Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 21, NUMBER 1 (2018) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

SOME STABILITY PROPERTIES RELATED TO INITIAL TIME DIFFERENCE FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Lipschitz stability and Mittag-Leffler stability with initial time difference for nonlinear nonautonomous Caputo fractional differential equation are defined and studied using Lyapunov like functions. Some sufficient conditions are obtained. The fractional order extension of comparison principles via scalar fractional differential equations with a parameter is employed. The relation between both types of stability is discussed theoretically and it is illustrated with examples.

MSC 2010: Primary 34A34; Secondary 34A08, 34A37

Key Words and Phrases: Caputo fractional differential equations, Lipschitz stability, Mittag-Leffler stability, initial data difference, Lyapunov functions

1. **Introduction**

Fractional calculus is the theory of integrals and derivatives of arbitrary non-integer order. The subject is as old as classical calculus and goes back to the 17-th century. It was realized that various processes with anomalous dynamics in science and engineering can be formulated mathematically using fractional differential operators because of its memory and hereditary properties [14], [21]. The qualitative theory of fractional differential equations (FrDE) has received a lot of attention. One of the main problems in the qualitative theory of differential equations is stability of the solutions. Some stability concepts were presented and studied by applying various

pp. 72–93 , DOI: 10.1515/fca-2018-0005 DE GRUYTER

methods such as the first and second method of Lyapunov ([1], [3], [16], [23]). One type of stability, useful in real world problems, is the so called Lipschitz stability. In 1986, F.M. Dannan, S. Elaydi ([13]) introduced the concept of Lipschitz stability for nonlinear ordinary differential equations. They mention that uniform Lipschitz stability lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation and uniform stability in variation on the other side. Furthermore, uniform Lipschitz stability neither implies asymptotic stability nor is it implied by it (see also [11]).

Recently, fractional calculus was used for the stability analysis of FrDE. However, there are several difficulties in applying Lyapunov's technique to stability analysis of FrDE which is connected with the type of derivative of the Lyapunov function:

– continuously differentiable Lyapunov functions: the Caputo derivative of Lyapunov functions of unknown solution is applied (see, for example, [8, 19, 20]). In this case the chain rule in fractional calculus can cause trouble in application.

– continuous Lyapunov functions: the Dini derivative of Lyapunov functions in the case of ordinary derivative is extended to the fractional Dini derivative of the Lyapunov function (see, for example, [17, 18, 22]). This derivative does not have a memory and it is independent on the initial time and this differs from the idea of fractional calculus.

In real life situations it may be impossible to have only a change in the space variable and to keep the initial time unchanged. However, this situation requires introducing and studying a new generalization of the classical concept of stability which involves a change in both the initial time and the initial values. This type of stability generalizes known stability in the literature (see, for example [2]). Recently Lyapunov functions are applied to study some types of stability with respect to initial time difference for FrDE by an application of a new type of Caputo fractional Dini derivative of the Lyapunov function ([5, 6, 7]).

In this paper Lipschitz stability and Mittag-Leffler stability for nonlinear nonautonomous Caputo fractional differential equations are studied using the Caputo fractional Dini derivative of the Lyapunov functions relative to initial data difference ([5, 6, 7]) along the given FrDE. The Lipschitz, uniformly Lipschitz, globally uniformly Lipschitz stability and Mittag-Leffler stability are appropriately defined relative to initial time difference for fractional differential equations. Several sufficient conditions for Lipschitz stability and Mittag-Leffler stability with initial data difference for nonlinear fractional differential equations via Lyapunov functions and comparison results for a scalar fractional differential equation with a parameter are obtained.

2. **Notes on Fractional Calculus**

Fractional calculus generalizes the derivatives and the integrals of a function to a non-integer order [14, 21] and there are several definitions of fractional derivatives and fractional integrals.

In many applications in science and engineering, the fractional order q is often less than 1, so we restrict $q \in (0,1)$ everywhere in the paper.

The most widely used definitions for fractional derivatives are the Riemann-Liouville (e.g., in calculus), the Caputo (e.g., in physics and numerical integration), and the Grunwald-Letnikov (e.g., in signal processing, engineering, and control) ones:

1: The *Riemann-Liouville (RL) fractional derivative* of order $q \in (0,1)$ is given by

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \qquad t \geq t_0, $$

where $\Gamma(.)$ denotes the Gamma function. The above definition of the fractional differentiation of Riemann-Liouville type leads to a conflict between the well-established mathematical theory of differential equations, such as the initial problem of the fractional differential equation, and the nonzero problem related to the Riemann-Liouville derivative of a constant.

2: The *Caputo fractional derivative* of order $q \in (0,1)$ is defined by

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m^{'}(s) ds, \quad t \geq t_0. \tag{2.1} $$

The Caputo and Riemann-Liouville formulations coincide when the initial conditions are zero. Also, the RL derivative is meaningful under weaker smoothness requirements.

The properties of the Caputo derivative are more similar to those of ordinary derivatives, such as the constant's property. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and the Caputo derivative is extensively used in real life applications.

3: The *Grünwald-Letnikov fractional derivative* of order $q \in (0,1)$ is given by

$$ {}_{t_0}^{GL}D_t^q m(t) = \lim_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \qquad t \geq t_0 $$

and Grünwald-Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \lim_{h\to 0+} \sup \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m\,(t-rh)\,, \qquad t \geq t_0\,, \tag{2.2} $$

where $\binom{q}{r} = \frac{q(q-1)\dots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

For a wide class of functions, the definitions of Grünwald-Letnikov fractional derivative and the Riemann-Liouville fractional derivative are equivalent (for example, if the functions are sufficiently smooth). This allows us to use Grünwald-Letnikov fractional derivative for the formulation of the problem and for proving theoretical results, and then one can turn to the Riemann-Liouville fractional derivative for applied problems.

The relation between both Caputo fractional derivative and Grünwald-Letnikov fractional derivative is:

$$ {}_{t_0}^{C}D^q m(t) = {}_{t_0}^{RL}D^q[m(t) - m(t_0)] = {}_{t_0}^{GL}D^q[m(t) - m(t_0)]. \tag{2.3} $$

Using (2.2) we define the Caputo fractional Dini derivative of a function as

$$ \begin{aligned} {}_{t_0}^{C}D_+^q m(t) = \lim_{h\to 0+} \sup \frac{1}{h^q} \Big[m(t) - m(t_0) \\ - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} (m\,(t-rh) - m(t_0)) \Big]. \end{aligned} \tag{2.4} $$

Definition 1. ([15]) We say the function $m(t) \in C^q([t_0,T],\mathbb{R}^n)$ if it is differentiable and the Caputo derivative ${}_{t_0}^{C}D^q m(t)$ exists and satisfies (2.1) for $t \in [t_0,T]$.

Remark 2.1. If $m(t) \in C^q([t_0,T],\mathbb{R}^n)$, then ${}_{t_0}^{C}D_+^q m(t) = {}_{t_0}^{C}D^q m(t)$.

3. Statement of the problem

Let $t_0 \in \mathbb{R}_+$ be an arbitrary initial time and consider the following initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative

$$ {}_{t_0}^{C}D^q x(t) = f(t, x(t)) \quad \text{for } t > t_0, \qquad x(t_0) = x_0, \tag{3.1} $$

where $0 < q < 1$, $x_0 \in \mathbb{R}^n$.

Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$, be an initial time and consider the following IVP for FrDE

$$ {}_{\tau_0}^{C}D^q x(t) = f(t, x(t)) \quad \text{for } t > \tau_0, \qquad x(\tau_0) = y_0, \tag{3.2} $$

where $y_0 \in \mathbb{R}^n$.

We will assume in the paper that the function $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is such that for any initial data $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the corresponding IVP for FrDE (3.1) has a solution $x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$. Note some sufficient conditions for global existence of solutions of (3.1) are given in [10, 9, 17].

We will make use of the following result:

Lemma 3.1. ([7]) *Let the function $x(t) \in C^q(\mathbb{R}_+, \mathbb{R}^n)$, $a \geq 0$, be a solution of the initial value problem for FrDE*

$$ {}_a^C D^q x(t) = f(t, x(t)) \quad \text{for } t > a, \qquad x(a) = x_0. \tag{3.3} $$

Then the function $\widetilde{x}(t) = x(t + \eta)$ satisfies the initial value problem for the FrDE

$$ {}_b^C D^q \widetilde{x}(t) = f(t + \eta, \widetilde{x}(t)) \quad \text{for } t > b, \qquad \widetilde{x}(b) = x_0. \tag{3.4} $$

where $b \geq 0$, $\eta = a - b$.

The relation between (3.1) and (3.2) is given by the following result:

Corollary 3.1. ([7]) *For any solution $x(t) = x(t; \tau_0, y_0)$ of (3.2) the function $\widetilde{x}(t) = x(t + \eta)$ is a solution of IVP for FrDE*

$$ {}_{t_0}^C D^q \widetilde{x}(t) = f(t + \eta, \widetilde{x}(t)) \quad \text{for } t > t_0, \qquad \widetilde{x}(t_0) = y_0, \tag{3.5} $$

where $\eta = \tau_0 - t_0$.

Remark 3.1. In the autonomous case, i.e. $f(t, x) \equiv F(x)$, from Corollary 1 it follows that we can study only the case of zero initial time and zero lower bound of the fractional derivative. At the same time changing the initial time of the IVP for nonautonomous case leads to a change of the lower limit of the fractional derivative and to a change of the equation.

The main goal is to compare the behavior of two solutions of Caputo fractional differential equations with different initial data, both initial time $\tau_0 \neq t_0$ and initial points $y_0 \neq x_0$. In real life situations it may not be possible to keep measurements with the expected initial time. So, when we study the influence of parameters, sometimes we need to consider two solutions which have not only different initial points, but also different initial times. The stability with respect to initial time difference (ITD) gives us an opportunity to compare solutions of FrDE when both initial time and position are different. We will study the Lipschitz stability with ITD and its connection with Mittag-Leffler stability with ITD of the system of Caputo fractional differential equations.

Definition 2. The given solution $x^*(t) = x(t; t_0, x_0)$ of FrDE (3.1) is called:

- *Lipschitz stable with initial time difference (ITD)*, if there exist $M \geq 1$, $\delta = \delta(t_0) > 0$ and $\sigma = \sigma(t_0) > 0$ such that for any initial value $y_0 \in \mathbb{R}^n : ||y_0 - x_0|| < \delta$ and any initial time $\tau_0 \in \mathbb{R}_+ : |\tau_0 - t_0| < \sigma$ the inequality $||y(t + \eta; \tau_0, y_0) - x^*(t)|| \leq M||y_0 - x_0||$ for $t \geq t_0$ holds, where $\eta = \tau_0 - t_0$ and $y(t; \tau_0, y_0)$ is a solution of (3.2).
- *eventually Lipschitz stable with ITD*, if there exist $M \geq 1$, $\delta = \delta(t_0) > 0$, $T = T(t_0) > 0$ and $\sigma = \sigma(t_0) > 0$ such that for any initial value $y_0 \in \mathbb{R}^n : ||y_0 - x_0|| < \delta$ and any initial time $\tau_0 \in \mathbb{R}_+ : |\tau_0 - t_0| < \sigma$ the inequality $||y(t + \eta; \tau_0, y_0) - x^*(t)|| \leq M||y_0 - x_0||$ for $t \geq t_0 + T$ holds, where $\eta = \tau_0 - t_0$ and $y(t; \tau_0, y_0)$ is a solution of (3.2)

Definition 3. The system of FrDE (3.1) is called:

- *uniformly Lipschitz stable with ITD*, if there exist constants $M \geq 1$ and $\delta, \sigma > 0$ such that for any initial values $x_0, y_0 \in \mathbb{R}^n$ and any initial times $t_0, \tau_0 \in \mathbb{R}_+$ the inequalities $||y_0 - x_0|| < \delta$ and $|\tau_0 - t_0| < \sigma$ imply $||y(t+\eta; \tau_0, y_0) - x(t; t_0, x_0)|| \leq M||y_0 - x_0||$ for $t \geq t_0$ where $\eta = \tau_0 - t_0$ and $x(t; t_0, x_0)$, $y(t; \tau_0, y_0)$ are solution of (3.1), (3.2), respectively;
- globally uniformly Lipschitz stable with ITD, if there exist constants $M \geq 1, \sigma > 0$ such that for any initial values $x_0, y_0 \in \mathbb{R}^n$ and any initial times $t_0, \tau_0 \in \mathbb{R}_+$ the inequalities $||y_0 - x_0|| < \infty$ and $|\tau_0 - t_0| < \sigma$ imply $||y(t + \eta; \tau_0, y_0) - x(t; t_0, x_0)|| \leq M||y_0 - x_0||$ for $t \geq t_0$ where $\eta = \tau_0 - t_0$ and $x(t; t_0, x_0)$, $y(t; \tau_0, y_0)$ are solution of (3.1), (3.2), respectively.

Definition 4. The solution $x^*(t) = x(t; t_0, x_0)$ of FrDE (3.1) is called Mittag-Leffler stable with ITD, if there exist $\lambda > 0$, $C > 0$, $\sigma = \sigma(t_0) > 0$ and constants $a, b > 0$, such that for any initial time $\tau_0 \in \mathbb{R}_+ : |\tau_0 - t_0| < \sigma$ the inequality $||y(t + \eta; \tau_0, y_0) - x^*(t)|| \leq C||y_0 - x_0||^a \{E_q(-\lambda(t - t_0)^q)\}^b$ for $t \geq t_0$ holds, where $\eta = \tau_0 - t_0$ and $y(t; \tau_0, y_0)$ is a solution of (3.2).

Note the concept of any type of stability with ITD is meaningful only in the case of non-autonomous systems (see Remark 3.1).

Remark 3.2. The concept of Lipschitz stability with ITD defined in Definition 2 generalizes Lipschitz stability for the zero solution of fractional equations in the literature [22] if $x^*(t) \equiv 0$ and $\tau_0 = t_0$.

Remark 3.3. In the case when $x^*(t) \equiv 0$ and $\tau_0 = t_0$ the concept of Mittag-Leffler stability with ITD defined in Definition 4 generalizes the definition of Mittag-Leffler stability in [20].

Remark 3.4. In the special case of $C \geq 1$ and $a = 1$ the Mittag-Leffler stability with ITD (Definition 4) implies the Lipschitz stability with ITD of $x^*(t) = x(t; t_0, x_0)$ of FrDE (3.1) (Definition 2).

Let $J \subset \mathbb{R}_+$, $\lambda > 0$. In our further consideration we will use the following sets:

$$\begin{aligned}
\mathcal{K}(J) &= \{a \in C[J, \mathbb{R}_+] : \ a(0) = 0, a(r) \text{ is strictly increasing in } J \\
&\quad \text{ and there exists a function } \ P_a \in C(R_+, \mathbb{R}_+) \\
&\quad \text{ such that } P_a(\alpha) \geq 1 \text{ for } \alpha \geq 1 \\
&\quad \text{and } a^{-1}(\alpha r) \leq r P_a(\alpha) \text{ for } \ \alpha \geq 1, r \geq 0\}; \\
\mathcal{M}(J) &= \{a \in C[J, \mathbb{R}_+] : \ a(0) = 0, a(r) \text{ is strictly increasing in } J \\
&\quad \text{and } \ a(r) \leq K_a r \text{ for some constant } \ K_a > 0\}; \\
S_\lambda &= \{x \in \mathbb{R}^n : \ ||x|| \leq \lambda\}, \\
B_\lambda &= \{u \in \mathbb{R} : \ |u| \leq \lambda\}.
\end{aligned}$$

Remark 3.5. The function $a(u) = K_1 u$, $K_1 \in (0, 1]$ is from the class $\mathcal{K}(\mathbb{R}_+)$ with $P_a(u) \equiv u$ and from the class $\mathcal{M}(\mathbb{R}_+)$. The function $b(u) = K_2 u^2$, $K_2 > 0$ is from the class $\mathcal{M}([0, 1])$.

We will use comparison results for the IVP for the scalar fractional differential equation with a parameter of the type

$$ {}_{t_0}^{C}D^q u(t) = g\left(t, u(t), \eta\right) \qquad \text{for } t > t_0, \qquad u(t_0) = u_0 \tag{3.6}$$

where $u, u_0 \in \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \times B_H \to \mathbb{R}$, $g(t, 0, 0) \equiv 0$, $\eta \in B_H$ is a parameter and $H > 0$ is a given number. We denote the solution of the IVP for the scalar FrDE (3.6) by $u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$. In the case of non-uniqueness of the solution we will assume the existence of a maximal one.

Definition 5. The zero solution of the scalar FrDE (3.6) with a parameter η is said to be

- *Lipschitz stable with respect to a parameter*, if for any $t_0 \in \mathbb{R}_+$ there exists $M \geq 1$, $\delta = \delta(t_0) > 0$ and $\sigma = \sigma(t_0) > 0$ such that for any $u_0 \in \mathbb{R} : |u_0| < \delta$ and any $|\eta| \leq \sigma$ the inequality $|u(t)| \leq M|u_0|$ for $t \geq t_0$, where $u(t) = u(t; t_0, u_0, \eta)$ is a solution of (3.6);

- *eventually Lipschitz stable with respect to a parameter*, if for any $t_0 \in \mathbb{R}_+$ there exists $M \geq 1$, $\delta = \delta(t_0) > 0$, $T = T(t_0) > 0$ and $\sigma = \sigma(t_0) > 0$ such that for any $u_0 \in \mathbb{R} : |u_0| < \delta$ and any $|\eta| \leq \sigma$ the inequality $|u(t)| \leq M|u_0|$ for $t \geq t_0 + T$, where $u(t) = u(t; t_0, u_0, \eta)$ is a solution of (3.6).

DEFINITION 6. The scalar FrDE (3.6) with a parameter η is said to be

– uniformly Lipschitz stable w.r.t. a parameter, if there exist constants $M \geq 1$ and $\delta, \sigma > 0$ such that for any $t_0 \in \mathbb{R}_+$ and for any $|\eta| \leq \sigma$ the inequality $|u_0| < \delta$ implies $|u(t)| \leq M|u_0|$ for $t \geq t_0$;

– globally uniformly Lipschitz stable w.r.t. a parameter, if there exist constants $M \geq 1, \sigma > 0$ such that for any $t_0 \in \mathbb{R}_+$ and for any $|\eta| \leq \sigma$ the inequality $|u_0| < \infty$ implies $|u(t)| \leq M|u_0|$ for $t \geq t_0$.

REMARK 3.6. Note that similar to Definition 2 and Definition 6, respectively, we can define eventually uniform Lipschitz stability with ITD, eventually globally uniformly Lipschitz stability with ITD for (3.1) and eventually uniform Lipschitz stable w.r.t. a parameter and eventually globally uniform Lipschitz stability w.r.t. a parameter for (3.6).

DEFINITION 7. The FrDE (3.6) is called Mittag-Leffler stable with respect to a parameter, if there exist constants $\lambda, C, H, a, b > 0$, such that for any $\eta \in B_H$ the inequality $||u(t; t_0, u_0, \eta)|| \leq C|u_0|^a \{E_q(-\lambda(t - t_0)^q)\}^b$ for $t \geq t_0$ holds, where $u(t; t_0, u_0, \eta)$ is a solution of (3.6).

EXAMPLE 1. Consider the IVP forthe scalar FrDE with a parameter

$$ {}_{t_0}^{C}D^q u(t) = (-\alpha + c\eta)u(t), \quad u(t_0) = u_0, $$

where $c > 0$ and $\alpha > 0$ are constants, $\eta \in \mathbb{R}$ is a parameter, $t_0 \in \mathbb{R}_+$ is an arbitrary number.

The above IVP is a special case of (3.6) with $g(t, u, \eta) = (-\alpha + c\eta)u$, $g(t, 0, 0) \equiv 0$ and it has a unique solution for any $\eta \in \mathbb{R}$ defined by

$$ u(t; t_0, u_0, \eta) = u_0 E_q((-\alpha + c\eta)(t - t_0)^q), \quad t \geq t_0. $$

Consider the positive constants $H < \frac{\alpha}{c}$ and $\lambda = \alpha - cH$. Then for $\eta \in B_H$ we have $-\alpha + c\eta \leq -\alpha + cH = -\lambda$ and the following estimate is true

$$ |u(t; t_0, u_0, \eta)| = |u_0| E_q((-\alpha + c\eta)(t - t_0)^q) \leq |u_0| E_q(-\lambda(t - t_0)^q), $$

i.e. the scalar FrDE (3.6) is Mittag-Leffler stable with respect to a parameter with $a = b = C = 1$.

□

We introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability properties with ITD for the system FrDE (3.1).

Definition 8. Let $I \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n$. We will say that the function $V(t,x) : I \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(I,\Delta)$ if $V(t,x)$ is continuous and locally Lipschitzian with respect to its second argument in $I \times \Delta$.

The application of Lyapunov functions for stability analysis with ITD requires an appropriate definition of the derivative of Lyapunov like function along the given nonlinear Caputo fractional differential equation. In [5, 6, 7], based on Eq. (2.4), we introduced the *generalized Caputo fractional Dini derivative w.r.t. ITD* of the function $V(t,x) \in \Lambda([t_0,\infty),\mathbb{R}^n)$ along the system of FrDE (3.1) for $t > t_0$, $\eta \in \mathbb{R} :\ t+\eta \geq 0$ and $x, y, x_0, y_0 \in \mathbb{R}^n$ by the equality

$$\begin{aligned}
&{}^{C}_{t_0}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0)\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[V(t,y-x) - V(t_0,y_0-x_0)\\
&- \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big(V\big(t-rh, y-x-h^q(f(t+\eta,y)-f(t,x))\big)\\
&\quad - V(t_0,y_0-x_0)\Big)\Big].
\end{aligned} \tag{3.7}$$

The generalized Caputo fractional Dini derivative w.r.t. ITD (3.7) was applied to study stability ([7]), practical stability ([5]), strict stability ([6]).

Example 2. We give some examples of Lyapunov functions and their generalized Caputo fractional Dini derivative w.r.t. ITD.

a) Lyapunov functions which do not depend on the time variable, i.e. $V(t,x) \equiv V(x)$ for $x \in \mathbb{R}$. Then for any $x, y, x_0, y_0 \in \mathbb{R}$, $t > t_0$, and $\eta \in \mathbb{R} :\ t+\eta \geq 0$ the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$\begin{aligned}
&{}^{C}_{t_0}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0)\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big(V(y-x) - V(y-x-h^q(f(t+\eta,y)-f(t,x)))h^q\Big)\\
&+ \Big(V(y-x) - V(y_0-x_0)\Big)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{3.8}$$

Special case. Let $V(x) = x^2$. Then

$$
\begin{aligned}
{}_{t_0}^{C}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0) &= 2(y-x)(f(t+\eta,y)-f(t,x)) \\
&\quad + \Big((y-x)^2-(y_0-x_0)^2\Big)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{3.9}
$$

b) Let $V(t,x)\equiv V(x)$ for $x\in\mathbb{R}$. Let $x(t)\in C^q([t_0,\infty),\mathbb{R})$ and $y(t)\in C^q([\tau_0,\infty),\mathbb{R})$ be solutions of (3.1) and (3.2), respectively. Then for $t>t_0$ and $\eta=\tau_0-t_0$ we have $t+\eta=t+\tau_0-t_0>\tau_0\geq 0$ and the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$
\begin{aligned}
&{}_{t_0}^{C}D^q_{(3.1)}V(t,x(t),y(t+\eta),\eta,x_0,y_0) \\
&= \lim_{h\to 0^+}\sup\frac{1}{h^q}\Big(V(y(t+\eta)-x(t)) \\
&- V\big(y(t+\eta)-x(t)-h^q(f(t+\eta,y(t+\eta))-f(t,x(t)))\big)\Big) \\
&+ \Big(V(y(t+\eta)-x(t))-V(y_0-x_0)\Big)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{3.10}
$$

c) Let $V(t,x)=m^2(t)x^2$ for $x\in\mathbb{R}$ where $m\in C^1(\mathbb{R}_+,\mathbb{R})$. Then for any $x,y,x_0,y_0\in\mathbb{R}$, $t>t_0$, and $\eta\in\mathbb{R}:\ t+\eta\geq 0$ the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$
\begin{aligned}
&{}_{t_0}^{C}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0) \\
&= 2(y-x)m^2(t)(f(t+\eta,y)-f(t,x))+(y-x)^2\Big({}_{t_0}^{GL}D^q_+m^2(t)\Big) \\
&- (y_0-x_0)^2m^2(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\
&= 2(y-x)m^2(t)(f(t+\eta,y)-f(t,x))+(y-x)^2\Big({}_{t_0}^{RL}D^q_tm^2(t)\Big) \\
&- (y_0-x_0)^2m^2(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{3.11}
$$

Special case. Let $m(t)=t^{\frac{q}{2}}$, $q\in(0,1), t_0=0$. Applying ${}_{t_0}^{RL}D^q_tt^k=\frac{\Gamma(k+1)}{\Gamma(k-q+1)}t^{k-q},\ k>-1, t>0$ we get

$$
{}_{t_0}^{C}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0)=2(y-x)t^q(f(t+\eta,y)-f(t,x))+(y-x)^2\Gamma(1+q).
$$

d) Let $V(t,x)=m^2(t)x^2$ for $x\in\mathbb{R}$ where $m\in C^1(\mathbb{R}_+,\mathbb{R})$. Let $x(t)\in C^q([t_0,\infty),\mathbb{R})$ and $y(t)\in C^q([\tau_0,\infty),\mathbb{R})$ be solutions of (3.1) and (3.2), respectively. Then for $t>t_0$ and $\eta=\tau_0-t_0$ we have $t+\eta=t+\tau_0-t_0>$

$\tau_0 \geq 0$ and the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$\begin{aligned} & {}_{t_0}^{C}D^q_{(3.1)}V(t,x(t),y(t+\eta),\eta,x_0,y_0) \\ &= 2(y(t+\eta)-x(t))m^2(t)(f(t+\eta,y(t+\eta))-f(t,x(t))) \\ &+ (y(t+\eta)-x(t))^2\Big({}_{t_0}^{RL}D^q_t m^2(t)\Big) - (y_0-x_0)^2 m^2(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}. \end{aligned} \tag{3.12}$$

e) Let $V(t,x_1,x_2) = m_1^2(t)x_1^2 + m_2^2(t)x_2^2$ for $x_1,x_2 \in \mathbb{R}$ where $m_1,m_2 \in C^1(\mathbb{R}_+,\mathbb{R})$. Then for any $x,y,x_0,y_0 \in \mathbb{R}^2$, $t > t_0$, and $\eta \in \mathbb{R}: \ t+\eta \geq 0$ the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$\begin{aligned} & {}_{t_0}^{C}D^q_{(3.1)}V(t,x,y,\eta,x_0,y_0) \\ &= 2(y_1-x_1)m_1^2(t)(f_1(t+\eta,y_1,y_2)-f_1(t,x_1,x_2)) \\ &+ 2(y_2-x_2)m_2^2(t)(f_2(t+\eta,y_1,y_2)-f_2(t,x_1,x_2)) \\ &+ (y_1-x_1)^2\Big({}_{t_0}^{RL}D^q_t m_1^2(t)\Big) + (y_2-x_2)^2\Big({}_{t_0}^{RL}D^q_t m_2^2(t)\Big) \\ &- \Big(m_1^2(t_0)(y_1^{(0)}-x_1^{(0)})^2 + m_2^2(t_0)(y_2^{(0)}-x_2^{(0)})^2\Big)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}. \end{aligned} \tag{3.13}$$

□

Remark 3.7. Note in some papers (see, for example [17, 18]) the derivative of Lyapunov functions with respect to system (3.1) is defined by

$$ {}^cD^qV(t,x) = \lim_{h\to 0^+}\sup\frac{1}{h^q}\Big[V(t,x) - V(t-h, x-h^q f(t,x))\Big]. \tag{3.14}$$

This derivative is called a fractional derivative of Lyapunov functions in Caputo's sense of order q with respect to system (3.1). This operator has no memory, which is different than the fractional derivative and it is independent on the initial time and it is not equivalent to the Caputo fractional derivative. In the general case if $x(t)$ is a solution of (3.1) then the inequality

$$ {}^cD^qV(t,x(t)) \neq {}_{t_0}^{c}D^q_t V(t,x(t)) \tag{3.15}$$

holds, where the operator ${}^cD^q$ is defined by (3.14) and the operator ${}_{t_0}^{c}D^q_t$ is defined by (2.4).

4. **Main Results**

First we recall the following comparison results giving us the relationship between Lyapunov functions, system FrDE (3.1) and the scalar FrDE (3.6).

LEMMA 4.1. ([7]) *Assume the following conditions are satisfied:*

1. *The functions* $x^*(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *are solutions of systems of FrDE (3.1) and (3.2) respectively,* $x^*(t) \in C^q([t_0, t_0+\theta], \mathbb{R}^n)$, $y(t) \in C^q([\tau_0, \tau_0+\theta], \mathbb{R}^n)$ *and* $y(t+\eta^*) - x^*(t) \in \Delta$ *for* $[t_0, t_0+\theta]$ *where* $t_0, \tau_0 \in \mathbb{R}_+ :\ \eta^* = \tau_0 - t_0$, $\Delta \subset \mathbb{R}^n$, $\theta > 0$ *is a given number.*
2. *The function* $G \in C[[t_0, t_0+\theta] \times \mathbb{R}, \mathbb{R}]$ *be such that for any* $\epsilon \in [0, H]$ *and* $v_0 \in \mathbb{R}$ *the scalar FrDE*
$$ {}_{t_0}^{C}D^q u = G(t,u) + \epsilon \quad \text{for } t > 0, \qquad u(t_0) = v_0 \tag{4.1}$$
has a solution $u(t; t_0, v_0, \epsilon) \in C^q([t_0, t_0+\theta], \mathbb{R})$ *where* $H, \Theta > 0$ *are given constants.*
3. *The function* $V \in \Lambda([t_0, t_0+\theta], \Delta)$ *and for* $t \in (t_0, t_0+\theta]$ *the inequality*
$$ {}_{t_0}^{C}D^q_{(3.1)} V(t,\ x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \le G(t, V(t, y(t+\eta^*) -\ x^*(t)))$$
holds.

Then $V(t_0, y_0 - x_0) \le u_0$ *implies* $V(t, y(t+\eta^*) -\ x^*(t)) \le u^*(t)$ *for* $t \in [t_0, t_0+\theta]$ *where* $u^*(t) = u(t; t_0, u_0, 0)$ *is the maximal solution of IVP for the scalar FrDE (4.1) with* $v_0 = u_0$ *and* $\epsilon = 0$.

COROLLARY 4.1. ([7]) *Let the conditions of Lemma 4.1 be satisfied for* $\theta = \infty$.

Then $V(t_0, y_0 - x_0) \le u_0$ *implies* $V(t, y(t+\eta^*) -\ x^*(t)) \le u^*(t)$ *for* $t \ge t_0$ *where* $u^*(t) = u(t; t_0, u_0, 0)$ *is the maximal solution of IVP for scalar FrDE (2.4) with* $v_0 = u_0$ *and* $\epsilon = 0$.

COROLLARY 4.2. *Let condition 1 of Lemma 4.1 be satisfied and the inequality*
$$ {}_{t_0}^{C}D^q_{(3.1)} V(t, x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \le \gamma V(t, y(t+\eta^*) - x^*(t)) + C\eta^*, \ \ t \ge t_0$$
holds where $C, \gamma \in \mathbb{R}$ *are constants.*

Then $V(t, y(t+\eta^*) - x^*(t)) \le [V(t_0, y_0 - x_0 + \frac{1}{\gamma}C\eta^*] E_q(\gamma(t-t_0)^q) - \frac{1}{\gamma}C\eta^*$ *for* $t \ge t_0$.

P r o o f. Consider the IVP for the scalar FrDE ${}_{t_0}^{C}D^q u = \gamma u + C\eta^*$, $u(t_0) = V(t_0, y_0 - x_0)$. Denote $v = u + \frac{1}{\gamma}C\eta^*$. Then ${}_{t_0}^{C}D^q v = \gamma v$, $v(t_0) = V(t_0, y_0 - x_0) + \frac{1}{\gamma}C\eta^*$. Then $v(t) = [V(t_0, y_0 - x_0) + \frac{1}{\gamma}C\eta^*] E_q(\gamma(t-t_0)^q)$

for $t \geq t_0$. Therefore, $u(t) = [V(t_0, y_0 - x_0) + \frac{1}{\gamma} C\eta^*] E_q(\gamma(t-t_0)^q) - \frac{1}{\gamma} C\eta^*$ for $t \geq t_0$. Applying Corollary 4.1 we obtain the claim in Corollary 4.2. □

We will obtain sufficient conditions for several types of stability such as Lipschitz (uniform Lipschitz) stability, globally uniformly Lipschitz stability, Mittag-Leffler stability with (ITD) by using continuous Lyapunov-like functions from the Λ class and the generalized Caputo fractional Dini derivative defined by (3.7). Lipshitz stability for fractional equations is studied in [22] using the derivative defined by (3.14) derivative and applying (3.15) as an equality instead of an inequality (see Remark 3.7).

THEOREM 4.1. *Let the following conditions be satisfied:*

1. *The function* $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$ *is a solution of system of FrDE (3.1), where* $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ *are given points.*
2. *The function* $g \in C[[t_0, \infty) \times \mathbb{R} \times B_H, \mathbb{R}]$, $g(t, 0, 0) \equiv 0$ *and for any parameter* $\eta \in B_H$ *the IVP for the scalar FrDE (10) has a solution* $u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$ *where* $H > 0$ *is a given number.*
3. *The zero solution of the scalar FrDE (3.6) is Lipschitz stable w.r.t. a parameter.*
4. *There exists a function* $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ *with Lipschitz constant* L *in* S_ρ *such that* $V(t_0, 0) = 0$ *and*
   (i) $b(\|x\|) \leq V(t, x)$ *for* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, *where* $\rho > 0$ *is a given number,* $b \in \mathcal{K}(\mathbb{R}_+)$;
   (ii) *for any* $y, y_0 \in \mathbb{R}^n$ *and* $\eta \in B_H$ *the inequality*

$$ {}_{t_0}^{C}D^q_{(3.1)} V(t,\ x^*(t), y, \eta, x_0, y_0) \leq g(t, V(t, y - x^*(t)), \eta) \text{ for } t > t_0 \quad (4.2)$$

holds.

Then the solution $x^*(t)$ *of the system of FrDE (3.1) is Lipschitz stable with ITD.*

P r o o f. From condition 3 it follows that there exist $M \geq 1$, $\delta_1 = \delta_1(t_0)$, $\sigma = \sigma(t_0)$ such that for any $u_0 \in \mathbb{R}:\ |u_0| < \delta_1$ and $|\eta| < \sigma$ the inequality

$$|u(t; t_0, u_0, \eta)| \leq M\ |u_0| \text{ for } t \geq t_0 \quad (4.3)$$

holds, where $u(t; t_0, u_0, \eta)$ is a solution of FrDE (3.6). Without loss of generality we assume $\sigma \leq H$.

Since $V(t_0, 0) = 0$ there exists a $\delta_2 = \delta_2(t_0, \delta_1) < \rho$ such that

$$V(t_0, x) < \delta_1 \quad \text{for} \quad \|x\| < \delta_2. \quad (4.4)$$

Without loss of generalization we can assume $\delta_2 \leq \delta_1$. The function $V(t, x)$ is Lipschitz on S_ρ and

$$|V(t_0, x)| = |V(t_0, x) - V(t_0, 0)| \leq L\,\|x\| \quad \text{for } \|x\| < \rho. \quad (4.5)$$

From $b \in \mathcal{K}(\mathbb{R}_+)$ it follows there exists a function $P_b \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$b^{-1}(\alpha r) \leq r P_b(\alpha) \quad \text{for} \quad \alpha \geq 1. \tag{4.6}$$

Choose $M_1 \geq 1$ such that $M_1 > ML$ and let $M_2 = P_b(M_1) \geq 1$.

Now let $y_0 \in \mathbb{R}^n$ and $\tau_0 \in \mathbb{R}_+$ be such that $\|y_0 - x_0\| < \delta_2$ and $|\eta^*| < \sigma$ where $\eta^* = \tau_0 - t_0$. Consider a solution $y(t) = y(t; \tau_0, y_0)$ of system of FrDE (3.2) with the chosen initial data (τ_0, y_0). Let $u_0^* = V(t_0, y_0 - x_0)$. Then from the choice of y_0 and Eq. (4.4) it follows that $u_0^* = V(t_0, y_0 - x_0) < \delta_1$. Therefore, applying (4.3) and (4.5) we obtain the inequality

$$|u^*(t)| \leq M|u_0^*| = MV(t_0, y_0 - x_0) \leq ML||y_0 - x_0|| \quad \text{for } t \geq t_0 \tag{4.7}$$

holds where $u^*(t) = u(t; t_0, u_0^*, \eta^*) \in C^q([t_0, \infty), \mathbb{R})$ is a solution of FrDE (3.6).

Using condition 4 (ii) and applying Lemma 2 with $G(t, u) = g(t, u, \eta^*)$, $\Delta = \mathbb{R}^n$, $\theta = \infty$ we get

$$V(t, y(t + \eta^*) - x^*(t)) \leq u^*(t) \text{ for } t \geq t_0. \tag{4.8}$$

From inequalities (4.7), (4.8), condition 4 (i) and the Lipschitz property of $V(t, x)$ we get

$$\begin{aligned} b(\|y(t + \eta^*) - x^*(t)\|) &\leq V(t, y(t + \eta^*) - x^*(t)) \leq u^*(t) \\ &\leq ML||y_0 - x_0|| < M_1||y_0 - x_0||. \end{aligned} \tag{4.9}$$

From the monotonicity property of the function $b(r)$ and inequalities (4.6), (4.9) we have

$$\|y(t + \eta^*) - x^*(t)\| \leq b^{-1}(M_1||y_0 - x_0||)$$

$$\leq ||y_0 - x_0||P_b(M_1) = M_2||y_0 - x_0||, \quad t \geq t_0.$$

□

Corollary 4.3. *Let the conditions of Theorem 1 be satisfied with* $b(u) = K_1 u$, $K_1 > 0$.

Then the solution $x^*(t)$ of the system of FrDE (3.1) is Lipschitz stable with ITD.

P r o o f. The proof is similar to the one in Theorem 1 with $M_1 \geq 1$: $M_1 > M\frac{L}{K_1}$ and $M_2 = M_1$. □

Theorem 4.2. *Let the conditions 1,2,4 (i) be satisfied, the zero solution of the scalar FrDE (3.6) is eventually Lipschitz stable w.r.t. a parameter and there exists $T = T(t_0) >$ such that (4.2) is satisfied for $t \geq t_0 + T$.*

Then the solution $x^*(t)$ of the system of FrDE (3.1) is eventually Lipschitz stable with ITD.

The proof of Theorem 4.2 is similar to the one in Theorem 4.1, so we omit it.

THEOREM 4.3. *Let the following conditions be satisfied:*

1. *The function $g \in C[[t_0,\infty) \times \mathbb{R} \times B_H, \mathbb{R}]$, $g(t,0,0) \equiv 0$ and for any parameter $\eta \in B_H$ the IVP for the scalar FrDE (3.6) has a solution $u(t;t_0,u_0,\eta) \in C^q([t_0,\infty),\mathbb{R})$ where $H > 0$ is a given number.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, S(\lambda))$ such that*
   (i) $b(\|x\|) \le V(t,x) \le a(\|x\|)$ for $(t,x) \in \mathbb{R}_+ \times S(\lambda)$,

   where $b \in \mathcal{K}([0,\lambda])$, $a \in \mathcal{M}([0,\lambda])$, $\lambda > 0$ is a given number;
   (ii) for any $t_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n : y - x \in S(\lambda)$, $y_0 - x_0 \in S(\lambda)$ and $\eta \in B_H$ the inequality

   $${}_{t_0}^{C}D^q_{(3.1)}V(t,\ x,y,\eta,x_0,y_0) \le g(t,V(t,y-x),\eta) \text{ for } t \ge t_0$$

   holds.
3. *The scalar FrDE (3.6) is uniformly Lipschitz stable w.r.t. a parameter (globally uniformly Lipschitz stable w.r.t. a parameter).*

Then the system of FrDE (3.1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable) with ITD.

P r o o f. Let the scalar FrDE (3.6) be uniformly Lipschitz stable w.r.t. a parameter. According to Definition 6 there exist constants $M \ge 1$, $\delta_1 > 0$, $\sigma > 0$ such that for any $t_0 \in \mathbb{R}_+$ and any $|\eta| < \sigma$ the inequality $|U_0| < \delta_1$ implies

$$|u(t;t_0,U_0,\eta)| \ \le M|U_0| \text{ for } t \ge t_0, \tag{4.10}$$

where $u(t;t_0,U_0,\eta)$ is a solution of FrDE (10) with initial data (t_0,U_0). From condition 2 (i) there exist a function $P_b \in C(\mathbb{R}_+,\mathbb{R}_+)$ and a positive constant K_a such that $b^{-1}(\alpha r) \le \alpha P_b(r)$ and $a(r) \le K_a r$ for $r \ge 0,]ga \ge 1$. Choose $M_1 \ge 1$ such that $M_1 > P_b(M)K_a$. Now let $\delta = \min\left\{\delta_1, \frac{\lambda}{M_1}, \frac{\delta_1}{K_a}\right\}$ and initial points x_0, $y_0 \in \ \mathbb{R}^n$ and $\tau_0, t_0 \in \ \mathbb{R}_+$ be such that

$$\|y_0 - x_0\| < \delta \text{ and } |\eta^*| < \sigma \tag{4.11}$$

where $\eta^* = \tau_0 - t_0$. Consider any solutions $x(t) = x(t;t_0,x_0)$ and $y(t) = y(t;\tau_0,y_0)$ of system of FrDE (3.1) and (3.2) correspondingly with the chosen initial data (τ_0,y_0) and (t_0,x_0) respectively. From the choices of the constants (4.11) we have $\|y_0 - x_0\| < \delta \le \frac{\lambda}{M_1} \le \lambda$, i.e. $y_0 - x_0 \in S(\lambda)$.

Let $u_0^* = V(t_0, y_0 - x_0)$. Then from Condition 2 (i) and the choice of x_0, y_0 it follows that $u_0^* = V(t_0, y_0 - x_0) \le a(\|y_0 - x_0\|) \le K_a \|y_0 - x_0\| < K_a\delta \le \delta_1$. Then according to inequality (4.10) it follows that

$$|u^*(t)| \le M|u_0| \text{ for } t \ge t_0 \tag{4.12}$$

where $u^*(t) = u(t; t_0, u_0^*, \eta^*) \in C^q([t_0, \infty), \mathbb{R})$ is a solution of FrDE (3.6). We will prove that if inequalities (4.11) holds then

$$\|y(t+\eta^*) - x(t)\| \le M_1 \|y_0 - x_0\| \quad \text{for } t \ge t_0. \tag{4.13}$$

Assume the opposite, i.e. there exists $t_1 > t_0$ such that

$$\begin{aligned}
&\|y(t+\eta^*) - x^*(t)\| \le M_1 \|y_0 - x_0\| \quad \text{for } t \in [t_0, t_1]\\
&\|y(t_1+\eta^*) - x^*(t_1)\| = M_1 \|y_0 - x_0\|\\
&\|y(t+\eta^*) - x^*(t)\| > M_1 \|y_0 - x_0\| \quad \text{for } t \in (t_1, t_1+\epsilon]
\end{aligned}$$

where $\epsilon > 0$ is a small enough number. Then for $t \in [t_0, t_1]$ the inequality $\|y(t+\eta^*) - x^*(t)\| \le M_1 \|y_0 - x_0\| < M_1\delta \le \lambda$ holds, i.e. $y(t+\eta^*) - x(t) \in S(\lambda)$ for $t_0 \le t \le t_1$. Then from Lemma 4.1 applied for $\theta = t_1 - t_0$, $\Delta = S(\lambda)$ and $G(t,u) = g(t,u,\eta^*)$ we get

$$V(t, y(t+\eta^*) - x(t)) \le u^*(t) \text{ for } t \in [t_0, t_1]. \tag{4.14}$$

From the choice of t^*, Condition 2 (i) and inequalities (4.12), (4.14) we obtain

$$\begin{aligned}
M_1 \|y_0 - x_0\| &= \|y(t_1+\eta^*) - x(t_1)\|) \le b^{-1}(V(t_1, y(t_1+\eta^*) - x(t_1)))\\
&\le b^{-1}(u^*(t^*)) \le b^{-1}(M|u_0^*|)\\
&= b^{-1}(MV(t_0, y_0 - x_0)) \le P_b(M)V(t_0, y_0 - x_0)\\
&\le P_b(M)a(||y_0 - x_0||) \le P_b(M)K_a||y_0 - x_0||\\
&< M_1||y_0 - x_0||.
\end{aligned} \tag{4.15}$$

The obtained contradiction proves the validity of (4.13). Therefore, according to Definition 3 the system of FrDE (3.1) is uniformly Lipschitz stable with ITD.

The proof of globally uniformly Lipschitz stable with ITD is analogous, so we omit it. □

THEOREM 4.4. *Let the following conditions be satisfied:*

1. *The condition 1 of Theorem 4.3 is satisfied.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, S(\lambda))$ such that*

   (i) $\lambda_1(t)\|x\|^2 \le V(t,x) \le \lambda_2(t)\|x\|^2$ *for* $(t,x) \in \mathbb{R}_+ \times S(\lambda)$,

   where $\lambda_1(t) \ge A_1 > 1, \lambda_2(t) \le A_2$ for $t \ge 0$, where $A_1, A_2 > 1$ are given constants;

(ii) for any $t_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n : y - x \in S(\lambda)$, $y_0 - x_0 \in S(\lambda)$ and $\eta \in B_H$ the inequality

$$ {}_{t_0}^{C}D^q_{(3.1)}V(t,\ x,y,\eta,x_0,y_0) \le (-\alpha + C\eta)V(t, y - x) \text{ for } t > t_0 \tag{4.16}$$

holds, where λ, H, C, α are given positive numbers.

Then the system of FrDE (3.1) is uniformly globally Lipschitz stable with ITD.

P r o o f. Let $g(t, u, \eta) = (-\alpha + C\eta)u$. Let $\sigma = \frac{\alpha}{C}$ and choose the initial times $t_0, \tau_0 \in \mathbb{R}_+ :\ |\eta^*| < \sigma$ where $\eta^* = \tau_0 - t_0$. Let the initial points $x_0, y_0 \in \mathbb{R}^n :\ ||x_0 - y_0|| < \infty$ and $u_0 = V(t_0, y_0 - x_0)$. According to Example 1 the solution of the comparison scalar FrDE (3.6) is $u(t; t_0, u_0, \eta) = u_0 E_q((-\alpha + C\eta)(t - t_0)^q)$. Therefore, $|u(t; t_0, u_0, \eta)| = |u_0| E_q((-\alpha + C\eta)(t - t_0)^q) \le |u_0|$, i.e. (3.6) is globally Lippshitz stable w.r.t. a parameter with $M = 1$. Also, Condition 2i of Theorem T3 is satisfied with $b(r) = A_1 r \in \mathcal{K}([0, \lambda])$, $P_b(r) = \frac{r}{A_1}$ and $a(r) = A_2 r \in \mathcal{M}([0, \lambda])$. According to Theorem 3 the system of FrDE (3.1) is uniformly globally Lipschitz stable with ITD. □

Corollary 4.4. *In the case when $\eta = 0$, i.e. the inequality*

$$ {}_{t_0}^{C}D^q_{(3.1)}V(t, x, y, x_0, y_0) \le -\alpha[V(t, y - x)]$$

holds, the result of Theorem 4.4 is reduced to the uniformly globally Lipschitz stability of the system of FrDE (3.1).

Theorem 4.5. *Let the following conditions be satisfied:*

1. *The function $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$ is a solution of system of FrDE (3.1), where $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ are given points.*
2. *The function $g \in C[[t_0, \infty) \times \mathbb{R} \times B_H, \mathbb{R}]$, $g(t, 0, 0) \equiv 0$ and for any parameter $\eta \in B_H$ the IVP for the scalar FrDE (3.6) has a solution $u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$ where $H > 0$ is a given number.*
3. *There exists a function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ such that $V(t_0, 0) = 0$ and*

   (i) $K_1\|x\|^a \le V(t, x) \le K_2\|x\|^{ab}$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where

   K_1, K_2, a and b are positive constants;

   (ii) for any $y, y_0 \in \mathbb{R}^n$ and $\eta \in B_H$ the inequality

$$ {}_{t_0}^{C}D^q_{(3.1)}V(t,\ x^*(t), y, \eta, x_0, y_0) \le (-\alpha + C\eta)V(t, y - x^*(t)) \text{ for } t > t_0 \tag{4.17}$$

   holds.

Then the solution $x^*(t)$ of the system of FrDE (3.1) is Mittag-Leffler stable with ITD.

P r o o f. Let $g(t,u,\eta) = (-\alpha + C\eta)u$. Let $\sigma = \frac{\alpha}{C}$ and choose the initial time $\tau_0 \in \mathbb{R}_+ :\ |\eta^*| < \frac{\alpha}{C}$ where $\eta^* = \tau_0 - t_0$. Let the initial point $y_0 \in \mathbb{R}^n$ and $u_0 = V(t_0, y_0 - x_0)$. According to Example 1 the solution of the comparison scalar FrDE (3.6) is $u(t; t_0, u_0, \eta) = u_0 E_q((-\alpha + C\eta)(t-t_0)^q)$.

Consider a solution $y(t) = y(t; \tau_0, y_0)$ of system of FrDE (3.2) with initial data (τ_0, y_0). Using condition $3i$ and due to Corollary 2 it follows that

$$\begin{aligned} K_1 \|y(t+\eta^*) - x(t)\|^a &\le V(t, y(t+\eta^*) - x(t)) \le u(t; t_0, u_0, \eta^*) \\ &\le V(t_0, y_0 - x_0) E_q((-\alpha + C\eta^*)(t-t_0)^q) \\ &\le K_2 ||y_0 - x_0||^{ab} E_q((-\alpha + C\eta^*)(t-t_0)^q). \end{aligned} \tag{4.18}$$

Thus we get

$$\|y(t+\eta^*) - x(t)\| \le (\frac{K_2}{K_1})^{\frac{1}{a}} ||y_0 - x_0||^b \left\{E_q((-\alpha + C\eta^*)(t-t_0)^q)\right\}^{\frac{1}{a}}$$

for $t \ge t_0$ where $C = (\frac{K_2}{K_1})^{\frac{1}{a}}$, $\lambda = \alpha - C\eta^* > 0$ with $|\eta^*| \le \frac{\alpha}{C}$ which implies that the solution $x^*(t)$ of the system of FrDE (3.1) is Mittag-Leffler stable with ITD. □

5. **Applications**

EXAMPLE 3. Consider the following scalar system of FrDE

$${}_0^C D^{0.9} x(t) = f(t, x(t)) \quad \text{for } t > 0, \qquad x(0) = x_0 \tag{5.1}$$

where $x_0 \in \mathbb{R}$ and the function $f(t,x) = g(t)x,\ t \ge 0, x \in \mathbb{R}$ with $g(t) = -\frac{0.01}{m^2(t)} - \frac{2}{\Gamma(0.1)} t^{-0.9} + 0.25 \frac{9}{\Gamma(0.1)} (t)^{0.1}\ {}_2F_1(1, 1.9, 1.1, -t)$ and $m^2(t) = 1 + \frac{1}{(t+1)^{0.9}} \in (1,2)$ is a decreasing function (see Figure 1), ${}_2F_1(a,b;c;z)$ is the hypergeometric function.

The equation (5.1) has a zero solution with $x_0 = 0$.

The functions $g(t)$ and respectively $f(t,x)$ are changing their signs for $t \ge 0$ (see Figure 2 for the graph of $g(t)$). It does not allow us to use the quadratic Lyapunov function for obtaining stability properties of the zero solution.

Define the function $V(t,x) = m^2(t)x^2$. According to Case d) of Example 1 and Eq.(3.12) the generalized Caputo fractional Dini derivative w.r.t. ITD is

$$\begin{aligned} &{}_0^C D^{0.9}_{(5.1)} V(t, 0, y, 0, \eta, y_0) \\ &= 2ym^2(t) f(t+\eta, y) + (y(t))^2 \Big({}_0^{RL} D_t^{0.9} m^2(t) \Big) \quad t > 0. \end{aligned} \tag{5.2}$$

For the fractional derivative we get

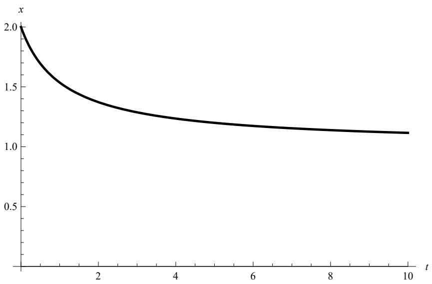

Figure 1. Graph of $m^2(t)$.

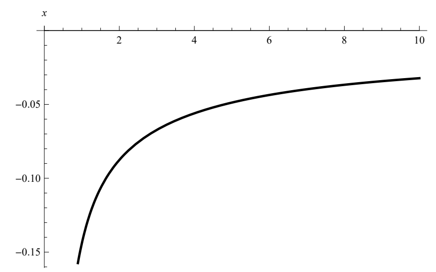

Figure 2. Graph of $g(t)$.

$$
\begin{aligned}
&{}_0^{RL}D_+^{0.9}\Big(1+\frac{1}{(t+1)^{0.9}}\Big)=\frac{1}{\Gamma(0.1)}t^{-0.9}+\ {}_0^{RL}D_+^{0.9}(t+1)^{-0.9}\\
&=\ {}_0^{C}D_+^{0.9}(t+1)^{-0.9}+\ {}_0^{RL}D_+^{q}1\\
&=-\frac{9}{\Gamma(0.1)}t^{0.1}\ {}_2F_1(1,1.9,1.1,-t)+\frac{2}{\Gamma(0.1)}t^{-0.9}.
\end{aligned}
$$

Then using the functions $F(t)=\frac{9}{\Gamma(0.1)}t^{0.1}\ {}_2F_1(1,1.9,1.1,-t)$ and $t^{-0.9}$ are positive decreasing (see Figure 3 for the graph of $F(t)$), the inequalities $m^2(t+\eta)\leq m^2(t)\leq 2$ and $\frac{2}{\Gamma(0.1)}t^{-0.9}-m^2(t)\frac{2}{\Gamma(0.1)}(t+\eta)^{-0.1}\leq\frac{2}{\Gamma(0.1)}\Big(t^{-0.9}-(t+0.1)^{-0.1}\Big)\leq 0$ for $\eta<0.1$ and $t\geq T=0.8$ (see Figure 4) we get

$$
\begin{aligned}
&{}_0^{C}D_{(5.1)}^{0.9}V(t,0,y+,0,0,y_0)\\
&=y^2\Big(-\frac{0.02m^2(t)}{m^2(t+\eta)}+0.5m^2(t)\frac{9}{\Gamma(0.1)}(t+\eta)^{0.1}\ {}_2F_1(1,1.9,1.1,-t-\eta)\\
&-\frac{9}{\Gamma(0.1)}t^{0.1}\ {}_2F_1(1,1.9,1.1,-t)+\frac{2}{\Gamma(1-q)}t^{-0.9}-m^2(t)\frac{2}{\Gamma(0.1)}(t+\eta)^{-0.9}\Big)\\
&\leq y^2\Big(-0.02+\frac{9}{\Gamma(0.1)}(t)^{0.1}\ {}_2F_1(1,1.9,1.1,-t)\\
&-\frac{9}{\Gamma(0.1)}t^{0.1}\ {}_2F_1(1,1.9,1.1,-t)+\frac{2}{\Gamma(0.1)}t^{-0.9}-m^2(t)\frac{2}{\Gamma(0.1)}(t+\eta)^{-0.9}\Big)\\
&\leq -0.02y^2=-0.01(2)y^2\leq -0.01m^2(t)y^2=-0.01V(t,y(t)),\quad t\geq 0.8.
\end{aligned}
$$

The solution of the comparison FrDE ${}_0^{C}D^q u(t)=-0.01u(t)$ for $t>0,\quad u(0)=u_0>0$ is $u(t)=u_0E_q(-0.01t^q)\leq u_0$. Therefore, from Theorem 2 the zero solution of FrDE (5.1) is eventually Lipschitz stable with ITD.

□

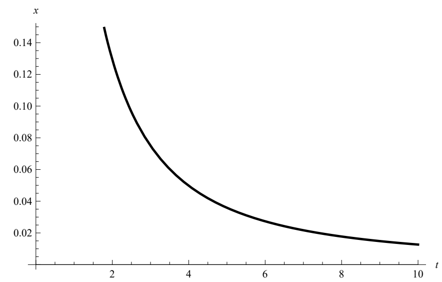

Figure 3. Graph of $F(t)$.

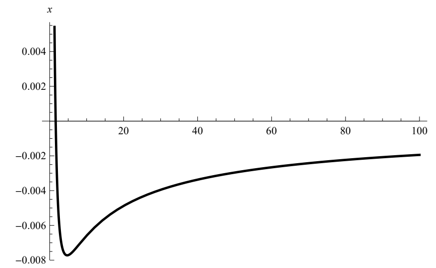

Figure 4. Graphs of $\frac{2}{\Gamma(0.1)}\Big(t^{-0.9} - (t+0.1)^{-0.9}\Big)$.

Acknowledgements

Research was partially supported by Fund MU17-FMI-007, University of Plovdiv "Paisii Hilendarski".

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Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **21**, No 1 (2018), pp. 72–93,
DOI: 10.1515/fca-2018-0005