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DISCUSSION PAPER

THE FLAW IN THE CONFORMABLE CALCULUS: IT IS CONFORMABLE BECAUSE IT IS *NOT* FRACTIONAL

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Abstract

We point out a major flaw in the so-called conformable calculus. We demonstrate why it fails at defining a fractional order derivative and where exactly these tempting conformability properties come from.

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Key Words and Phrases: conformable derivative; fractional calculus

1. **Introduction**

Khalil et al. proposed a definition for the fractional derivative in [5] using what they called the "conformable derivative". This concept was quickly adopted by T. Abdeljawad in [1] where he claims to have developed some tools of fractional calculus.

The conformable derivative is local by its very definition. Moreover, we proved rigorously in [2] that the conformable derivative of a function f does not exist at any point $x > 0$, unless f is differentiable at x. The term "conformable" is supposedly attributed to the properties this proposed definition provides.

We point out the flaw in Khalil et al.'s definition and uncover the real source of this conformability through reviewing the statements and proofs in [1, 5]. Analogous remarks apply to the statements and proofs in [3] and [4]. It turns out that the reason behind the conformability of this derivative is, ironically, the same reason it is not fractional.

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We would like to emphasize here that we are not reviewing the aforementioned work to provide useful formulae to work with. On the contrary, our real purpose is to discourage researchers from using it, by making it clear from the mathematical point of view why the conformable derivative is not fractional.

We have shown in ([2], Section 5) the disadvantages of using the conformable definition in solving fractional differential equations. It breaks the fractional equation and replaces it with an ordinary equation that may no longer properly describe the underlying fractional phenomenon. This is probably the reason it produces a substantially larger error compared with the Caputo fractional derivative when used to solve fractional models (see [2], Section 6).

We discuss concrete examples that illustrate how the conformable derivative is incapable of giving the fractional derivative obtainable from the classical Riemann-Liouville or Caputo derivatives. More examples are provided to show how the conformable operator produces functions with a much different behaviour than the classical fractional derivatives. The latter are known to be successful at describing many fractional phenomena (see e.g. [6]).

2. **The problems in the statements and proofs in** [5]

The results in [5] are all based on the following definition:

Definition 2.1. ([5], Definition 2.1) Given a function $f : [0, \infty[\to \mathbb{R}$, then the " conformable fractional derivative" of f of order α is defined by

$$T_\alpha f(t) := \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in]0, 1[$.

Definition 2.1 is flawed. Once we establish this, it will be immediately seen that the proofs in [5] are unnecessarily involved. More importantly, the results therein will be found insignificant as they follow directly from the traditional integer-order calculus.

We proved the following theorem in [2]:

Theorem 2.1. ([2], Theorem 1) *Fix* $0 < \alpha < 1$ *and let* $t > 0$. *A function* $f : [0, \infty[\longrightarrow \mathbb{R}$ *has a "conformable fractional derivative" of order* α *at* t *if and only if it is differentiable at* t*, in which case we have the pointwise relation*

$$T_\alpha f(t) = t^{1-\alpha} f'(t). \tag{2.2}$$

We note here the problems with Definition 2.1 in the light of Theorem 2.1:

REMARK 2.1. The limit (2.1) does not exist unless $\lim_{\epsilon\to 0}(f(t+\epsilon)-f(t))/\epsilon$ exists. In other words, there does not exist a function differentiable in the sense of Definition 2.1 that is not differentiable. In fact, the false claim in [1, 3, 5, 4] that the "conformable" derivative may exist at a point where the function is not differentiable is the only excuse for the results in these papers.

REMARK 2.2. The identity (2.2) is the reason $T_\alpha f$ demonstrates "conformability". The conformability comes precisely from the integer-order derivative, the factor f', in (2.2).

REMARK 2.3. The derivative $T_\alpha f$ is *not* a fractional (order) derivative. It is exactly the integer-order derivative times the root function $t^{1-\alpha}$.

Therefore, Definition 2.1 is to be understood as follows:

DEFINITION 2.2. (What Definition 2.1 in [5] really suggests) Given a function $f : [0,\infty[\to \mathbb{R}$, then f is α-differentiable at $t > 0$, if it is differentiable at t, and its α-derivative $T_\alpha f(t) := t^{1-\alpha} f'(t),\ t > 0$.

Now, we show how this correct understanding of what Definition 2.1 proposes trivializes the results in [5].

THEOREM 2.2. ([5], Theorem 2.1) *If a function $f : [0,\infty[\to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in]0,1]$, then f is continuous at t_0.*

If f is α-differentiable at $t_0 > 0$, then it is differentiable at t_0. It is well-known that if a function is differentiable at some point, then it is continuous thereat.

The next theorem explains why $T_\alpha f$ is described as conformable. We show why the statements are trivial and how the conformability comes from (2.2).

THEOREM 2.3. ([5], Theorem 2.2) *Let $\alpha \in]0,1]$ and f, g be α-differentiable. Then:*

(1): $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$.
(2): $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
(3): $T_\alpha(f) = 0$ *for all constant functions* f.

(4): $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(5): $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(6): *If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}f'(t)$.*

If f, g are α-differentiable, then they are in fact differentiable, and we have $T_\alpha(f)(t) = t^{1-\alpha}f'(t)$, and $T_\alpha(g)(t) = t^{1-\alpha}g'(t)$.

Let us start with **(1)**. Since f, g are differentiable, then so is $af + bg$. By Theorem 2.1, $af + bg$ is α-differentiable and we have

$$T_\alpha(af + bg) = t^{1-\alpha}(af + bg)' = at^{1-\alpha}f' + bt^{1-\alpha}g' = aT_\alpha(f) + bT_\alpha(g).$$

The proofs of items **(2)** through **(5)** are as trivial as the proof of **(1)**.

The statement **(6)** is inaccurate. The truth is f is α-differentiable at $t > 0$ if and only if f is differentiable at t. Thus, if f is α-differentiable at $t > 0$, then $T_\alpha(f)(t) = t^{1-\alpha}f'(t)$. We do not need to require f to be differentiable. Differentiability is already implied by assuming f is α-differentiable.

Theorem 2.4. ([5], Theorem 2.3) *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function such that:*

(i): *f is continuous on $[a, b]$,*
(ii): *f is α-differentiable for some $\alpha \in]0, 1[$,*
(iii): *$f(a) = f(b)$.*

Then, there exists $c \in]0, 1[$ such that $f^{(\alpha)}(c) = 0$.

The condition (ii) implies that f is differentiable on $]a, b[$ and $f^{(\alpha)}(t) = t^{1-\alpha}f'(t)$ for all $t \in]a, b[$. We know from the classical Rolle's theorem that there exists $c \in]a, b[$ such that $f^{(\alpha)}(c) = c^{1-\alpha}f'(c) = 0$.

Theorem 2.5. ([5], Theorem 2.4) *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ satisfy*

(i): *f is continuous on $[a, b]$,*
(ii): *f is α-differentiable for some $\alpha \in]0, 1[$.*

Then, there exists $c \in]0, 1[$ such that $f^{(\alpha)}(c) = \dfrac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Once again, by Theorem 2.1, the condition (ii) implies that f is differentiable on $]a, b[$ and $f^{(\alpha)} = t^{1-\alpha}f'$ on $]a, b[$. Now, apply the classical Cauchy mean value theorem to the functions f and $t \mapsto \frac{t^\alpha}{\alpha}$ on the interval

$]a,b[$, we already know that there is $c\in\,]a,b[$ such that

$$f^{(\alpha)}(c)=\frac{f'(c)}{c^{\alpha-1}}=\frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}.$$

Let $a>0$. Proposition 2.1 in [5] introduces absolutely no novelty because, by (2.2), and the fact that $t\mapsto t^{1-\alpha}$ is locally bounded, $f^{(\alpha)}$ is bounded on $[a,b]$ if and only if f' is bounded on $[a,b]$. And if f' is bounded on $[a,b]$, then f is Lipschitz on $[a,b]$, not only uniformly continuous.

The remark that follows Proposition 2.1 in [5] is false. If an α-differentiable function f on $]a,b[$ is uniformly continuous on $[a,b]$, then its α-derivative is not necessarily bounded therein. A counterexample is $f(t)=t^{\frac{1}{4}}$ which is Lipschitz on $[0,1]$, but $f^{\frac{1}{2}}(t)=t^{-\frac{1}{4}}/4$ is unbounded on $]0,1[$.

Take a look at:

Definition 2.3. ([5], Definition 3.1) Let $a\geq 0$. The α-integral of a function f is $I_{\alpha}^{a}(f)(t):=\displaystyle\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$.

The example given in [5], right after Definition 3.1, seems to try to sell I_α^a as the antiderivative of T_α. Of course, $T_{\frac{1}{2}}(\sin t)=\sqrt{t}\cos t$, and $I_{\frac{1}{2}}^{0}\left(\sqrt{t}\cos t\right)=\int_0^t\cos x dx=\sin t$. The truth is $\int_a^t\frac{f^{(\alpha)}(x)}{x^{1-\alpha}}dx=\int_a^t f'(x)dx=f(t)-f(a)$. For example, $I_{\frac{1}{2}}^{0}\left(\sqrt{t}\sin t\right)=I_{\frac{1}{2}}^{0}\left(T_{\frac{1}{2}}(-\cos t)\right)=\int_0^t\sin x dx=1-\cos t$.

3. **The problems in the statements and proofs in** [1]

We proceed to demonstrate the flaws in the definitions suggested in [1]. We prove that the tools of calculus proposed there lack the novelty, as they are trivial consequences of the traditional calculus. The ideas in [1] are all based on the following definition:

Definition 3.1. ([1], Definition 2.1) The (left) fractional derivative starting from a of a function $f:[a,\infty[\to\infty$ of order $0<\alpha\leq 1$ is defined by

$$T_\alpha^a f(t):=\lim_{\epsilon\to 0}\frac{f(t+\epsilon(t-a)^{1-\alpha})-f(t)}{\epsilon},\quad t>a. \tag{3.3}$$

The (right) fractional derivative of order $0<\alpha\leq 1$ of a function $f:]-\infty,b]\to\infty$ is defined by

$${}^{\,b}_{\alpha}Tf(t):=-\lim_{\epsilon\to 0}\frac{f(t+\epsilon(b-t)^{1-\alpha})-f(t)}{\epsilon},\quad t<b. \tag{3.4}$$

If $T_\alpha^a f(t)$ exists on $]a,b[$ then $T_\alpha^a f(a) := \lim_{t\to a^+} T_\alpha f(t)$. If ${}_\alpha^b T f(t)$ exists on $]a,b[$ then ${}_\alpha^b T f(b) := \lim_{t\to b^-} {}_\alpha^b T f(t)$.

It is also noted in [1] that if f is differentiable, then

$$T_\alpha^a f(t) = (t-a)^{1-\alpha} f'(t) \quad \text{and} \quad {}_\alpha^b T f(t) = -(b-t)^{1-\alpha} f'(t).$$

The following theorem is given in [2]:

THEOREM 3.1. ([2], Theorem 3) *Suppose* $h :]-1,1[\times\mathbb{R} \longrightarrow \mathbb{R}$ *is such that* $\lim_{\epsilon\to 0} h(\epsilon, t_0) \neq 0$ *for some* $t_0 \in \mathbb{R}$. *Then a function* $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable at* t_0 *if and only if the limit*

$$\tilde{\psi}(t_0) := \lim_{\epsilon\to 0} \frac{\psi\left(t_0 + \epsilon h(\epsilon, t_0)\right) - \psi(t_0)}{\epsilon}$$

exists, in which case $\tilde{\psi}(t_0) = \psi(t_0)\psi'(t_0),\ \psi(t) = \lim\limits_{\epsilon\to 0} h(\epsilon, t)$.

Let us see the problems in Definition 3.1:

REMARK 3.1. According to Theorem 3.1, the limit in (3.3) exists at $t > a$ if and only if $\lim\limits_{\epsilon\to 0} (f(t+\epsilon) - f(t))/\epsilon$ exists. Similarly, the limit in (3.4) exists at $t < b$ if and only if $f'(t)$ exits. This means that neither $T_\alpha^a f(t)$ nor ${}_\alpha^b T f(t)$ exits unless f is differentiable at t. In fact, by Theorem 3.1, Definition 3.1 reads:

$$\begin{aligned} T_\alpha^a f(t) :=&\quad (t-a)^{1-\alpha} f'(t),\ t > a \\ {}_\alpha^b T f(t) :=&\quad (b-t)^{1-\alpha} f'(t),\ t < b, \end{aligned} \tag{3.5}$$

provided $f'(t)$ exists.

REMARK 3.2. Unlike with the classical fractional derivatives of Riemann-Liouville and Caputo, Definition 3.1 does not work for functions defined on $\mathbb{R}$. Indeed, by Remark 3.1, if f is defined on $\mathbb{R}$, then both derivatives $T_\alpha^{-\infty} f(t)$ and ${}_\alpha^\infty T f(t)$ are ill-defined at every $t \in \mathbb{R}$, which is unacceptable.

REMARK 3.3. There is no geometric or physical motivation that justifies the negative sign in the definition of the operator ${}_\alpha^b T$. Furthermore, the case $\alpha = 1$ is supposed to give the left first order integer derivative, but Remark 3.1 implies

$${}_1^b T f(t) = -f'(t),\ t < b,$$

which is neither the left nor the right derivative of f at t.

Remark 3.4. There is an obvious inconsistency in Definition 3.1 when it comes to defining $T^a_\alpha f(a)$ and ${}^b_\alpha T f(b)$. Let $a < b$. We have clarified in Remark 3.1 that if a function is not differentiable at some point $t \in]a, b[$, then the pointwise criterion of Definition 3.1 does not allow it to be α-differentiable at t. This, however, excludes the endpoints a and b. We are going to discuss $T^a_\alpha f(a)$ and the analogue applies to ${}^b_\alpha T f(b)$. Unjustifiably, Definition 3.1 allows the derivative $T^a_\alpha f(a)$ to exist regardless of the existence of the right derivative of f at a. Precisely, by Remark 3.1, if $T^a_\alpha f$ exists on $]a, a+\delta[$, for some $\delta > 0$, then

$$T^a_\alpha f(a) = \lim_{t\to a^+} T^a_\alpha f(t) = \lim_{t\to a^+} (t-a)^{1-\alpha} f'(t).$$

Therefore, according to Definition 3.1, $T^a_\alpha f(a)$ exists if and only if f' exists on $]a, a+\delta[$, and $\lim_{t\to a^+}(t-a)^{1-\alpha} f'(t)$ exists. This is evidently a weaker condition than the existence of f' on $]a, a+\delta[$ and $\lim_{t\to a^+} f'(t)$. It is also independent of the existence of the right derivative $f'_+(a)$ of f at a. Many examples are given in [1] for functions differentiable on $]a, b[$ such that $T^a_\alpha f(a)$ exists, but $f'_+(a)$ does not. This may lead to the false intuition that the operator T^a_α is well-defined on a larger class of functions than the derivative. The reality is there exist smooth functions on $]a, b[$ such that $f'(a)$ exists but $T^a_\alpha f(a)$ does not. Consider for instance $g(t) := \begin{cases} x^2 \sin \frac{1}{x^3}, & x \neq 0; \\ 0, & \text{x=0}. \end{cases}$. We have $g \in C^\infty(\mathbb{R} \setminus \{0\})$ and $g'(0) = 0$, yet $\lim_{x\to 0^+} x^{1-\alpha} g'(x)$ does not exist for any $0 \leq \alpha \leq 1$.

Another issue with $T^a_\alpha f(a)$ is that its existence depends on the domain. For example, $h(x) := \sin\sqrt{t - t_0}$ is not differentiable at $t = t_0$, and consequently, by Remark 3.1, $T^c_\alpha h(t_0)$ does not exist. But this is true only if h is considered on the domain $[c, \infty[$ with any $c < t_0$. If $c = t_0$, however, then $T^c_\alpha h(t_0)$ magically exists and equals 0, for every $0 < \alpha \leq \frac{1}{2}$.

Remark 3.5. The identities (3.5) prove that the derivative in Definition 3.1 is not fractional and that the conformability comes from the integer-order derivative factor. What is worse is that the derivative in Definition 3.1 fails to give the fractional derivative for some functions whose fractional derivative exist and can be easily calculated using the Riemann-Liouville or Caputo definition.

See the following examples:

Example 3.1. Consider the function $f_1(t) := \begin{cases} 1, & 0 \leq t \leq 1; \\ 0, & 1 < t \leq 2. \end{cases}$ It is easily verifiable that $(T^0_\alpha f_1)(1)$ does not exist. But the Riemann-Liouville fractional derivative $D^\alpha_{0^+} f_1$ exists at $t = 1$, and

$$D^\alpha_{0^+} f_1(1) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{d\xi}{(1-\xi)^\alpha} = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}.$$

Example 3.2. Consider the function $f_2(t) := |t-1|$ on $[0,2]$. Again, $(T^0_\alpha f_2)(1)$ does not exist. Nevertheless, the Caputo fractional derivative ${}^C D^\alpha_{0^+} f_2$ exists at $t=1$, and

$${}^C D^\alpha_{0^+} f_2(1) = \frac{-1}{\Gamma(1-\alpha)} \int_0^1 \frac{d\xi}{(1-\xi)^\alpha} = \frac{-1}{(1-\alpha)\Gamma(1-\alpha)}.$$

Remark 3.6. Pointwise multiplication of the derivative f' of a function f defined on $[a,\infty[$ by the function $(t-a)^{1-\alpha}$ does not give the physical properties we hope from a fractional derivative. We show this by comparing $T^a_\alpha f(t) = (t-a)^{1-\alpha} f'$ to the Riemann-Liouville and Caputo fractional derivatives for the sine and hyperbolic sine functions. Similar differences show up with the cosine and hyperbolic cosine functions. Notice here that the Riemann-Liouville fractional derivative coincides with the Caputo derivative for each of these functions. We see the great difference in behaviour between T^a_α and the classical fractional operators:

Example 3.3. Let $g_1(t) = \sin t$. Then $\left(T^0_\alpha g_1\right)(t) = t^{1-\alpha} \cos t$. We can calculate

$$\left(D^\alpha_{0^+} g_1\right)(t) = \left({}^C D^\alpha_{0^+} g_1\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\cos(t-\xi)}{\xi^\alpha} d\xi.$$

Notice that $\left(T^0_\alpha g_1\right)(t)$ grows unboundedly with t. Contrarily, the fractional derivatives $D^\alpha_{0^+} g_1$ and ${}^C D^\alpha_{0^+} g_1$ are bounded. To see this, let $t > 1$. We have

$$\left| \int_0^1 \frac{\cos(t-\xi)}{\xi^\alpha} d\xi \right| \le \int_0^1 \frac{1}{\xi^\alpha} d\xi = \frac{1}{1-\alpha}. \tag{3.6}$$

Also, integrating by parts,

$$\int_1^t \frac{\cos(t-\xi)}{\xi^\alpha} d\xi = \sin(t-1) - \alpha \int_1^t \frac{\sin(t-\xi)}{\xi^{1+\alpha}} d\xi, \tag{3.7}$$

and we have

$$\left| \int_1^t \frac{\sin(t-\xi)}{\xi^{1+\alpha}} d\xi \right| \le \int_1^t \frac{1}{\xi^{1+\alpha}} d\xi = \frac{1}{\alpha}\left(1 - \frac{1}{t^\alpha}\right) < \frac{1}{\alpha}. \tag{3.8}$$

The boundedness of $D^\alpha_{0^+} g_1$, ${}^C D^\alpha_{0^+} g_1$ follows from (3.6), (3.7), and (3.8). See Figure 1.

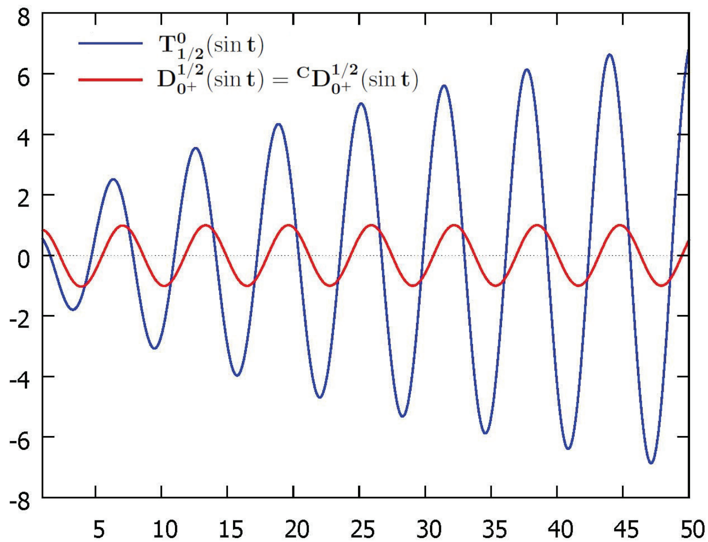

FIGURE 1. The behavior of $T^0_\alpha g_1$ is very different from that of $D^\alpha_{0^+} g_1$, ${}^C D^\alpha_{0^+} g_1$

EXAMPLE 3.4. Let $g_2(t) = \sinh t$. Then $\left(T^0_\alpha g_2\right)(t) = t^{1-\alpha} \cosh t$. On the other hand,

$$\left(D^\alpha_{0^+} g_2\right)(t) = \left({}^C D^\alpha_{0^+} g_1\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\cosh(t-\xi)}{\xi^\alpha} d\xi.$$

The function $T^0_\alpha g_2$ grows much faster than the fractional derivative. To prove this, we compute

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\left(D^\alpha_{0^+} g_2\right)(t)}{\left(T^0_\alpha g_2\right)(t)} = \lim_{t\to\infty} \frac{\int_0^t \frac{\cosh(t-\xi)}{\xi^\alpha} d\xi}{t^{1-\alpha}\cosh t} = \lim_{t\to\infty} \frac{\int_0^t \frac{1}{\xi^\alpha}(\cosh \xi - \tanh t \sinh \xi) d\xi}{t^{1-\alpha}} \\
=&\frac{1}{1-\alpha}\left(\lim_{t\to\infty} \frac{1}{\cosh t} - \lim_{t\to\infty} \frac{\int_0^t \frac{\sinh \xi}{\xi^\alpha} d\xi}{t^{-\alpha}\cosh^2 t}\right) = \frac{-1}{1-\alpha} \lim_{t\to\infty} \frac{\int_0^t \frac{\sinh \xi}{\xi^\alpha} d\xi}{t^{-\alpha}\cosh^2 t} \\
=&\frac{-1}{1-\alpha} \lim_{t\to\infty} \frac{1}{-\alpha \frac{\cosh^2 t}{t \sinh t} + 2\cosh t} = 0.
\end{aligned}$$

See Figure 2.

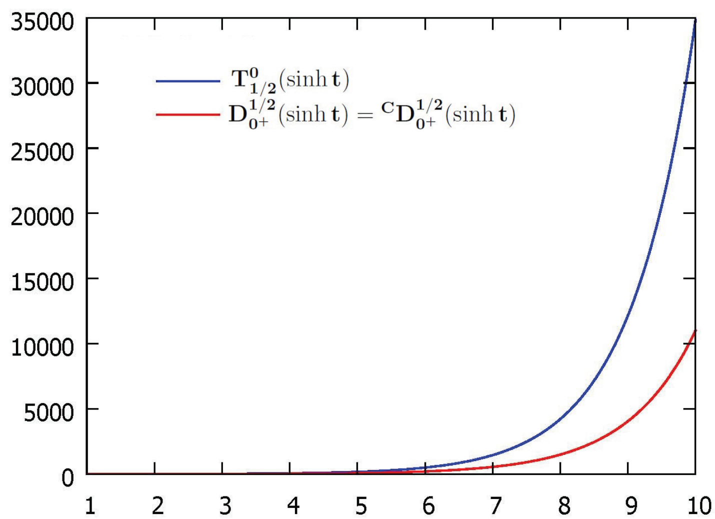

FIGURE 2. The behavior of $T^0_\alpha g_2$ is very different from that of $D^\alpha_{0+} g_2$, ${}^C D^\alpha_{0+} g_2$

Remarks 3.1 through 3.6 show the insignificance of the results in [1]. We illustrate how the proofs presented in [1] reduce to trivial exercises of calculus. For example:

THEOREM 3.2. ([1], Theorem 2.11) *Assume $f, g :]a, \infty[\to \mathbb{R}$ are (left) α-differentiable functions, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t)$. Then h is (left) α-differentiable, and for all $t > a$ such that $g(t) \neq 0$ we have*

$$(T^a_\alpha h)(t) = (T^a_\alpha f)(g(t)).(T^a_\alpha g)(t).g(t)^{\alpha-1}. \tag{3.9}$$

First of all, the conclusion (3.9) of Theorem 3.2 is incorrect. It is correct if $a = 0$. We consider this case.

As noted in Remark 3.1, if f, g are (left) α-differentiable on $]a, \infty[$, then they are actually differentiable on $]a, \infty[$. Moreover, by the identities (3.5),

$$\begin{aligned}(T^0_\alpha f)(g(t)).(T^0_\alpha g)(t).g(t)^{\alpha-1} &= g(t)^{1-\alpha}.f'(g(t)).t^{1-\alpha}g'(t).g(t)^{\alpha-1} \\ &= t^{1-\alpha}f'(g(t)).g'(t) \\ &= t^{1-\alpha}(f(g(t)))' = (T^0_\alpha h)(t).\end{aligned}$$

Both the statement and proof of the next theorem ([1], Theorem 4.1) we investigate are incorrect. This implies that Proposition 4.2 and Examples 4.1 through 4.3 in [1] are also incorrect.

THEOREM 3.3. ([1], Theorem 4.1) *Assume f is an infinitely α-differentiable function, for some $0<\alpha<1$ at a neighborhood of a point t_0. Then f has the fractional power series expansion:*

$$f(t)=\sum_{k=0}^{\infty}\frac{(T_{\alpha}^{t_0}f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!},\ \ t_0<t<t_0+R^{\frac{1}{\alpha}},\ R>0, \tag{3.10}$$

where $(T_{\alpha}^{t_0}f)^{(k)}$ means the application of $T_{\alpha}^{t_0}$ k times.

The proof in [1] begins with writing

$$f(t)=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+c_3(t-t_0)^{3\alpha}+...,\tag{3.11}$$

and proceeds by applying $T_{\alpha}^{t_0}$ to both sides of (3.11), then evaluating both sides at t_0, and repeating the process k times. The coefficients c_k are inaccurately calculated:

$$c_k=\frac{(T_{\alpha}^{t_0}f)^{(k)}(t_0)}{\alpha^k k!}.$$

By Remark 3.1, the assumption that f is an infinitely α-differentiable is equivalent to assuming f is infinitely differentiable.

Using (3.5), we get

$$\begin{aligned}
(T_{\alpha}^{t_0}f)^{(1)}(t) &= (t-t_0)^{1-\alpha}f'(t),\\
(T_{\alpha}^{t_0}f)^{(2)}(t) &= (1-\alpha)(t-t_0)^{1-2\alpha}f'(t)+(t-t_0)^{2-2\alpha}f''(t),\\
(T_{\alpha}^{t_0}f)^{(3)}(t) &= (1-\alpha)(1-2\alpha)(t-t_0)^{1-3\alpha}f'(t)+3(1-\alpha)(t-t_0)^{2-3\alpha}f''(t)\\
&\quad +(t-t_0)^{3-3\alpha}f'''(t),\\
(T_{\alpha}^{t_0}f)^{(4)}(t) &= (1-\alpha)(1-2\alpha)(1-3\alpha)(t-t_0)^{1-4\alpha}f'(t)\\
&\quad +(1-\alpha)(3(1-\alpha)+4(1-2\alpha))(t-t_0)^{2-4\alpha}f''(t)\\
&\quad +6(1-\alpha)(t-t_0)^{3-4\alpha}f'''(t)++(t-t_0)^{4-4\alpha}f^{(4)}(t),\\
&\ \ \vdots\\
(T_{\alpha}^{t_0}f)^{(k)}(t) &= \sum_{j=1}^{k-1}a_{j,k}(\alpha)\frac{f^{(j)}(t)}{(t-t_0)^{k\alpha-j}}+\frac{f^{(k)}(t)}{(t-t_0)^{k\alpha-k}},\ k\geq 2, \qquad (3.12)
\end{aligned}$$

where $a_{1,k}=\prod_{j=1}^{k-1}(1-j\alpha)$, and $a_{j,k}(\alpha)$, $2\leq j\leq k-1$, are also constants that depend only on α. At first glance we observe the following.

REMARK 3.7. Since $\lim_{t\to t_0{}^+}(t-t_0)^{1-\alpha}f' = 0$, for any f continuously differentiable and every $\alpha < 1$, then, the coefficient of $(t-t_0)^\alpha$ in the series (3.10) is zero for any smooth function. Similarly, if $\alpha < \frac{1}{2}$, then $\lim_{t\to t_0{}^+}(T_\alpha^{t_0} f)^{(1)}(t) = \lim_{t\to t_0{}^+}(T_\alpha^{t_0} f)^{(2)}(t) = 0$. Consequently, the coefficient of $(t-t_0)^\alpha$ and that of $(t-t_0)^{2\alpha}$ are both zero for any smooth f. Generally, given $\alpha \in]0,1[$, there exists $n > 1$ such that $\alpha < \frac{1}{n}$, and, strangely enough, the coefficients of $(t-t_0)^{k\alpha}$, $1 \le k \le n$ are all zero, regardless of the function f.

We prove in Proposition 3.1 below that the proof presented in [1] is incorrect and the series (3.10) does not make sense.

We immediately realize from (3.12) that the infinite differentiability of f does not guarantee that the series (3.10) makes sense. We need infinitely many more smallness restrictions on the derivatives of f near t_0. Precisely, we have the following proposition.

PROPOSITION 3.1. *Given a function f and $\alpha \in]0,1[$, the expansion (3.10) does not make sense, unless the infinitely many limits*

$$\lim_{t\to t_0{}^+} \frac{\sum_{j=1}^{k-1} a_{j,k}(\alpha)(t-t_0)^{j-1} f^{(j)}(t)}{(t-t_0)^{k\alpha-1}}, \quad k > \frac{1}{\alpha}, \tag{3.13}$$

exist.

Examples 4.1 through 4.3, in [1] that apply the generally incorrect expansion (3.11) are conveniently for functions of the form $f(t) = g\left(\left(\frac{t-t_0}{\alpha}\right)^\alpha\right)$. They seem to work because $(T_\alpha^{t_0} f)^{(k)}(t) = g^{(k)}((t-t_0)^\alpha)$. In fact, the series (3.11) that works for the function $t \mapsto e^{\frac{(t-t_0)^\alpha}{\alpha}}$ of Example 4.1 in [1] fails, for any $\alpha \in]0,1[$, for the infinitely differentiable function $t \mapsto e^{\frac{(t-s)^\alpha}{\alpha}}$ on $]s,\infty[$, with $s \neq t_0$. Verifiably, if $\alpha > \frac{1}{2}$, then $(T_\alpha^{t_0} h)^{(2)}(t_0)$ does not exit because

$$\lim_{t\to t_0{}^+} \frac{h'(t)}{(t-t_0)^{2\alpha-1}} = \lim_{t\to t_0{}^+} \frac{(t-s)^{\alpha-1}e^{\frac{(t-s)^\alpha}{\alpha}}}{(t-t_0)^{2\alpha-1}} = +\infty.$$

If $\alpha > \frac{1}{3}$, then $(T_\alpha^{t_0} h)^{(3)}(t_0)$ does not exit since

$$\lim_{t\to t_0{}^+} \frac{(1-\alpha)(1-2\alpha)h'(t) + 3(1-\alpha)(t-t_0)h''(t)}{(t-t_0)^{3\alpha-1}} = (1-2\alpha)\infty.$$

In fact, if $\alpha > \frac{1}{n}$ for some $n \ge 2$, one can show that $(T_\alpha^{t_0} h)^{(k)}(t_0)$ does not exist for any $k \ge n$. Hence, for the smooth function h, the coefficients, c_k, of $(t-t_0)^{k\alpha}$ in the expansion (3.11) are $c_k = \begin{cases} 0, & k < 1/\alpha; \\ \pm\infty, & k > 1/\alpha. \end{cases}$, $k \geq 2$. Even more, the series (3.11) fails for the simplest analytic function e^t. An analogous argument applies to Examples 4.2 and 4.3 in [1].

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