag{3.4}$$

where $\bar{q} = q_1 + \cdots + q_{n-1}$. Apply the change of variables $(1-\bar{q})/q_i = z_i$ for $i = 1, \ldots, n-1$, in multivariate integration. Thus, we have

$$q_i = \frac{\prod_{j\in\mathcal{I}_{n-1,i}} z_j}{\prod_{j=1}^{n-1} z_j + \sum_{k=1}^{n-1}\prod_{j\in\mathcal{I}_{n-1,k}} z_j}, \qquad i = 1, \ldots, n-1, \tag{3.5}$$

$$J = \frac{\prod_{j=1}^{n-1} z_j^{n-2}}{\left(\prod_{j=1}^{n-1} z_j + \sum_{k=1}^{n-1}\prod_{j\in\mathcal{I}_{n-1,k}} z_j\right)^n},$$

where $\mathcal{I}_{n-1,k} = \{1, \ldots, k-1, k+1, \ldots, n-1\}$ for $k = 1, \ldots, n-1$, and J is the Jacobian of the transformation. Note that

$$1 - \bar{q} = \frac{\prod_{j=1}^{n-1} z_j}{\left(\prod_{j=1}^{n-1} z_j + \sum_{k=1}^{n-1}\prod_{j\in\mathcal{I}_{n-1,k}} z_j\right)}. \tag{3.6}$$

By putting (3.5) and (3.6) in (3.4) we have

$$I = \int_0^\infty \mathrm{d}z_1 \cdots \int_0^\infty \mathrm{d}z_{n-1} \prod_{i=1}^{n-1} z_j^{-\nu\beta_j - 1} \left(1 + \sum_{j=1}^{n-1} \frac{1}{z_j^\nu}\right)^{-\bar{\beta}}. \tag{3.7}$$

Apply the change of variables $z_i^\nu = t_i$ for $i = 1, ..., n-1$, in multivariate integration. Then, we have $z_i = t_i^{1/\nu}$ for $i = 1, ..., n-1$, and $\nu^{1-n}\prod_{i=1}^{n-1} t_i^{1/\nu - 1}$ is the Jacobian of this transformation. From (3.7)

$$I = \frac{1}{\nu^{n-1}} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_{n-2}\, I_{n-2}(t_1, \ldots, t_{n-2}), \tag{3.8}$$

where

$$I_{n-2}(t_1, \ldots, t_{n-2}) = \int_0^\infty \mathrm{d}t_{n-1} \prod_{i=1}^{n-1} t_j^{-\beta_j - 1} \left(1 + \sum_{j=1}^{n-1} \frac{1}{t_j}\right)^{-\bar{\beta}}. \tag{3.9}$$

Observe that $I_{n-2}(t_1, \ldots, t_{n-2})$ in (3.9) can be rewritten as

$$\begin{aligned} & I_{n-2}(t_1, \ldots, t_{n-2}) \qquad (3.10)\\ &= \int_0^\infty \mathrm{d}t_{n-1} \prod_{j=1}^{n-1} t_j^{\bar{\beta}-\beta_j-1} \left(\prod_{j=1}^{n-1} t_j + \sum_{k=1}^{n-1} \prod_{j\in\mathcal{I}_{n-1,k}} t_j\right)^{-\bar{\beta}} \\ &= \prod_{i=1}^{n-2} t_i^{-\beta_i - 1} \\ &\quad \times \int_0^\infty \mathrm{d}t_{n-1} \left(1 + t_{n-1}\frac{\prod_{i=1}^{n-2} t_i + \sum_{k=1}^{n-2}\prod_{i\in\mathcal{I}_{n-2,k}} t_i}{\prod_{i=1}^{n-2} t_i}\right)^{-\bar{\beta}} t_{n-1}^{\bar{\beta}-\beta_{n-1}-1}. \end{aligned}$$

With the change of variable $z = t_{n-1}(\prod_{i=1}^{n-2} t_i + \sum_{k=1}^{n-2} \prod_{i\in\mathcal{I}_{n-2,k}} t_i)/\prod_{i=1}^{n-2} t_i$ and by recalling the Mellin trasform of $(1+z)^{-\bar{\beta}}$, we recover

$$\begin{aligned} &I_{n-2}(t_1,\dots,t_{n-2}) \\ &= \frac{\prod_{i=1}^{n-2} t_i^{\bar{\beta}-\beta_i-\beta_{n-1}-1}}{(\prod_{i=1}^{n-2} t_i + \sum_{k=1}^{n-2} \prod_{i\in\mathcal{I}_{n-2,k}} t_i)^{\bar{\beta}-\beta_{n-1}}} \frac{\Gamma(\bar{\beta}-\beta_{n-1})\Gamma(\beta_{n-1})}{\Gamma(\bar{\beta})}. \end{aligned} \tag{3.11}$$

Now, replace $I_{n-2}(t_1,\dots,t_{n-2})$ in (3.8) with the closed form (3.11). This leads us to

$$I = \frac{1}{\nu^{n-1}} \frac{\Gamma(\bar{\beta}-\beta_{n-1})\Gamma(\beta_{n-1})}{\Gamma(\bar{\beta})} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_{n-3} I_{n-3}(t_1,\dots,t_{n-3}), \tag{3.12}$$

where

$$\begin{aligned} &I_{n-3}(t_1,\dots,t_{n-3}) \\ &= \int_0^\infty \mathrm{d}t_{n-2} \prod_{j=1}^{n-2} t_j^{\bar{\beta}-\beta_{n-1}-\beta_j-1} \left(\prod_{j=1}^{n-2} t_j + \sum_{k=1}^{n-2} \prod_{j\in\mathcal{I}_{n-2,k}} t_j \right)^{-(\bar{\beta}-\beta_{n-1})}. \end{aligned} \tag{3.13}$$

By comparing the integral in (3.13) with that in (3.10), we observe that the former has the same expression of the latter with $\bar{\beta}$ replaced by $\bar{\beta} - \beta_{n-1}$. Thus, by recurring to the same arguments employed to compute $I_{n-2}(t_1,\dots,t_{n-2})$ we recover

$$\begin{aligned} I_{n-3}(t_1,\dots,t_{n-3}) = &\frac{\prod_{i=1}^{n-3} t_i^{\bar{\beta}-\beta_{n-1}-\beta_{n-2}-\beta_i-1}}{(\prod_{i=1}^{n-3} t_i + \sum_{k=1}^{n-3} \prod_{i\in\mathcal{I}_{n-3,k}} t_i)^{\bar{\beta}-\beta_{n-1}-\beta_{n-2}}} \\ &\times \frac{\Gamma(\bar{\beta}-\beta_{n-1}-\beta_{n-2})\Gamma(\beta_{n-2})}{\Gamma(\bar{\beta}-\beta_{n-1})}. \end{aligned} \tag{3.14}$$

Replacing $I_{n-3}(t_1,\dots,t_{n-3})$ in (3.12) with the closed form (3.14) we get

$$\begin{aligned} I = &\frac{1}{\nu^{n-1}} \frac{\Gamma(\bar{\beta}-\beta_{n-1}-\beta_{n-2})\Gamma(\beta_{n-1})\Gamma(\beta_{n-2})}{\Gamma(\bar{\beta})} \\ &\times \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_{n-4} I_{n-4}(t_1,\dots,t_{n-4}), \end{aligned} \tag{3.15}$$

where

$$I_{n-4}(t_1,\dots,t_{n-4}) = \int_0^\infty \mathrm{d}t_{n-3} \prod_{j=1}^{n-3} t_j^{\bar{\beta}-\beta_{n-1}-\beta_{n-2}-\beta_j-1} \tag{3.16}$$

$$\times \left(\prod_{j=1}^{n-3} t_j + \sum_{k=1}^{n-3} \prod_{j \in \mathcal{I}_{n-3,k}} t_j \right)^{-(\bar{\beta}-\beta_{n-1}-\beta_{n-2})},$$

which indeed has the same expression of I_{n-2} and I_{n-3} with suitable updates of $\bar{\beta}$. The result follows by iterating from $i = 4$ up to $i = n-1$ the computation of

$$I = \frac{1}{\nu^{n-1}} \frac{\Gamma(\bar{\beta} - \sum_{k=n-i+2}^{n-1} \beta_k) \prod_{k=n-i+2}^{n-1} \Gamma(\beta_k)}{\Gamma(\bar{\beta})} \times \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_{n-i} I_{n-i}(t_1, \ldots, t_{n-i}) \tag{3.17}$$

with

$$I_{n-i}(t_1, \ldots, t_{n-i}) = \int_0^\infty \mathrm{d}t_{n-i+1} \prod_{j=1}^{n-i+1} t_j^{\bar{\beta}-\sum_{k=n-i+2}^{n-1}\beta_k-\beta_j-1} \times \left(\prod_{j=1}^{n-i+1} t_j + \sum_{k=1}^{n-i+1} \prod_{j \in \mathcal{I}_{n-i+1,k}} t_j \right)^{-(\bar{\beta}-\sum_{k=n-i+2}^{n-1}\beta_k)}. \tag{3.18}$$

We obtain the closed form expression

$$I_{n-i}(t_1, \ldots, t_{n-i}) = \frac{\prod_{j=1}^{n-i} t_j^{\bar{\beta}-\sum_{k=n-i+1}^{n-1}\beta_k-\beta_j-1}}{(\prod_{i=1}^{n-i} t_i + \sum_{k=1}^{n-i} \prod_{i \in \mathcal{I}_{n-i,k}} t_i)^{\bar{\beta}-\sum_{k=n-i+1}^{n-1}\beta_k}} \times \frac{\Gamma(\bar{\beta} - \sum_{k=n-i+i}^{n-1} \beta_k)\Gamma(\beta_{n-i+1})}{\Gamma(\bar{\beta} - \sum_{k=n-i+2}^{n-1} \beta_k)} \tag{3.19}$$

with $\sum_{k=1}^{n-i} \prod_{i \in \mathcal{I}_{n-i,k}} t_i = 1$ for $i = n-1$. The last replacement with $I_1(t_1)$ gives

$$I = \frac{1}{\nu^{n-1}} \frac{\Gamma(\beta_1 + \beta_n)\Gamma(\beta_2) \cdot \ldots \cdot \Gamma(\beta_{n-1})}{\Gamma(\bar{\beta})} \int_0^\infty \mathrm{d}t_1 (1+t_1)^{-(\beta_1+\beta_n)} t_1^{\beta_n-1} \tag{3.20}$$

from which the claimed result follows by observing that

$$\int_0^\infty \mathrm{d}t_1 t_1^{\beta_n-1}(1+t_1)^{-(\beta_1+\beta_n)} = \Gamma(\beta_n)\Gamma(\beta_1)/\Gamma(\beta_1+\beta_n).$$

□

REMARK 3.1. Alternatively, in (3.7) use the transformation $z_1^{-1} = t_1, \ldots, z_n^{-1} = t_n$. Then, we have (cf. [9] no. 4.638/3, p. 649)

$$\begin{aligned} I &= \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_{n-1} \frac{\prod_{i=1}^{n-1} t_j^{\nu\beta_j - 1}}{(1 + t_1^\nu + \cdots + t_{n-1}^\nu)^{\bar{\beta}}} \\ &= \frac{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_{n-1})}{\nu^{n-1}} \, \frac{\Gamma(\bar{\beta} - \beta_1 - \cdots - \beta_{n-1})}{\Gamma(\bar{\beta})}, \end{aligned} \tag{3.21}$$

which is in agreement with (3.3).

REMARK 3.2. On the n-dimensional simplex Δ_n the pdf of the random vector $(Q_1, \ldots, Q_n)$, where $\sum_n Q_n = 1$ a.s., writes

$$\begin{aligned} &\mathbb{P}\left((Q_1, \ldots, Q_n) \in \mathrm{d}(q_1, \ldots, q_n)\right) \\ &= \frac{\nu^{n-1}\,\Gamma(\bar{\beta})}{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_n)} \, (q_1^\nu + \ldots + q_n^\nu)^{-\bar{\beta}} \prod_{i=1}^n q_i^{\nu\beta_i - 1} \\ &= \frac{\nu^{n-1}}{B(\boldsymbol{\beta})} \, \prod_{i=1}^n q_i^{-1} \left(\frac{q_i^\nu}{\sum_{i=1}^n q_i^\nu} \right)^{\beta_i}, \end{aligned} \tag{3.22}$$

with $B(\boldsymbol{\beta}) = \prod_{i=1}^n \Gamma(\beta_i) / \Gamma(\sum_{i=1}^n \beta_i)$. In short we write $(Q_1, \ldots, Q_n) \sim \mathrm{GDIR}(\nu, \boldsymbol{\beta})$.

Notice that for $\nu = 1$ the Dirichlet($\boldsymbol{\beta}$) is obtained. In this case the random vector $(Q_1, \ldots, Q_n)$ is uniformly distributed on Δ_n for $\beta_i = 1$, $i \in \mathbb{N}^*$. If instead we only let $\beta_i = 1$,

$$\mathbb{P}\left((Q_1, \ldots, Q_n) \in \mathrm{d}(q_1, \ldots, q_n)\right) = \frac{\nu^{n-1}\,(n-1)!}{(q_1^\nu + \ldots + q_n^\nu)^n} \, \prod_{i=1}^n q_i^{\nu-1} \tag{3.23}$$

which is symmetric but clearly not uniform. If $\beta_i = 1/\nu$ (again symmetric) we obtain

$$\mathbb{P}\left((Q_1, \ldots, Q_n) \in \mathrm{d}(q_1, \ldots, q_n)\right) = \frac{\nu^{n-1}\,\Gamma(n/\nu)}{\Gamma(1/\nu)^n} \, (q_1^\nu + \cdots + q_n^\nu)^{-n/\nu}. \tag{3.24}$$

REMARK 3.3. The alternative generalized Dirichlet distribution considered in this section (i.e. that with pdf (3.1)) can be derived by the same procedure described in Section 2 with $(Z_i)^\nu$ distributed as Gamma($\beta_i, 1$), $i = 1, \ldots, n-1$. Note that the random variable X such that X^ν, $\nu > 0$, is Gamma($\alpha, 1$)-distributed, $\alpha > 0$, is a special case of the *generalized Gamma* distribution (see e.g. [11], Section 8.7). In particular, X has pdf

$$f_X(x) = \nu \frac{x^{\nu\alpha-1}e^{-x^\nu}}{\Gamma(\alpha)} \mathbf{1}_{\mathbb{R}_+}, \tag{3.25}$$

and Laplace transform (from (2.3.23) of [15] and the definition of Wright functions)

$$\mathbb{E}e^{-zX} = \frac{\nu z^{-\nu\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-z^{-\nu})^k}{k!} \Gamma(\nu(k+\alpha)). \tag{3.26}$$

Remark 3.4. The generalized Dirichlet pdf (3.1) turns out to be a reasonably good approximation of the fractional Dirichlet pdf (2.5) for $\beta_i < 1$ (see for example Fig. 3). A partial explanation is that for $\lambda = 1, \beta < 1$, and $\nu \in (0, 1]$ the fractional Gamma pdf (2.1) has a rather similar shape to the generalized Gamma pdf (3.25), as Fig. 1 shows for $\beta = 0.2$ and $\beta = 0.4$. For $\beta_i > 1$, the fractional Dirichlet pdf exhibits a behaviour different from the generalized Dirichlet pdf (see for example Fig. 2). Indeed, Fig. 1 shows a different shape of the fractional Gamma pdf compared to the generalized Gamma pdf for $\beta = 2$ and $\beta = 3$.

Proposition 3.1 (Conjugate distribution). *The generalized Dirichlet distribution GDIR$(\nu, \boldsymbol{\beta})$ (with pdf (3.22)) is the conjugate prior to a reparametrized Multinomial distribution with pmf*

$$\frac{N}{x_1! \cdot \ldots \cdot x_n!} \left(\prod_{i=1}^{n} q_i^{\nu x_i} \right) (q_1^\nu + \ldots + q_n^\nu)^{-\sum_{i=1}^n x_i}, \tag{3.27}$$

where $N \in \mathbb{N}^+$, $x_i \in \{0, \ldots, N\}$, $i = 1, \ldots, n$, $n \in \mathbb{N}^+$, $q_1 + \ldots + q_n = 1$, $\nu > 0$. In particular, if the prior is GDIR$(\nu, \boldsymbol{\beta})$ and the likelihood is as in (3.27), then the posterior becomes GDIR$(\nu, \boldsymbol{\beta} + \boldsymbol{x})$.

P r o o f. The proof is a straightforward application of Bayes theorem. The reparametrization in (3.27) is such that $p_i = q_i^\nu / (\sum_{i=1}^n q_i^\nu)$, $i = 1, \ldots, n$, are the event probabilities (i.e. $\sum_{i=1}^n p_i = 1$). □

3.1. **Representation in terms of Dirichlet random variables.** In order to derive a meaningful representation in terms of Dirichlet random variables for the random vector $\boldsymbol{Q}$, we first recall the definitions of two related classes of random vectors (see [10]).

Definition 3.1 (Liouville distribution of the first kind). Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be an absolutely continuous random vector supported on the

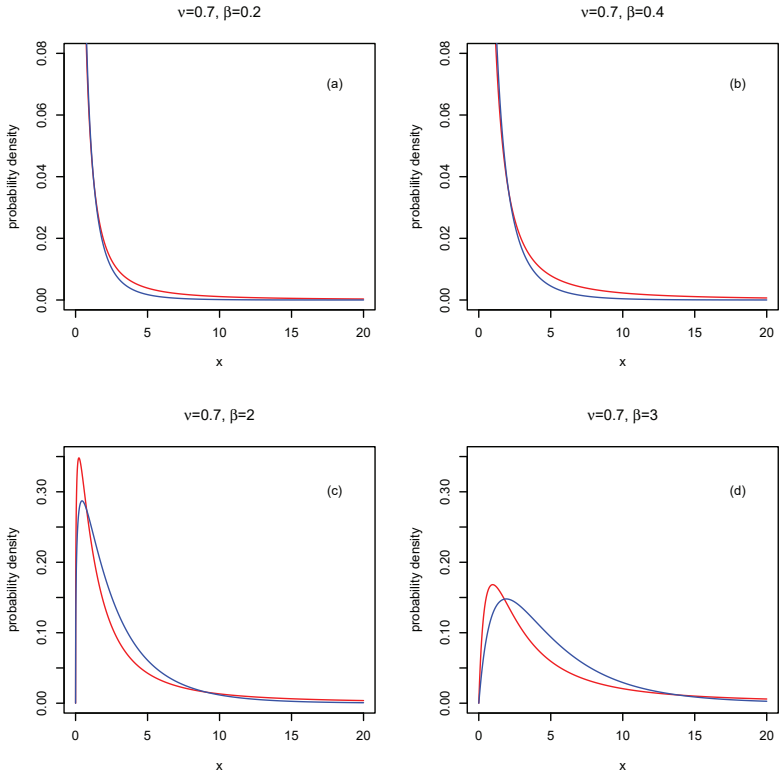

FIGURE 1. Comparison of the fractional Gamma pdf (2.1) (red line) versus the generalized Gamma pdf (3.25) (blue line) for $\nu = 0.7$ and $\beta = 0.2$ in (a) and $\beta = 0.4$ in (b), as in Fig. 3, $\beta = 2$ in (c) and $\beta = 3$ in (d), as in Fig. 2.

n-dimensional positive orthant, i.e. $R^n = \{(x_1, \ldots, x_n) \colon x_i > 0 \text{ for each } i = 1, \ldots, n\}$. It is said to have *Liouville distribution of the first kind* if its joint pdf writes

$$f_{\boldsymbol{X}}(x_1, \ldots, x_n) \propto f\left(\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i^{a_i - 1}, \tag{3.28}$$

where $a_i > 0$, $i = 1, \ldots, n$, and f is a positive continuous function satisfying $\int_{\mathbb{R}_+} y^{a-1} f(y) \mathrm{d}y < \infty$, with $a = a_1, \ldots, a_n$. Further, we write $\boldsymbol{X} \sim L_n^{(1)}[f(\cdot); a_1, \ldots, a_n]$.

DEFINITION 3.2 (Liouville distribution of the second kind). Let $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ be an absolutely continuous random vector supported on $S_n = \{(z_1, \ldots, z_n)\colon z_i > 0 \text{ for each } i = 1, \ldots, n, \sum_{i=1}^n z_i < 1\}$. It is said to have *Liouville distribution of the second kind* if its joint pdf writes

$$f_{\boldsymbol{Z}}(z_1, \ldots, z_n) \propto g\left(\sum_{i=1}^n z_i\right) \prod_{i=1}^n z_i^{c_i - 1}, \tag{3.29}$$

where $c_i > 0$, $i = 1, \ldots, n$, and g is a positive continuous function satisfying $\int_{\mathbb{R}_+} y^{c-1} g(y) \mathrm{d}y < \infty$, with $c = c_1, \ldots, c_n$. Further, we write $\boldsymbol{Z} \sim L_n^{(2)}[g(\cdot); c_1, \ldots, c_n]$.

REMARK 3.5. If we let $f(t) = (1+t)^{-(a+a_{n+1})}$, $t > 0$, $a_{n+1} > 0$, in (3.28), then $\boldsymbol{X}$ is distributed as an inverted Dirichlet.

If, in (3.29), we choose $g(t) = (1-t)^{c_{n+1}-1}$, $0 < t < 1$, $a_{n+1} > 0$, we have that $\boldsymbol{Z}$ is distributed as a Dirichlet.

Proposition 3.1 of [10] tells us what is the relationship between Liouville distributions of the first and of the second kind (and hence between the Dirichlet and the inverted Dirichlet). Specifically, if $\boldsymbol{Z} \sim L_n^{(2)}[g(\cdot); c_1, \ldots, c_n]$ and we consider the transformation

$$X_i = \frac{Z_i}{1 - \sum_{i=1}^n Z_i}, \qquad i = 1, \ldots, n, \tag{3.30}$$

then $\boldsymbol{X} \sim L_n^{(1)}[f(\cdot); c_1, \ldots, c_n]$, where

$$f(t) = (1+t)^{-(c+1)} g\left(\frac{t}{1+t}\right), \qquad t > 0. \tag{3.31}$$

Plainly, the converse relation is true as well: inverting (3.31) (letting $h = t/(1+t)$) we have

$$g(h) = \left(\frac{1}{1-h}\right) f\left(\frac{h}{1-h}\right), \qquad 0 < h < 1. \tag{3.32}$$

As a simple example, considering $f(t) = (1+t)^{-(c+c_{n+1})}$, $t > 0$ (inverted Dirichlet), we readily obtain $g(1-h)^{c_{n+1}-1}$ (Dirichlet).

Now, by exploiting the above definition we prove the following distributional representation for $\boldsymbol{Q}$.

PROPOSITION 3.2. *Let $\boldsymbol{Q} = (Q_1, \ldots, Q_{n-1})$ be distributed with pdf (3.1). Then the random vector $\boldsymbol{M} = (M_1, \ldots, M_{n-1})$ such that*

$$M_i = \frac{\left(\frac{Q_i}{1-\sum_{i=1}^{n-1} Q_i}\right)^{\nu}}{1 + \left(\frac{Q_i}{1-\sum_{i=1}^{n-1} Q_i}\right)^{\nu}}, \qquad i = 1, \ldots, n-1, \tag{3.33}$$

is distributed as a Dirichlet($\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$).

Conversely, if $\boldsymbol{M} \sim$ Dirichlet($\boldsymbol{\beta}$) we have that $\boldsymbol{Q} = (Q_1, \ldots, Q_{n-1})$ such that

$$Q_i = \frac{\left(\frac{M_i}{1-\sum_{i=1}^{n-1} M_i}\right)^{\frac{1}{\nu}}}{1 + \left(\frac{M_i}{1-\sum_{i=1}^{n-1} M_i}\right)^{\frac{1}{\nu}}}, \qquad i = 1, \ldots, n-1, \tag{3.34}$$

is distributed with pdf (3.1).

P r o o f. Let us define the random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_{n-1})$ such that

$$Y_i = \left(\frac{Q_i}{1 - \sum_{i=1}^{n-1} Q_i}\right)^{\nu}, \qquad i = 1, \ldots, n-1, \tag{3.35}$$

and let $\beta^* = \beta_1 + \ldots + \beta_{n-1}$. Combining the transformations in the proof of Theorem 3.3 and of Remark 3.1 we see that $\boldsymbol{Y}$ has pdf

$$f_{\boldsymbol{Y}}(y_1, \ldots, y) = \frac{\Gamma(\beta^* + \beta_n)}{\Gamma(\beta_1) \cdot \ldots \cdot \Gamma(\beta_n)} \left(1 + \sum_{i=1}^{n-1} y_i\right)^{-\beta^* - \beta_n} \prod_{i=1}^{n-1} y_i^{\beta_i - 1}, \tag{3.36}$$

and hence $\boldsymbol{Y} \sim L_{n-1}^{(1)}[(1+\cdot)^{-(\beta^* + \beta_n)}; \beta_1, \ldots, \beta_{n-1}]$ (i.e. an inverted Dirichlet distribution).

By using the mentioned transformations between Liouville distributions of first and second kinds we have that $\boldsymbol{M} \sim L_{n-1}^{(2)}[(1-\cdot)^{\beta_n - 1}; \beta_1, \ldots, \beta_{n-1}]$ (Dirichlet), where

$$M_i = \frac{Y_i}{1 + \sum_{i=1}^{n-1} Y_i}, \qquad i = 1, \ldots, n-1, \tag{3.37}$$

and (3.33) follows.

Finally, a rewriting of the components of $\boldsymbol{Q}$ in terms of those of $\boldsymbol{Y}$, leads easily to (3.34). □

4. Monte Carlo simulations

The simulation of the random variables $\boldsymbol{Q}$ for the fractional Dirichlet distribution is straightforward based on the construction presented in Section 2. First, one needs to generate random variables with density (2.1) and one can use the mixture representation discussed in [4]

$$X_i \stackrel{d}{=} U_i^{1/\nu} V_\nu,$$

where U_i is Gamma(β_i, λ)-distributed and V_ν is strictly positive-stable distributed with $\exp(-s^\nu)$ as the Laplace transform of the pdf. Summing the X_i to get W and dividing X_i by W gives Q_i.

Remark 4.1. For $\beta = 1$, there is an alternative representation [13, 6]:

$$X \stackrel{d}{=} \Xi Z^{1/\nu},$$

where Ξ is Exp(λ)-distributed and Z is Cauchy-distributed.

The behaviour of the fractional Dirichlet distribution in the case $N = 2$ is shown in Fig. 2 for $\nu = 0.7$, $\beta_1 = 2$, $\beta_2 = 3$. In this case, the generalized Dirichlet distribution is not a good approximation.

This is not the case for $N = 2$, $\nu = 0.7$, $\beta_1 = 0.2$ and $\beta_2 = 0.4$ where the generalized Dirichlet distribution is a reasonably good approximation of the fractional Dirichlet distribution. This is represented in Fig. 3.

For larger values of the parameters β_i, one gets a unimodal distribution in both cases as shown in Fig. 4 for $N = 2$, $\nu = 0.95$, $\beta_1 = 10$, $\beta_2 = 30$.

In Fig. 5, the heavy character of the right tail of the generalized Dirichlet distribution is highlighted by the log-log plot.

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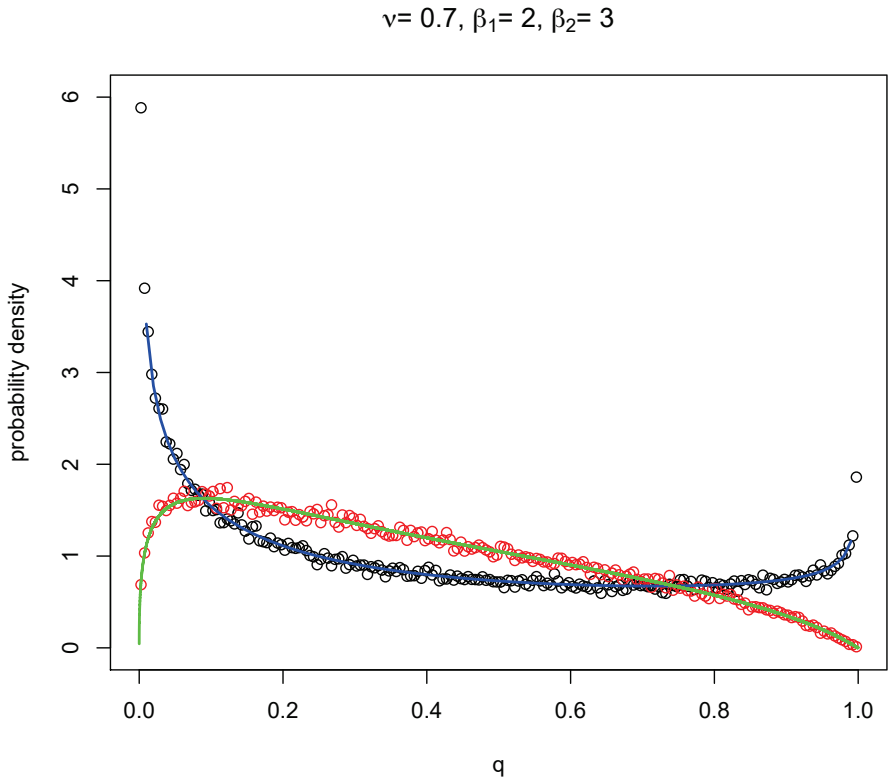

FIGURE 2. Pdf's for $\nu = 0.7$, $\beta_1 = 2$, $\beta_2 = 3$. The black circles represent the results of a Monte Carlo simulation for the fractional Dirichlet distribution and the blue line is the corresponding theoretical value from equation (2.5). The red circles come from a Monte Carlo simulation of the generalized Dirichlet distribution and the green curve is the plot of the theoretical pdf.

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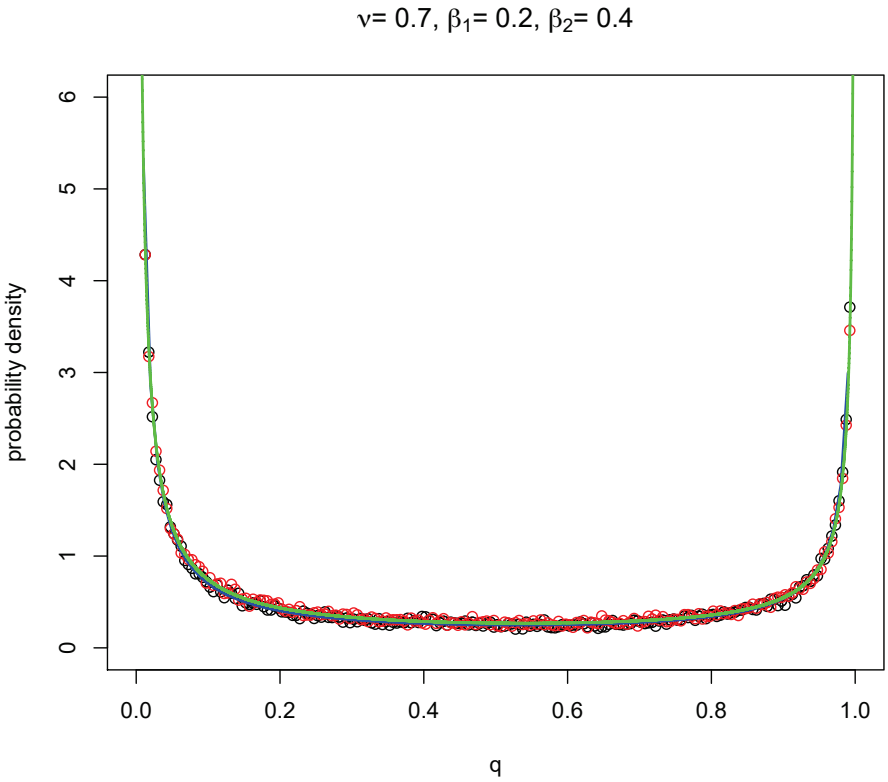

FIGURE 3. Pdf's for $\nu = 0.7$, $\beta_1 = 0.2$, $\beta_2 = 0.4$. Circles and curves have the same meaning as in Fig. 2.

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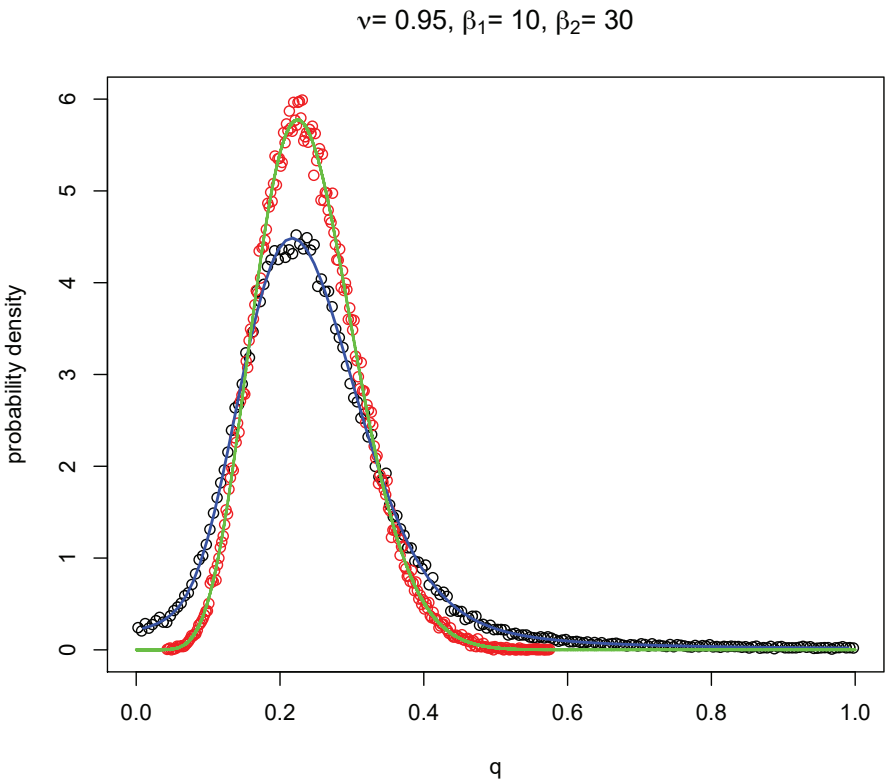

FIGURE 4. Pdf's for $\nu = 0.95$, $\beta_1 = 10$, $\beta_2 = 30$. Circles and curves have the same meaning as in Fig. 2.

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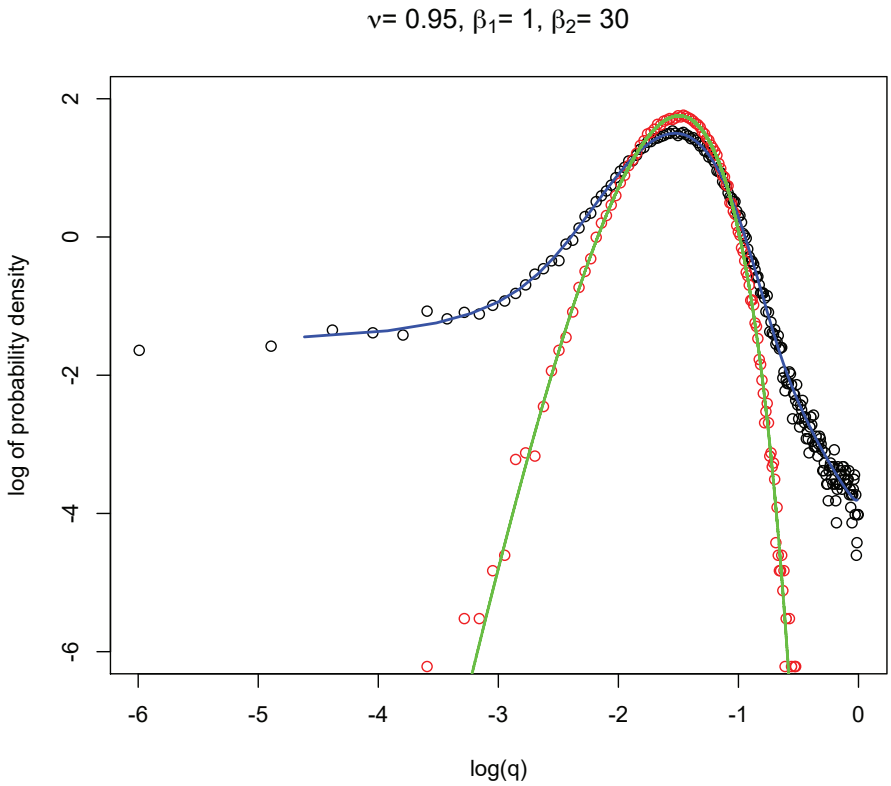

FIGURE 5. Double logarithmic plot of the pdf's for $\nu = 0.95$, $\beta_1 = 10$, $\beta_2 = 30$. Circles and curves have the same meaning as in Fig. 2.

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