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RESEARCH PAPER

TIME OPTIMAL CONTROLS FOR FRACTIONAL DIFFERENTIAL SYSTEMS WITH RIEMANN-LIOUVILLE DERIVATIVES

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Abstract

Time optimal control problems governed by Riemann-Liouville fractional differential system are considered in this paper. Firstly, the existence results are obtained by using the theory of semigroup and Schauder's fixed point. Secondly, the new approach of establishing time minimizing sequences twice is applied to acquire the time optimal pairs without the Lipschitz continuity of nonlinear function. Moreover, the reflexivity of state space is removed with the help of compact method. Finally, an example is given to illustrate the main conclusions. Our work essentially improves and generalizes the corresponding results in the existing literature.

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Key Words and Phrases: Riemann-Liouville fractional derivatives, time optimal controls, mild solution

1. Introduction

In the last dozen years or so, fractional differential equations have been served as mathematical models for describing various phenomena in the field of physics, biology, engineering, etc. For more details, we refer to the books [12, 15, 24], the recent papers [1, 2, 3, 4, 7, 13, 17, 21, 27, 28, 34] and the reference therein. The advantages of fractional derivatives over integer derivatives are the memory and genetic properties. On the other

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hand, Heymans and Podlubny [8] indicated that the initial conditions of differential systems with Riemann-Liouville fractional derivative are more accordant with practical circumstances in the field of viscoelasticity than that with Caputo fractional derivative. So, lot's of scholars have done much research in this area, see [5, 16, 18, 19, 20, 23, 30] and the references therein. Fan [5], Li and Peng [16] and Mei et al. [23] investigated the Riemann-Liouville fractional differential systems by using the theory of fractional resolvent. Liu and Li [18] established the sufficient conditions for the approximate controllability of Riemann-Liouville fractional differential systems by the iterative and approximate method.

Time optimal control is a classical and important topic in the theory of optimal controls for both finite and infinite dimensional systems. To our knowledge, the time optimal pairs have been derived provided that the nonlinear function is Lipschitz continuous and both the state space X and the control space Y are reflexive (see e.g. [10, 11, 14, 26, 29, 32]). The Lipschitz continuity guarantees the existence and uniqueness of mild solution of the corresponding differential systems, and the reflexivity of the spaces X and Y ensure the weak convergence of solution sequences and control sequences, respectively.

Inspired by the above mentioned papers, it is our intension to deal with the time optimal control problems subjected to the following differential system with the Riemann-Liouville fractional derivative

$$\begin{cases} {}^{L}D^{\gamma}y(t) = Ay(t) + g(t, y(t)) + B(t)u(t), \ t \in (0, c], \\ I^{1-\gamma}y(t)|_{t=0} = y_0 \in X, \\ u \in U_{ad}, \end{cases} \tag{1.1}$$

where $0 < \gamma < 1$, $y(t) \in X$ and $u(t) \in Y$. The linear operator $A : D(A) \subseteq X \to X$ generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$. $B \in L^{\infty}([0, c], \mathcal{L}(Y, X))$. The admissible set U_{ad} for control functions and the nonlinear function $g : [0, c] \times X \to X$ will be given in Section 2. The following two improvements are made in this article. One is that the Lipschitz continuity of g is removed without imposing any other conditions, and the solvability of (1.1) is acquired in a new space $C_{1-\gamma}([0, c], X)$ using the theory of semigroup. Inspired by Zhu and Huang [35], the new idea of setting up time minimizing sequences twice is used to compensate the lack of uniqueness of mild solutions. The other is that the reflexivity of X is no longer required by making full use of the compact method. So, our work essentially improves some related results on this topic.

This paper is structured as follows. Section 2 presents the preliminaries and basic assumptions for system (1.1). We establish the solvability of system (1.1) in Section 3. Section 4 solves the time optimal control problems subjected to system (1.1). An example is proposed to illustrate our main results in Section 5.

2. **Preliminaries and Basic Assumptions**

Throughout this paper, let X be a Banach space and Y be a separable reflexive Banach space. $\mathbb{R}$ and $\mathbb{R}^+$ are the sets of real numbers and nonnegative real numbers, respectively. The set of all continuous functions from $[0,c]$ to Banach space X with $\|y\|_C = \sup\{\|y(t)\|, t \in [0,c]\}$ is denoted by $C([0,c],X)$, and the Banach space $C_{1-\gamma}([0,c],X) = \{y : \cdot^{1-\gamma} y(\cdot) \in C([0,c],X), 0 < \gamma < 1\}$ with $\|y\|_{C_{1-\gamma}} = \sup\{\|t^{1-\gamma}y(t)\|, t \in [0,c]\}$, where $t^{1-\gamma}y(t)|_{t=0} = \lim\limits_{t\to 0^+} t^{1-\gamma}y(t)$. We also denote by $L^p([0,c],X)$ the space of Bochner integrable functions from $[0,c]$ to Banach space X with $\|f\|_{L^p} = (\int_0^c \|f(t)\|^p dt)^{1/p}$, where $1 \le p < \infty$. Let $L^\infty([0,c],X)$ be the set of all essentially bounded functions on $[0,c]$ with values in X and $\|f\|_\infty = esssup\{\|f(t)\|, t \in [0,c]\}$, and $\mathcal{L}(X,Y)$ be the space of all linear and continuous operators from X to Y with the operator norm $\|\cdot\|$. $\mathcal{L}(X)$ represents the space $\mathcal{L}(X,X)$ especially.

Let $f : [0,\infty) \to X$ be an appropriate abstract function. The Riemann-Liouville fractional integral and fractional derivative of order $0 < \gamma < 1$ are defined by

$$I^\gamma f(t) = \int_0^t g_\gamma(t-\tau) f(\tau) \mathrm{d}\tau,$$

and

$$^L D^\gamma f(t) = \frac{d}{dt}\int_0^t g_{1-\gamma}(t-\tau) f(\tau) \mathrm{d}\tau,$$

respectively, provided the right sides exist, where $g_\gamma(t) := \frac{t^{\gamma-1}}{\Gamma(\gamma)}, t > 0$.

Now, we give the mild solution of system (1.1) in the space $C_{1-\gamma}([0,c],X)$ using the Laplace transformation, some proper density function as well as the definition of Riemann-Liouville fractional derivatives. For details, see the recent paper [18].

Definition 2.1. A function $y \in C_{1-\gamma}([0,c],X)$ is called the mild solution of (1.1) if

$$y(t) = t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 + \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,y(\tau)) + B(\tau)u(\tau)]\mathrm{d}\tau,$$

for each $t \in (0,c]$ and $u \in U_{ad}$, where

$$\mathcal{S}_\gamma(t) = \gamma\int_0^\infty \theta h_\gamma(\theta) T(t^\gamma\theta)\mathrm{d}\theta,$$

$$h_\gamma(\theta) = \frac{1}{\gamma}\theta^{-1-\frac{1}{\gamma}}\omega_\gamma(\theta^{-\frac{1}{\gamma}}) \ge 0,\ \theta \in (0,\infty),$$

$$\omega_\gamma(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\gamma-1}\frac{\Gamma(n\gamma+1)}{n!}\sin(\pi n\gamma),\ \theta \in (0,\infty).$$

REMARK 2.1. The function $\omega_\gamma(\cdot)$ is an one-side stable probability density defined on $(0,\infty)$ (see [22]), whose Laplace transformation satisfies

$$\int_0^\infty e^{-\lambda\theta}\omega_\gamma(\theta)\mathrm{d}\theta = e^{-\lambda^\gamma},\ \gamma \in (0,1).$$

The function h_γ is a probability density function defined on $(0,\infty)$ satisfying

$$\int_0^\infty h_\gamma(\theta)\mathrm{d}\theta = 1, \qquad \int_0^\infty \theta^v h_\gamma(\theta)\mathrm{d}\theta = \frac{\Gamma(1+v)}{\Gamma(1+\gamma v)},\ v \in [0,1].$$

We now list all the assumptions which will be applied in the whole paper.

(HA) The linear closed and densely defined operator A on X generates a compact C_0 semigroup $\{T(t)\}_{t>0}$, and set $M := \sup_{t\in[0,c]}\|T(t)\| < +\infty$.

(Hg) (1) $g(t,y)$ is measurable in t on $[0,c]$ for all $y \in X$, and continuous in y on X for a.e. $t \in [0,c]$.

(2) For all $y \in X$ and a.e. $t \in [0,c]$, there exist a function $\eta \in L^p([0,c],\mathbb{R}^+)$ and a constant ρ with $p > \frac{1}{\gamma}$ and $0 < \rho < \frac{\Gamma(1+\gamma)}{cM}$ such that

$$\|g(t,y)\| \le \eta(t) + \rho t^{1-\gamma}\|y\|. \tag{2.1}$$

(HB) $B \in L^\infty([0,c],\mathcal{L}(Y,X))$.

The set U_{ad} for control functions is defined as

$$U_{ad} = \{u \in L^p([0,c],Y) : u(t) \in U(t),\ a.e.\ t \in [0,c]\},$$

where $p > \frac{1}{\gamma}$, and the multivalued map $U : [0,c] \to P_f(Y)$ (the set of all nonempty closed and convex subset of Y) satisfies the following condition (HU).

(HU) (1) $U(\cdot)$ is graph measurable.

(2) For a.e. $t \in [0,c]$, there exists a function $m \in L^p([0,c],\mathbb{R}^+)$ with $p > \frac{1}{\gamma}$ such that

$$\|U(t)\| = \sup\{\|\mu\| : \mu \in U(t)\} \le m(t).$$

A noteworthy fact in [9] is that (HU) implies that $U_{ad} \neq \emptyset$, and obviously U_{ad} is bounded, closed and convex. Moreover, $\|u\|_{L^p} \le \|m\|_{L^p}$ and $Bu \in L^p([0,c],X)$ for $p > \frac{1}{\gamma}$ and $u \in U_{ad}$.

The following properties play an important role in this paper.

LEMMA 2.1. *Let $0 < \gamma < 1$ and (HA) be satisfied. Then, the operator $\mathcal{S}_\gamma(t)$ $(t \ge 0)$ defined in Definition 2.1 satisfies:*

(1) $\{\mathcal{S}_\gamma(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$*, and for each* $x \in X$ *and* $t \geq 0$*, there holds:*

$$\|\mathcal{S}_\gamma(t)x\| \leq \frac{\gamma M}{\Gamma(1+\gamma)}\|x\|. \tag{2.2}$$

(2) for each $x \in X$*,* $\mathcal{S}_\gamma(\cdot)x \in C([0,c],X)$*.*

(3) for each $t > 0$*,* $\mathcal{S}_\gamma(t)$ *is a compact operator on* X*.*

(4) the operator $\mathcal{S}_\gamma(t)$ *is continuous in the uniform operator topology for* $t > 0$*.*

(5) $\lim\limits_{h\to 0^+} \|\frac{1}{\Gamma(\gamma)}\mathcal{S}_\gamma(t+h) - \mathcal{S}_\gamma(t)\mathcal{S}_\gamma(h)\| = 0,\ t > 0,$

$\lim\limits_{h\to 0^+} \|\frac{1}{\Gamma(\gamma)}\mathcal{S}_\gamma(t) - \mathcal{S}_\gamma(h)\mathcal{S}_\gamma(t-h)\| = 0,\ t > 0.$

P r o o f. For the properties (1)(2) and (3), we refer to [33] for details.

We now verify property (4). For $0 < t_1 < t_2$, there exist positive numbers ϱ and N such that

$$\begin{aligned}
& \|\mathcal{S}_\gamma(t_2) - \mathcal{S}_\gamma(t_1)\| \\
\leq\ & \gamma\int_0^\infty \theta h_\gamma(\theta)\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\|\mathrm{d}\theta \\
\leq\ & \gamma\int_0^\varrho \theta h_\gamma(\theta)\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\|\mathrm{d}\theta + \gamma\int_\varrho^N \theta h_\gamma(\theta)\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\|\mathrm{d}\theta \\
& +\gamma\int_N^\infty \theta h_\gamma(\theta)\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\|\mathrm{d}\theta \\
\leq\ & 2M\gamma\int_0^\varrho \theta h_\gamma(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\gamma)}\sup_{\theta\in[\varrho,N]}\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\| \\
& +2M\gamma\int_N^\infty \theta h_\gamma(\theta)\mathrm{d}\theta.
\end{aligned}$$

The compactness of $T(t)$ $(t > 0)$ yields $\|T(t_2^\gamma\theta) - T(t_1^\gamma\theta)\| \to 0$ as $t_1 \to t_2$ and $\theta \in [\varrho, N]$. This together with the arbitrariness of ϱ and N as well as the fact $\int_0^\infty \theta h_\gamma(\theta)\mathrm{d}\theta = \frac{1}{\Gamma(1+\gamma)}$ gives that $\|\mathcal{S}_\gamma(t_2) - \mathcal{S}_\gamma(t_1)\| \to 0$ as $t_1 \to t_2$.

For property (5), a similar manner as did in [6] gives the conclusion. This completes the proof. □

LEMMA 2.2. *If* (HA) *holds, then the operator* $F : L^p([0,c],X) \to C_{1-\gamma}([0,c],X)$ *given by*

$$(Fh)(\cdot) = \int_0^\cdot (\cdot - \tau)^{\gamma-1}\mathcal{S}_\gamma(\cdot - \tau)h(\tau)\mathrm{d}\tau$$

is compact for $p > \frac{1}{\gamma}$*.*

P r o o f. For fixed $r > 0$, let $B_{L^p}(r) = \{h \in L^p([0,c],X) : \|h\|_{L^p} \leq r\}$. We will show that the set $\{Fh : h \in B_{L^p}(r)\} \subseteq C_{1-\gamma}([0,c],X)$ is precompact, that is $\{\cdot^{1-\gamma}Fh(\cdot) : h \in B_{L^p}(r)\} \subseteq C([0,c],X)$ is precompact.

Firstly, we verify that $\{\cdot^{1-\gamma}Fh(\cdot) : h \in B_{L^p}(r)\}$ is equicontinuous. Let $0 \leq t_1 < t_2 \leq c$ and $h \in B_{L^p}(r)$. If $t_1 = 0$,

$$\begin{aligned}
& \|t_2^{1-\gamma}Fh(t_2) - t_1^{1-\gamma}Fh(t_1)\| \\
\leq\ & \frac{\gamma M}{\Gamma(1+\gamma)} t_2^{1-\gamma} \|h\|_{L^p([0,t_2])} \|(t_2 - \cdot)^{\gamma-1}\|_{L^{\frac{p}{p-1}}([0,t_2])} \\
\leq\ & \frac{\gamma r M}{\Gamma(1+\gamma)} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} t_2^{1-\frac{1}{p}} \to 0,
\end{aligned}$$

as $t_2 \to 0$. If $t_1 > 0$, for $\varrho > 0$ small enough with $t_1 - \varrho > 0$, one has

$$\begin{aligned}
& \|t_2^{1-\gamma}Fh(t_2) - t_1^{1-\gamma}Fh(t_1)\| \\
\leq\ & t_2^{1-\gamma} \int_0^{t_1-\varrho} (t_2-\tau)^{\gamma-1} \|\mathcal{S}_\gamma(t_2-\tau) - \mathcal{S}_\gamma(t_1-\tau)\| \|h(\tau)\| \mathrm{d}\tau \\
& + t_2^{1-\gamma} \int_{t_1-\varrho}^{t_1} (t_2-\tau)^{\gamma-1} \|\mathcal{S}_\gamma(t_2-\tau) - \mathcal{S}_\gamma(t_1-\tau)\| \|h(\tau)\| \mathrm{d}\tau \\
& + t_2^{1-\gamma} \int_{t_1}^{t_2} (t_2-\tau)^{\gamma-1} \|\mathcal{S}_\gamma(t_2-\tau) h(\tau)\| \mathrm{d}\tau \\
& + t_2^{1-\gamma} \int_0^{t_1} |(t_2-\tau)^{\gamma-1} - (t_1-\tau)^{\gamma-1}| \|\mathcal{S}_\gamma(t_1-\tau) h(\tau)\| \mathrm{d}\tau \\
& + [t_2^{1-\gamma} - t_1^{1-\gamma}] \int_0^{t_1} (t_1-\tau)^{\gamma-1} \|\mathcal{S}_\gamma(t_1-\tau) h(\tau)\| \mathrm{d}\tau \\
\leq\ & c^{1-\gamma} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} r [t_2^{\frac{p\gamma-1}{p-1}} - (t_2 - t_1 + \varrho)^{\frac{p\gamma-1}{p-1}}]^{1-\frac{1}{p}} \\
& \sup_{\tau\in[0,t_1-\varrho]} \|\mathcal{S}_\gamma(t_2-\tau) - \mathcal{S}_\gamma(t_1-\tau)\| \\
& + c^{1-\gamma} \frac{2\gamma M r}{\Gamma(1+\gamma)} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} [(t_2 - t_1 + \varrho)^{\frac{p\gamma-1}{p-1}} - (t_2 - t_1)^{\frac{p\gamma-1}{p-1}}]^{1-\frac{1}{p}} \\
& + c^{1-\gamma} \frac{\gamma M r}{\Gamma(1+\gamma)} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} (t_2 - t_1)^{\gamma-\frac{1}{p}} \\
& + c^{1-\gamma} \frac{\gamma M r}{\Gamma(1+\gamma)} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} [t_1^{\frac{p\gamma-1}{p-1}} + (t_2 - t_1)^{\frac{p\gamma-1}{p-1}} - t_2^{\frac{p\gamma-1}{p-1}}]^{1-\frac{1}{p}} \\
& + [t_2^{1-\gamma} - t_1^{1-\gamma}] \frac{\gamma M r}{\Gamma(1+\gamma)} \Big(\frac{p-1}{p\gamma-1}\Big)^{1-\frac{1}{p}} t_1^{\gamma-\frac{1}{p}} \to 0,
\end{aligned}$$

as $t_2 \to t_1$, due to the uniform continuity of $\mathcal{S}_\gamma(t), t > 0$ and the arbitrariness of ϱ.

Secondly, for each $t \in [0, c]$, we prove that $\{t^{1-\gamma}Fh(t) : h \in B_{L^p}(r)\}$ is precompact in X. If $t = 0$, the conclusion is obvious. If $t \in (0, c]$, for each

$\varepsilon > 0$ with $t-2\varepsilon > 0$, define the set $\{t^{1-\gamma}F^{\varepsilon}h(t) : h \in B_{L^p}(r)\}$ in X, where

$$F^{\varepsilon}h(t) \;=\; \Gamma(\gamma)\mathcal{S}_{\gamma}(\varepsilon)\int_0^{t-\varepsilon}(t-\tau)^{\gamma-1}\mathcal{S}_{\gamma}(t-\tau-\varepsilon)h(\tau)\mathrm{d}\tau.$$

Taking into account the compactness of $\mathcal{S}_{\gamma}(\varepsilon)$, we obtain that the set $\{t^{1-\gamma}F^{\varepsilon}h(t) : h \in B_{L^p}(r)\}$ is precompact in X. Moreover, for each $h \in B_{L^p}(r)$, we have

$$\begin{aligned}
&\|t^{1-\gamma}Fh(t) - t^{1-\gamma}F^{\varepsilon}h(t)\| \\
\leq\;& c^{1-\gamma}\int_0^{t-2\varepsilon}(t-\tau)^{\gamma-1}\|[\mathcal{S}_{\gamma}(t-\tau) - \Gamma(\gamma)\mathcal{S}_{\gamma}(\varepsilon)\mathcal{S}_{\gamma}(t-\tau-\varepsilon)]h(\tau)\|\mathrm{d}\tau \\
&+c^{1-\gamma}\int_{t-2\varepsilon}^{t}\|(t-\tau)^{\gamma-1}\mathcal{S}_{\gamma}(t-\tau)h(\tau)\mathrm{d}\tau\| \\
&+c^{1-\gamma}\Gamma(\gamma)\int_{t-2\varepsilon}^{t-\varepsilon}\|(t-\tau)^{\gamma-1}\mathcal{S}_{\gamma}(\varepsilon)\mathcal{S}_{\gamma}(t-\tau-\varepsilon)h(\tau)\|\mathrm{d}\tau \\
\leq\;& c^{1-\gamma}(\frac{p-1}{p\gamma-1})^{1-\frac{1}{p}}r[t^{\frac{p\gamma-1}{p-1}} - (2\varepsilon)^{\frac{p\gamma-1}{p-1}}]^{1-\frac{1}{p}} \\
&\sup_{\tau\in[0,t-2\varepsilon]}\|\mathcal{S}_{\gamma}(t-\tau) - \Gamma(\gamma)\mathcal{S}_{\gamma}(\varepsilon)\mathcal{S}_{\gamma}(t-\tau-\varepsilon)\| \\
&+c^{1-\gamma}(\frac{p-1}{p\gamma-1})^{1-\frac{1}{p}}\frac{\gamma Mr}{\Gamma(1+\gamma)}(2\varepsilon)^{\gamma-\frac{1}{p}} \\
&+c^{1-\gamma}(\frac{p-1}{p\gamma-1})^{1-\frac{1}{p}}\frac{M^2r}{\Gamma(\gamma)}[(2\varepsilon)^{\frac{p\gamma-1}{p-1}} - \varepsilon^{\frac{p\gamma-1}{p-1}}]^{1-\frac{1}{p}} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$ by using the property (5) of Lemma 2.1. Then, the set $\{t^{1-\gamma}Fh(t) : h \in B_{L^p}(r)\}$ is precompact in X owning to the fact that the precompact set $\{t^{1-\gamma}F^{\varepsilon}h(t) : h \in B_{L^p}(r)\}$ in X is close arbitrarily to it.

Finally, applying Ascoli-Arzela theorem, one gets that $\{\cdot^{1-\gamma}Fh(\cdot) : h \in B_{L^p}(r)\}$ is precompact in $C([0,c],X)$, which means that $\{Fh(\cdot) : h \in B_{L^p}(r)\}$ is precompact in $C_{1-\gamma}([0,c],X)$. Then, we can come to the conclusion that F is compact. This completes the proof. □

3. **The solvability of fractional differential system** (1.1)

In this section, we derive the solvability of system (1.1) in the space $C_{1-\gamma}([0,c],X)$ by Schauder's fixed point theorem.

THEOREM 3.1. *Let all the hypotheses listed in Section 2 be fulfilled. Then, for each $u \in U_{ad}$, system* (1.1) *possesses at least one mild solution in $C_{1-\gamma}([0,c],X)$.*

P r o o f. For each $y_0 \in X$ and $u \in U_{ad}$, consider the operator $\mathcal{Q}$: $C_{1-\gamma}([0,c],X) \to C_{1-\gamma}([0,c],X)$ as follows:

$$\mathcal{Q}y(t) = t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 + \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,y(\tau)) + B(\tau)u(\tau)]\mathrm{d}\tau.$$

It is not difficult to verify that $\mathcal{Q}$ is well defined. Then, the solvability of system (1.1) will be transformed into a fixed point problem of $\mathcal{Q}$. For clarity, we proceed into the following steps.

Step 1. Let $r > 0$ and $B_{C_{1-\gamma}}(r) = \{y \in C_{1-\gamma}([0,c],X) : \|y\|_{C_{1-\gamma}} \le r\}$. We show that $\mathcal{Q}B_{C_{1-\gamma}}(r) \subseteq B_{C_{1-\gamma}}(r)$ provided that

$$r > \frac{M\gamma\|y_0\| + M\gamma[\big(\frac{(p-1)c}{p\gamma-1}\big)^{1-\frac{1}{p}}(\|\eta\|_{L^p} + \|Bu\|_{L^p})]}{\Gamma(1+\gamma) - M\rho c}.$$

In fact, for each $y \in B_{C_{1-\gamma}}(r)$, one has

$$\begin{aligned}
& t^{1-\gamma}\|\mathcal{Q}y(t)\| \\
\le\ & \frac{M\gamma}{\Gamma(1+\gamma)}\|y_0\| + \frac{M\gamma}{\Gamma(1+\gamma)}[\big(\frac{(p-1)c}{p\gamma-1}\big)^{1-\frac{1}{p}}(\|\eta\|_{L^p} + \|Bu\|_{L^p}) + \frac{\rho r c}{\gamma}] \\
\le\ & r,
\end{aligned}$$

that is,

$$\||\mathcal{Q}y\|_{C_{1-\gamma}} = \sup_{t\in[0,c]} t^{1-\gamma}\|\mathcal{Q}y(t)\| \le r.$$

Step 2. We show that $\mathcal{Q}$ is continuous on $B_{C_{1-\gamma}}(r)$. To this end, let $\{y_n\}_{n\ge1} \subseteq B_{C_{1-\gamma}}(r)$ with $\lim_{n\to\infty} y_n = y \in B_{C_{1-\gamma}}(r)$, that is,

$$\|y_n - y\|_{C_{1-\gamma}} = \sup_{t\in[0,c]} t^{1-\gamma}\|y_n(t) - y(t)\| \to 0$$

as $n \to \infty$. This yields $t^{1-\gamma}y_n(t) \to t^{1-\gamma}y(t)$ as $n \to \infty$ uniformly for $t \in [0,c]$. Note that, for a.e. $\tau \in [0,t]$,

$$g(\tau, y_n(\tau)) = g(\tau, \tau^{\gamma-1}\tau^{1-\gamma}y_n(\tau)) \to g(\tau, \tau^{\gamma-1}\tau^{1-\gamma}y(\tau)) = g(\tau, y(\tau))$$

as $n \to \infty$, and

$$\|g(\tau, y_n(\tau)) - g(\tau, y(\tau))\| \ \le\ 2(\eta(\tau) + \rho r),$$

where $\eta(\cdot) \in L^p([0,c], \mathbb{R}^+)$. Then, by applying the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
& t^{1-\gamma}\|\mathcal{Q}y_n(t) - \mathcal{Q}y(t)\| \\
\le\ & t^{1-\gamma}\int_0^t (t-\tau)^{\gamma-1}\|\mathcal{S}_\gamma(t-\tau)[g(\tau,y_n(\tau)) - g(\tau,y(\tau))]\|\mathrm{d}\tau
\end{aligned}$$

$$\leq \frac{M\gamma}{\Gamma(1+\gamma)}\Big(\frac{(p-1)c}{p\gamma-1}\Big)^{1-\frac{1}{p}}\Big(\int_0^c \|g(\tau,y_n(\tau))-g(\tau,y(\tau))\|^p \mathrm{d}\tau\Big)^{\frac{1}{p}} \to 0,$$

as $n\to\infty$ uniformly for each $t\in[0,c]$, which means that

$$\|\mathcal{Q}y_n-\mathcal{Q}y\|_{C_{1-\gamma}} = \sup_{t\in[0,c]} t^{1-\gamma}\|\mathcal{Q}y_n(t)-\mathcal{Q}y(t)\| \to 0,\ n\to\infty.$$

Step 3. We check the compactness of $\mathcal{Q}$ on $C_{1-\gamma}([0,c],X)$. The definition of $\mathcal{Q}$ yields that the compactness of $\mathcal{Q}$ is reduced to the compactness of $\tilde{\mathcal{Q}}$ on $C_{1-\gamma}([0,c],X)$, where

$$\tilde{\mathcal{Q}}y(t)=\int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)g(\tau,y(\tau))\mathrm{d}\tau,\ t\in[0,c]$$

for each $y\in C_{1-\gamma}([0,c],X)$. It should be point out that $(Hg)(2)$ implies that $g(\cdot,y(\cdot))\in L^p([0,c],X)$. A similar manner utilized in Lemma 2.2 gives the compactness of $\tilde{\mathcal{Q}}$.

Now, it is obvious that the conclusion of Theorem 3.1 holds by using the Schauder's fixed point theorem. □

Remark 3.1. By virtue of Theorem 3.1, for each $u\in U_{ad}$, let $y^u\in C_{1-\gamma}([0,c],X)$ be any one of the corresponding mild solutions of system (1.1). Then, $\|y^u\|_{C_{1-\gamma}}\leq R$, where

$$R:=\frac{M\gamma}{\Gamma(1+\gamma)}[\|y_0\|+\Big(\frac{(p-1)c}{p\gamma-1}\Big)^{1-\frac{1}{p}}(\|\eta\|_{L^p}+\|B\|_\infty\|m\|_{L^p})]E_\gamma(Mc\rho),$$

which is independent of u, and $E_\gamma(z)=\sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(n\gamma+1)}$ is the Mittag-Leffler function. In fact, for each $t\in[0,c]$, one has

$$\begin{aligned}
& t^{1-\gamma}\|y^u(t)\| \\
\leq\ & \|\mathcal{S}_\gamma(t)y_0\|+t^{1-\gamma}\int_0^t (t-\tau)^{\gamma-1}\|\mathcal{S}_\gamma(t-\tau)[g(\tau,y^u(\tau))+B(\tau)u(\tau)]\|\mathrm{d}\tau \\
\leq\ & \frac{M\gamma}{\Gamma(1+\gamma)}[\|y_0\|+\Big(\frac{(p-1)c}{p\gamma-1}\Big)^{1-\frac{1}{p}}(\|\eta\|_{L^p}+\|Bu\|_{L^p})] \\
& +\frac{M\gamma}{\Gamma(1+\gamma)}c^{1-\gamma}\int_0^t (t-\tau)^{\gamma-1}\rho\|\tau^{1-\gamma}y^u(\tau)\|\mathrm{d}\tau.
\end{aligned}$$

By using Corollary 2 of [31], one can obtain that

$$t^{1-\gamma}\|y^u(t)\|\leq\frac{M\gamma}{\Gamma(1+\gamma)}[\|y_0\|+\Big(\frac{(p-1)c}{p\gamma-1}\Big)^{1-\frac{1}{p}}(\|\eta\|_{L^p}+\|B\|_\infty\|m\|_{L^p})]E_\gamma(Mc\rho).$$

This means that

$$\|y^u\|_{C_{1-\gamma}} \leq \frac{M\gamma}{\Gamma(1+\gamma)}[\|y_0\|+\Big(\frac{(p-1)c}{p\gamma-1}\Big)^{1-\frac{1}{p}}(\|\eta\|_{L^p([0,c])}$$

$$+ \quad \|B\|_\infty \|m\|_{L^p([0,c])})]E_\gamma(Mc\rho).$$

REMARK 3.2. For simplicity, we denote by

$$S(u) = \{y^u \in B_{C_{1-\gamma}}(R) : y^u \textit{ is mild solution of } (1.1) \textit{ corresponding to the control } u \in U_{ad}\},$$

$$\mathcal{A}_d = \{(y^u, u) : u \in U_{ad},\ y^u \in S(u)\}.$$

REMARK 3.3. A pair (y^u, u) is said to be feasible for the system (1.1) if and only if $(y^u, u) \in \mathcal{A}_d$.

4. **Time optimal control problems subjected to system** (1.1)

Let W_T be a bounded, closed and convex subset in X. Define the subsets as follows:

$$\mathcal{A}_d^{W_T} = \{(y^u, u) \in \mathcal{A}_d : t^{1-\gamma} y^u(t) \in W_T \text{ for some } t \in [0, c]\};$$

$$U_0 = \{u \in U_{ad} : (y^u, u) \in \mathcal{A}_d^{W_T} \text{ for some } y^u \in S(u)\};$$

$$S_u^{W_T} = \{y^u \in S(u) : u \in U_0,\ (y^u, u) \in \mathcal{A}_d^{W_T}\}.$$

Suppose that $\mathcal{A}_d^{W_T} \neq \emptyset$. For each $(y^u, u) \in \mathcal{A}_d^{W_T}$, we define the transition time $t_{(y^u,u)}$ as the first time such that $t_{(y^u,u)}^{1-\gamma} y^u(t_{(y^u,u)}) \in W_T$. The set W_T is called the target set.

REMARK 4.1. In general case, the subset $\mathcal{A}_d^{W_T}$ is defined as:

$$\mathcal{A}_d^{W_T} = \{(y^u, u) \in \mathcal{A}_d : y^u(t) \in W_T \text{ for some } t \in [0, c]\}.$$

However, since the solution is obtained in the space $C_{1-\gamma}([0, c], X)$ and $y^u(t)$ is indeed unbounded near the zero, a rescaling technique is necessary and a reasonable definition of $\mathcal{A}_d^{W_T}$ in case of Riemann-Liouville derivatives is given as above.

REMARK 4.2. For each $(y^u, u) \in \mathcal{A}_d^{W_T}$, the definition of transition time gives

$$t_{(y^u,u)} = \min \mathcal{T}(y^u, u),$$

where $\mathcal{T}(y^u, u) := \{t : t \in [0, c],\ t^{1-\gamma} y^u(t) \in W_T\}$. We claim that $t_{(y^u,u)}$ is well defined. In fact, if the set $\mathcal{T}(y^u, u)$ contains finite elements, the proof is trivial. Otherwise, let $\tilde{t} = \inf \mathcal{T}(y^u, u)$. This gives that

$$\lim_{n\to\infty} t_n = \tilde{t}$$

for some decreasing $\{t_n\}_{n\geq 1} \subseteq \mathcal{T}(y^u,u)$, that is, $t_n^{1-\gamma}y^u(t_n) \in W_T$. The fact $\cdot^{1-\gamma}y^u(\cdot) \in C([0,c],X)$ yields

$$\lim_{n\to\infty} t_n^{1-\gamma}y^u(t_n) = \tilde{t}^{1-\gamma}y^u(\tilde{t}).$$

This together with the closeness of W_T gives

$$\tilde{t}^{1-\gamma}y^u(\tilde{t}) \in W_T.$$

This means that $\tilde{t} = \min \mathcal{T}(y^u,u)$, and $t_{(y^u,u)} = \tilde{t}$.

Based on the above definitions and notations, now we consider the time optimal control problem (P): Find $(y_*,u_*) \in \mathcal{A}_d^{W_T}$ such that

$$t_{(y_*,u_*)} = \min_{(y^u,u)\in\mathcal{A}_d^{W_T}} t_{(y^u,u)}.$$

If the control u_*, the time $t_{(y_*,u_*)}$ and the pair (y_*,u_*) exist solving problem (P), we call them time optimal control, optimal time and time optimal pair, respectively.

THEOREM 4.1. *Assume that all the hypotheses given in Section 2 are satisfied. Then, problem (P) possesses at least one time optimal pair.*

P r o o f. In view of Theorem 3.1, there exists at least one $y^u \in B_{C_{1-\gamma}}(R)$ such that $(y^u,u) \in \mathcal{A}_d$ for each $u \in U_{ad}$. We will proceed in the following two steps to derive the main result.

Step 1. For each $u \in U_0$, set $t_u = \inf_{y^u\in S_u^{W_T}} t_{(y^u,u)}$. We now need to check that $t_u^{1-\gamma}\hat{y}^u(t_u) \in W_T$ for some $\hat{y}^u \in S_u^{W_T}$. It is trivial in situation in which the set $S_u^{W_T}$ has finite elements. Otherwise, there is a monotone decreasing sequence $\{t_{(y_n^u,u)}\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} t_{(y_n^u,u)} = t_u, \tag{4.1}$$

where $(y_n^u,u) \in \mathcal{A}_d^{W_T}$ for each $n \geq 1$. Moreover, the definition of $t_{(y_n^u,u)}$ gives

$$t_{(y_n^u,u)}^{1-\gamma} y_n^u(t_{(y_n^u,u)}) \in W_T. \tag{4.2}$$

The fact $y_n^u \in S(u)$ yields

$$y_n^u(t) = t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 + \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,y_n^u(\tau)) + B(\tau)u(\tau)]\mathrm{d}\tau \tag{4.3}$$

for each $n \geq 1$ and $t \in (0,c]$. Exploiting the compactness of $\mathcal{S}_\gamma(t), t > 0$, a similar method used in Lemma 2.2, we can infer that $\{y_n^u\}_{n\geq 1}$ is precompact in $C_{1-\gamma}([0,c],X)$. Then, there is a subsequence of $\{y_n^u\}_{n\geq 1}$, still relabled by it, and a function $\hat{y}^u \in B_{C_{1-\gamma}}(R)$, such that

$$\|y_n^u - \hat{y}^u\|_{C_{1-\gamma}} = \sup_{t\in[0,c]} t^{1-\gamma}\|y_n^u(t) - \hat{y}^u(t)\| \to 0 \tag{4.4}$$

as $n \to \infty$. This together with (Hg) gives that

$$g(\tau, y_n^u(\tau)) \to g(\tau, \hat{y}^u(\tau)) \text{ and } \|g(\tau, y_n^u(\tau))\| \quad \leq \quad \eta(\tau) + \rho R$$

for a.e. $\tau \in [0,t]$, where $\eta(\cdot) \in L^p([0,c],\mathbb{R}^+)$. Now, taking $n \to \infty$ to both sides of (4.3) and using Lebesgue dominated convergence theorem yield

$$\hat{y}^u(t) \quad = \quad t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 + \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,\hat{y}^u(\tau)) + B(\tau)u(\tau)]\mathrm{d}\tau \tag{4.5}$$

for each $t \in (0,c]$. This gives that

$$\hat{y}^u \in S(u). \tag{4.6}$$

It is worth noticing that (4.1) and (4.4) lead to

$$t_{(y_n^u,u)}^{1-\gamma} y_n^u(t_{(y_n^u,u)}) \to t_u^{1-\gamma}\hat{y}^u(t_u) \tag{4.7}$$

as $n \to \infty$. In fact,

$$\begin{aligned}\|t_{(y_n^u,u)}^{1-\gamma} y_n^u(t_{(y_n^u,u)}) - t_u^{1-\gamma}\hat{y}^u(t_u)\| \quad &\leq \quad \|t_{(y_n^u,u)}^{1-\gamma} y_n^u(t_{(y_n^u,u)}) - t_{(y_n^u,u)}^{1-\gamma}\hat{y}^u(t_{(y_n^u,u)})\| \\ &\quad +\|t_{(y_n^u,u)}^{1-\gamma}\hat{y}^u(t_{(y_n^u,u)}) - t_u^{1-\gamma}\hat{y}^u(t_u)\|.\end{aligned}$$

It follows from (4.4) that $\|t_{(y_n^u,u)}^{1-\gamma} y_n^u(t_{(y_n^u,u)}) - t_{(y_n^u,u)}^{1-\gamma}\hat{y}^u(t_{(y_n^u,u)})\| \to 0$ as $n \to \infty$. The fact $\cdot^{1-\gamma}\hat{y}^u(\cdot) \in C([0,c],X)$ and (4.1) give $\|t_{(y_n^u,u)}^{1-\gamma}\hat{y}^u(t_{(y_n^u,u)}) - t_u^{1-\gamma}\hat{y}^u(t_u)\| \to 0$ as $n \to \infty$. (4.2) and (4.7), together with the closeness of W_T give rise to the fact that

$$t_u^{1-\gamma}\hat{y}^u(t_u) \in W_T. \tag{4.8}$$

Combining this with (4.6) yields $\hat{y}^u \in S_u^{W_T}$.

Step 2. Put $t_* = \inf_{u\in U_0} t_u$, where t_u is the optimal time for fixed u in Step 1. Our task now is to seek an $u_* \in U_0$ and $y_* \in S_{u_*}^{W_T}$ such that $t_*^{1-\gamma}y_*(t_*) \in W_T$. The proof is trivial provided that U_0 contains finite elements, or there exists a monotone decreasing sequence $\{t_{u_n}\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} t_{u_n} = t_*. \tag{4.9}$$

According to Step 1, for each $n \geq 1$, we can deduce that there is a function $\hat{y}^{u_n} \in S_{u_n}^{W_T}$ such that

$$t_{u_n}^{1-\gamma}\hat{y}^{u_n}(t_{u_n}) \in W_T, \tag{4.10}$$

and

$$\begin{aligned}\hat{y}^{u_n}(t) &= t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 \\ &+ \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,\hat{y}^{u_n}(\tau)) + B(\tau)u_n(\tau)]\mathrm{d}\tau \end{aligned}\tag{4.11}$$

for each $n \geq 1$ and $t \in (0,c]$. Note that $B(\cdot)u_n(\cdot) \in L^p([0,c],X)$. So, an argument similar with the one employed in Lemma 2.2 gives rise to the precompactness of $\{\hat{y}^{u_n}\}_{n\geq 1}$ in $C_{1-\gamma}([0,c],X)$. Then, a subsequence of $\{\hat{y}^{u_n}\}_{n\geq 1}$ can be extracted, and still denoted by it, which satisfies

$$\lim_{n\to\infty}\hat{y}^{u_n} = y_* \tag{4.12}$$

for some $y_* \in C_{1-\gamma}([0,c],X)$. It is notable that $\{u_n\}_{n\geq 1} \subseteq U_{ad} \subseteq L^p([0,c],Y)$ and $\|u_n\|_{L^p} \leq \|m\|_{L^p}$. The reflexivity of Y and the boundedness of $\{u_n\}_{n\geq 1}$ imply that a subsequence of $\{u_n\}_{n\geq 1}$, still relabled by it, satisfies

$$u_n \rightharpoonup u_* \tag{4.13}$$

as $n \to \infty$ for some $u_* \in L^p([0,c],Y)$. Owing to the fact that U_{ad} is convex and closed, we can infer that $u_* \in U_{ad}$ by using Mazur's lemma. Thanks to Lemma 2.2, it is easy to deduce that

$$\begin{aligned}&\int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)B(\tau)u_n(\tau)\mathrm{d}\tau \\ \to &\int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)B(\tau)u_*(\tau)\mathrm{d}\tau\end{aligned}\tag{4.14}$$

as $n \to \infty$ since (4.13) holds. Now, making $n \to \infty$ to both sides of (4.11) gives

$$y_*(t) = t^{\gamma-1}\mathcal{S}_\gamma(t)y_0 + \int_0^t (t-\tau)^{\gamma-1}\mathcal{S}_\gamma(t-\tau)[g(\tau,y_*(\tau)) + B(\tau)u_*(\tau)]\mathrm{d}\tau \tag{4.15}$$

for $t \in (0,c]$, which implies that $y_* \in S(u_*)$. We now turn back to (4.12). Together with (4.9), we have

$$t_{u_n}^{1-\gamma}\hat{y}^{u_n}(t_{u_n}) \to t_*^{1-\gamma}y_*(t_*) \tag{4.16}$$

as $n \to \infty$, which means that $t_*^{1-\gamma}y_*(t_*) \in W_T$ due to the closeness of W_T and (4.10), and it is straight forward to see that $y_* \in S_{u_*}^{W_T}$. This completes the proof. □

Remark 4.3. The new approach of constructing time minimizing sequences twice is applied to make up the lack of uniqueness of the mild solution. Thus, we can remove the Lipschitz continuity of nonlinear terms without any additional conditions. What is more, since the reflexivity of the state space X is no longer satisfied, we take full advantage of the compact method, and thus the time optimal pairs are still acquired. Therefore, the results here essentially generalize those in [10, 11, 14, 26, 29, 32], and the references therein, where the Lipschitz continuity of nonlinear function and the reflexivity of X are all required.

5. **Applications**

The following system concerned with the fractional Riemann-Liouville derivative will be considered here to illustrate our main results.

$$\begin{cases} \frac{\partial^{\frac{3}{4}}\omega(t,\theta)}{\partial t^{\frac{3}{4}}} = \frac{\partial^2}{\partial\theta^2}\omega(t,\theta) + \frac{1}{4}t^{\frac{1}{4}}\sin(\omega(t,\theta)) + \int_0^\pi \zeta(\theta,\tau)u(t,\tau)\mathrm{d}\tau, \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in(0,1], \theta\in[0,\pi], \\ \omega(t,0)=\omega(t,\pi)=0, \\ I^{\frac{1}{4}}\omega(t,\theta)|_{t=0}=\omega_0(\theta). \end{cases} \tag{5.1}$$

Let $X=Y=L^2([0,\pi],\mathbb{R})$. Define the operator $A: D(A)\subseteq X\to X$ as

$$Ax=x''$$

with the domain

$D(A)=\{x\in X; x, x'$ are absolutely continuous, $x''\in X, x(0)=x(\pi)=0\}$.

Then,

$$Ax=\sum_{n=1}^{\infty}-n^2(x,\xi_n)\xi_n,\ x\in D(A),$$

where $\xi_n(\theta)=\sqrt{\frac{2}{\pi}}\sin(n\theta), n=1,2,\cdots$ is an orthonormal basis of X. By virtue of [25], we infer that A generates a compact and analytic semigroup $\{T(t)\}_{t>0}$ in X, and

$$T(t)x=\sum_{n=1}^{\infty}e^{-n^2t}(x,\xi_n)\xi_n,\ x\in X.$$

Obviously, $\|T(t)\|\le 1$.

Now, for every $t\in(0,1],\ \theta\in[0,\pi]$, let $y(t)(\theta)=\omega(t,\theta)$, $g(t,y(t))(\theta)=\frac{1}{4}t^{\frac{1}{4}}\sin(y(t)(\theta))$, $u\in L^2([0,1]\times[0,\pi],\mathbb{R})$, $u(t)(\theta)=u(t,\theta)$. $\zeta\in C([0,\pi]\times[0,\pi],\mathbb{R})$, and $B(t)u(t)(\theta)=\int_0^\pi\zeta(\theta,\tau)u(t,\tau)\mathrm{d}\tau$.

Define admissible control set

$$U(t)=\{u(t)(\cdot)\in Y: \|u(t)(\cdot)\|_Y\le N_1\},$$

with $N_1 > 0$. Then, fractional system (5.1) can be reformulated as the abstract fractional system (1.1), and we can prove that all the conditions listed in Section 2 are satisfied. In fact, $c = 1$ and $M = \sup_{t\in[0,1]} \|T(t)\| = 1$. Moreover,

$$\begin{aligned}\|g(t,y(t))(\cdot)\|_X &= \left(\int_0^\pi \|\frac{1}{4}t^{\frac{1}{4}}\sin(y(t)(\theta))\|^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \le \frac{1}{4}t^{\frac{1}{4}}\left(\int_0^\pi \|y(t)(\theta)\|^2\mathrm{d}\theta\right)^{\frac{1}{2}} \\ &= \frac{1}{4}t^{\frac{1}{4}}\|y(t)(\cdot)\|_X,\end{aligned}$$

with $\eta(t) = 0$ and $\rho = \frac{1}{4} < \frac{\Gamma(\frac{7}{4})}{Mc}$, and

$$\begin{aligned}&\|B(t)u(t)(\cdot)\|_X \le \sqrt{\pi}\sup_{0\le\theta\le\pi}\left|\int_0^\pi \zeta(\theta,\tau)u(t,\tau)\mathrm{d}\tau\right| \\ &\le \sqrt{\pi}\sup_{0\le\theta\le\pi}\left|\left(\int_0^\pi |\zeta(\theta,\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^\pi |u(t,\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \le a\|u(t)(\cdot)\|_Y\end{aligned}$$

for each $t \in [0,1]$, where $a = \pi \sup_{0\le\theta,\tau\le\pi} |\zeta(\theta,\tau)|$. Define the target set

$$W_T = \{x \in X : \|x\|_X \le N_2\}$$

with $N_2 > 0$. If the set $\mathcal{A}_d^{W_T} \neq \emptyset$, then it follows from Theorem 4.1 that there exists a time optimal pair (y_*, u_*) such that the transition time $t_{(y,u)}$ attains its minimum.

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