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RESEARCH PAPER

EXACT AND NUMERICAL SOLUTIONS OF THE FRACTIONAL STURM–LIOUVILLE PROBLEM

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Abstract

In the paper, we discuss the regular fractional Sturm-Liouville problem in a bounded domain, subjected to the homogeneous mixed boundary conditions. The results on exact and numerical solutions are based on transformation of the differential fractional Sturm-Liouville problem into the integral one. First, we prove the existence of a purely discrete, countable spectrum and the orthogonal system of eigenfunctions by using the tools of Hilbert-Schmidt operators theory. Then, we construct a new variant of the numerical method which produces eigenvalues and approximate eigenfunctions. The convergence of the procedure is controlled by using the experimental rate of convergence approach and the orthogonality of eigenfunctions is preserved at each step of approximation. In the final part, the illustrative examples of calculations and estimation of the experimental rate of convergence are presented.

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1. Introduction

In the paper, we study (from a theoretical and numerical point of view) the fractional Sturm-Liouville problem (FSLP) with the homogeneous mixed boundary conditions. The Sturm-Liouville problem in a fractional version can be derived by using different approaches. The first one

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consists of replacing the integer order derivative in the classical Sturm-Liouville problem by a fractional order derivative [1, 14]. However, this approach does not lead to orthogonal systems of eigenfunctions. The second approach is connected with the application of the calculus of variations [16, 17]. In this case, the obtained fractional differential equations can be interpreted as fractional Euler-Lagrange equations [2, 16, 18, 21, 22, 27]. They contain the differential operator, which is a composition of the left and the right fractional derivative [16, 17]. This feature leads to fundamental difficulties in calculating eigenvalues and deriving the exact solutions in a closed form, even in a simple case of a fractional oscillator problem in a bounded domain. Explicit solutions and eigenvalues are known, so far, only for a few FSLP, like fractional oscillator problem on unbounded domain [28], and for some singular cases like fractional Legendre and Jacobi problems [17, 31, 32], and fractional Bessel equation [23].

The FSLP, along with its eigenfunction's system and eigenvalues, is connected to a fractional diffusion [19, 20] in a bounded domain. The term 'fractional diffusion' is to be understood as the application of fractional derivatives in description of processes of anomalous diffusion. Such diffusive processes appear in many fields of science and engineering, e.g., heat conduction in materials, fluid pressure in porous media, human migration, movement of proteins in cells, transport of lipids on cell membranes, transport on social networks, bacterial motility, and others. Classical results show that in order to solve diffusion equations in a bounded domain, one needs to apply the suitable orthogonal systems of functions, usually connected to a respective Sturm-Liouville problem. Therefore FSLPs, determined in bounded domains, are an emerging meaningful area of the fractional differential equations theory. The orthogonal eigenfunctions' systems of FSLPs are and will be a useful tool in solving partial fractional differential equations connected to anomalous diffusion processes. Preliminary results are given in papers [17, 18, 19, 20] and show that by applying the eigenfunctions' systems, we can study fractional diffusion problems with variable diffusivity and calculate the explicit solutions or establish the existence-uniqueness results and analyze the properties of solutions. Similar to the classical Sturm-Liouville theory, it appears that the existence of the purely discrete spectrum and the associated orthogonal eigenfunctions' system is strongly connected to the singularity of FSLP, or in case of a regular FSLP, to the choice of boundary conditions. In paper [20], the proof was given for the regular FSLP subjected to the homogeneous Dirichlet conditions, now we shall present the result on the regular FSLP subjected to the homogeneous mixed boundary conditions. This new result is developed by converting the differential FSLP to the equivalent integral one and by applying the results of the Hilbert-Schmidt operators theory.

As we have mentioned, the problem of finding analytical solutions of the FSLP, containing both the left and right fractional derivative, is still a challenge for scientists. Therefore, numerical methods of solving the FSLP are being simultaneously developed. During the last few years, several numerical algorithms (based on direct or indirect methods) have been proposed to obtain approximate solutions of the fractional Euler-Lagrange equations [5, 6, 7, 8, 9, 10, 11, 30].

The same problem appears in calculating the eigenvalues of the FSLP. The most common approach to determine eigenvalues and eigenfunctions for Sturm-Liouville operators of integer and fractional order is to use a numerical method. The numerical solution of the Sturm-Liouville problem of integer order can be found in literature i.e. the Pruess, shooting and finite difference methods [4, 26]. In paper [29], the control volume method is used to determine the eigenvalues of the classical Sturm-Liouville problem. However, for FSLPs involving both the left and the right derivative, the adequate set of numerical tools still requires further and extensive work.

In our previous paper [12], we developed the numerical method for solving a fractional eigenvalue problem - the version of the FSLP with the homogeneous mixed boundary conditions. The proposed numerical scheme was based on the discretization of Caputo derivatives involving the boundary conditions. This approach allowed us to approximate the eigenfunctions keeping their orthogonality at each step of approximation. Moreover, the convergence was controlled by using the experimental rates of convergence formulas both for the eigenvalues and for the eigenvectors. The obtained rate of convergence was close to 1.

In paper [3], the FSLP with Dirichlet boundary conditions was considered. The authors analysed two approaches to the FSLP: discrete and continuous. They investigated the numerical solution of the FSLP by using the truncated Grunwald-Letnikov fractional derivative.

Now, we will construct a numerical scheme to calculate the approximate eigenvalues and eigenfunctions, by applying the approach presented in papers [7, 9, 11]. First, we transform the FSLP into an intermediate integral equation and then we discretize the obtained equation by using the numerical quadrature rule.. This method allows us to obtain the numerical scheme for which the experimental rate of convergence in all the considered examples tends to 2α. As we study the FSLP with order $\alpha > 1/2$, we clearly see that the new numerical method gives better convergence than the one introduced in [12], while the orthogonality of the approximate eigenfunctions is also kept at each step of the procedure.

The paper is organized as follows. Section 2 presents the analyzed problem and recalls basic definitions and main properties of fractional differential and integral operators. In Section 3 the exact solution of the FSLP is depicted, while in Section 4 the numerical solution is given. Finally, in Section 5, we show numerical results for two examples of the FSLP, and we conclude the paper with a section containing brief conclusions.

2. Preliminaries

We recall the left and right Caputo fractional derivatives of order $\alpha \in (0,1)$ (see e.g. [13, 15, 24])

$$ {}^{c}D^{\alpha}_{a+}\, y(x) := I^{1-\alpha}_{a^{+}} y'(x), \tag{2.1}$$

$$ {}^{c}D^{\alpha}_{b-}\, y(x) := -I^{1-\alpha}_{b^{-}} y'(x), \tag{2.2}$$

and the left and right Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ ([13, 15, 24])

$$ D^{\alpha}_{a+}\, y(x) := \frac{d}{dx} I^{1-\alpha}_{a^{+}} y(x), \tag{2.3}$$

$$ D^{\alpha}_{b-}\, y(x) := -\frac{d}{dx} I^{1-\alpha}_{b^{-}} y(x), \tag{2.4}$$

where the operators I^{α}_{a+} and I^{α}_{b-} are respectively the left and the right fractional Riemann-Liouville integrals of order $\alpha > 0$ defined by

$$ I^{\alpha}_{a+} y(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{y(t)}{(x-t)^{1-\alpha}} dt, \quad \text{for } x > a, \tag{2.5}$$

$$ I^{\alpha}_{b-} y(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{y(t)}{(t-x)^{1-\alpha}} dt, \quad \text{for } x < b. \tag{2.6}$$

We also recall the composition rules of fractional operators for the case of order $\alpha \in (0,1]$

$$ I^{\alpha}_{a+}\, {}^{c}D^{\alpha}_{a+} y(x) = y(x) - y(a) \tag{2.7}$$

$$ I^{\alpha}_{b-}\, {}^{c}D^{\alpha}_{b-} y(x) = y(x) - y(b) \tag{2.8}$$

and for the Riemann-Liouville derivatives

$$ D^{\alpha}_{a+} I^{\alpha}_{a+} y(x) = y(x) \tag{2.9}$$

$$ D^{\alpha}_{b-} I^{\alpha}_{b-} y(x) = y(x). \tag{2.10}$$

All the above rules are fulfilled for all $x \in [a,b]$ when function y is a continuous one.

Now, we shall quote the general formulation of the fractional eigenvalue problem, introduced and investigated in papers [16, 17].

DEFINITION 2.1. Let $\alpha \in (0,1)$. With the notation

$$\mathcal{L}_q := {}^cD^{\alpha}_{b-} p(x) \;\; D^{\alpha}_{a+} + q(x), \tag{2.11}$$

consider the fractional Sturm-Liouville equation (FSLE)

$$\mathcal{L}_q y_\lambda(x) = \lambda w(x) y_\lambda(x), \tag{2.12}$$

where $p(x) \neq 0, w(x) > 0 \quad \forall x \in [a,b]$ and p, q, w are real-valued continuous functions in $[a,b]$ and boundary conditions are:

$$c_1 I^{1-\alpha}_{a+} y_\lambda(x) \mid_{x=a} + c_2 p(x) D^{\alpha}_{a+} y_\lambda(x) \mid_{x=a} = 0, \tag{2.13}$$

$$d_1 I^{1-\alpha}_{a+} y_\lambda(x) \mid_{x=b} + d_2 p(x) D^{\alpha}_{a+} y_\lambda(x) \mid_{x=b} = 0 \tag{2.14}$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. The problem of finding number λ such that the BVP has a non-trivial solution will be called the regular fractional Sturm-Liouville eigenvalue problem (FSLP). When $p(a) = p(b) = 0$, the above eigenvalue problem will be called the singular fractional Sturm-Liouville problem. The parameter λ in Eq. (2.12), called an eigenvalue, exists for a non-trivial solution of the above FSLP, whereas the solution is called an eigenfunction associated to λ.

Let us observe that in the case $\alpha = 1$ we have: ${}^cD^1_{a^+}\, y = y'$ and ${}^cD^1_{b^-}\, y = -y'$, hence Eq. (2.12) takes the classical form

$$-\left(p\,(x)\, y'\,(x)\right)' + q\,(x)\, y\,(x) = \lambda w\,(x)\, y\,(x) \tag{2.15}$$

with the boundary conditions:

$$c_1 y_\lambda(a) + c_2 p(a) y'_\lambda(a) = 0,$$
$$d_1 y_\lambda(b) + d_2 p(a) y'_\lambda(b) = 0.$$

The aim of this paper is to study FSLP subjected to a such a set of boundary conditions that leads to a purely discrete countable spectrum and to the orthogonal eigenfunctions' system constituting the basis in the respective weighted Hilbert space.

3. **Exact solutions**

In this section, we shall formulate the FSLP with an equation containing the fractional differential operator (2.11). We investigate the eigenvalues and eigenfunctions' system connected to the FSLE in the case of order α fulfilling condition: $1 \geq \alpha > 1/2$

$$\mathcal{L}_q f(x) = \lambda w(x) f(x) \tag{3.1}$$

subject to the mixed boundary conditions in the fractional version and on the space of continuous functions

$$f(a) = 0 \quad D^{\alpha}_{a+} f(b) = {}^cD^{\alpha}_{a+} f(b) = 0 \quad f \in C[a,b]. \tag{3.2}$$

Let us observe that the above regular FSLP is a special case of the general FSLP given in Definition 2.1 when constants $c_2 = d_1 = 0$.

We propose a transformation of the introduced differential Sturm-Liouville problem to the integral one on the subspace of continuous functions defined below:

$$C_B[a,b] := \{f \in C[a,b];\ \ f(a) = 0 \quad D^{\alpha}_{a+}f(b) = {}^{c}D^{\alpha}_{a+}f(b) = 0\}. \tag{3.3}$$

We start by introducing the following integral operator

$$T_w f(x) := I^{\alpha}_{a+} \frac{1}{p(x)} I^{\alpha}_{b-} w(x) f(x) \tag{3.4}$$

and we note that on the $C_B[a,b]$, subspace of continuous functions, the following relation is valid

$$T_w \frac{1}{w(x)} \mathcal{L}_q f(x) = \left(1 + T_w \frac{q(x)}{w(x)}\right) f(x). \tag{3.5}$$

In addition, it is easy to check that

$$T_w \frac{q(x)}{w(x)} f(x) = T_q f(x), \tag{3.6}$$

therefore the intermediate integral form of the equation (2.12) looks as follows

$$(1 + T_q)\, f(x) = \lambda T_w f(x) \quad f \in C_B[a,b]. \tag{3.7}$$

In order to invert the integral operator on the left-hand side, we estimate the norm of the T_q operator in the $C[a,b]$ space with the supremum norm $||\cdot||$ and obtain

$$\begin{aligned} ||T_q|| = \sup_{||f||\le 1} \frac{||T_q f||}{||f||} &= \sup_{||f||\le 1} \frac{||I^{\alpha}_{a+}\frac{1}{p} I^{\alpha}_{b-} q f||}{||f||} \\ &\le ||I^{\alpha}_{a+} \frac{1}{p(x)} I^{\alpha}_{b-} q|| \\ \le ||q|| \cdot ||\frac{1}{p}|| \cdot ||I^{\alpha}_{a+} I^{\alpha}_{b-} 1|| &\le ||q|| \cdot ||\frac{1}{p}|| \cdot \frac{(b-a)^{2\alpha}}{(2\alpha - 1)(\Gamma(\alpha))^2}. \end{aligned} \tag{3.8}$$

Let us denote the parameter ξ as follows:

$$\xi := ||q|| \cdot ||\frac{1}{p}|| \cdot \frac{(b-a)^{2\alpha}}{(2\alpha - 1)(\Gamma(\alpha))^2}, \tag{3.9}$$

then the above calculations lead to the following lemma.

LEMMA 3.1. *Let $1 \ge \alpha > 1/2$ and $q, \dfrac{1}{p} \in C[a,b]$ and $\xi < 1$. Then, on the $C[a,b]$-space we have: $||T_q|| < 1$.*

When the norm of the T_q-operator is smaller than 1, we can invert operator $1+T_q$.

LEMMA 3.2. *Let* $1 \geq \alpha > 1/2$, $q, \dfrac{1}{p} \in C[a,b]$ *and function* $f \in C_B[a,b]$ *fulfill equation (2.12). Then, the following equality is valid*

$$f(x) = \lambda(1+T_q)^{-1}T_w f(x) = \lambda \sum_{k=0}^{\infty}(-1)^k (T_q)^k T_w f(x), \tag{3.10}$$

and the function $(1+T_q)^{-1}T_w f \in C_B[a,b]$, *i.e. it obeys boundary conditions (3.2).*

P r o o f. Let us denote

$$T := (1+T_q)^{-1}T_w = \sum_{k=0}^{\infty}(-1)^k (T_q)^k T_w \tag{3.11}$$

and observe that for any function $f \in C[a,b]$ and $k \in \mathbb{N}$

$$||(T_q)^k T_w f|| \leq ||T_q||^k \cdot ||T_w f|| \leq \xi^k \cdot ||T_w f||.$$

Thus, series $\sum_{k=0}^{\infty}(-1)^k (T_q)^k T_w f$ is uniformly convergent on interval $[a,b]$ and its sum Tf is a continuous function obeying boundary condition $Tf(a) = 0$. Now, we shall check the second boundary condition (the right-sided). Applying the theorem on integrating the series term by term w obtain the equality

$$D^{\alpha}_{a+}Tf(x) = D^{\alpha}_{a+}I^{\alpha}_{a+}\frac{1}{p}I^{\alpha}_{b-}q(x)\left(\frac{w(x)}{q(x)}f(x) + \sum_{k=1}^{\infty}(-1)^k (T_q)^{k-1}T_w f(x)\right) \tag{3.12}$$

$$= \frac{1}{p(x)}I^{\alpha}_{b-}q(x)\left(\frac{w(x)}{q(x)}f(x) + \sum_{k=1}^{\infty}(-1)^k (T_q)^{k-1}T_w f(x)\right),$$

and we recall that for any $f \in C[a,b]$ we have $I^{\alpha}_{b-}y(x)|_{x=b} = 0$. Thence, at the boundary we have for $k \in \mathbb{N}$

$$I^{\alpha}_{b-}w(x)f(x)|_{x=b} = 0, \quad I^{\alpha}_{b-}\left(q(x)(T_q)^{k-1}T_w f(x)\right)|_{x=b} = 0, \tag{3.13}$$

and we conclude that

$$D^{\alpha}_{a+}Tf(b) = 0.$$

□

For functions $f \in C_B[a,b]$ we can prove the equivalence of the differential and the integral form of the fractional Sturm-Liouville problem. Namely, the following lemma is valid.

LEMMA 3.3. *Let $1 \geq \alpha > 1/2$, $q, \dfrac{1}{p} \in C[a,b]$ and $\xi < 1$. Then, the following equivalence is valid on the $C_B[a,b]$-space*

$$\mathcal{L}_q f(x) = \lambda w(x) f(x) \Longleftrightarrow T f(x) = \frac{1}{\lambda} f(x). \tag{3.14}$$

P r o o f. First, assuming that $f \in C_B[a,b]$ is an eigenfunction corresponding to eigenvalue λ

$$\frac{1}{w(x)} \mathcal{L}_q f(x) = \lambda f(x)$$

we obtain the equality

$$T_w \frac{1}{w(x)} \mathcal{L}_q f(x) = \lambda T_w f(x) \tag{3.15}$$

which leads to the integral equation

$$(1 + T_q) f(x) = \lambda T_w f(x). \tag{3.16}$$

Because $\xi < 1$ by assumption we can apply Lemma 3.2 which means we can invert the $(1+T_q)$-operator on the $C_B[a,b]$-space

$$\frac{1}{\lambda} f(x) = (1 + T_q)^{-1} T_w f(x) = T f(x). \tag{3.17}$$

It proves the first part of the equivalence statement.

Now, we assume that the continuous function $f \in C_B[a,b]$ is an eigenfunction of the integral FSLP corresponding to eigenvalue $\frac{1}{\lambda}$:

$$T f(x) = \frac{1}{\lambda} f(x). \tag{3.18}$$

We calculate the composition $\mathcal{L}_q T$ applying the theorem on integrating of series term by term

$$\begin{aligned} \mathcal{L}_q T f(x) &= w(x) f(x) + q(x) T_w f(x) \qquad (3.19) \\ &+ \mathcal{L}_q I^{\alpha}_{a+} \frac{1}{p} I^{\alpha}_{b-} q(x) \left(\sum_{k=1}^{\infty} (-1)^k q(x) \left[(T_q)^{k-1} + (T_q)^k \right] T_w f(x) \right) \\ &= w(x) f(x) + q(x) T_w f(x) + \sum_{k=0}^{\infty} (-1)^{k+1} q(x) (T_q)^k T_w f(x) \\ &+ \sum_{k=1}^{\infty} (-1)^k q(x) (T_q)^k T_w f(x) \\ &= w(x) f(x) + q(x) T_w f(x) - q(x) T_w f(x) = w(x) f(x). \end{aligned}$$

From the above result and from equation (3.18) we obtain the implication

$$\mathcal{L}_q Tf(x) = w(x)f(x) = \frac{1}{\lambda}\mathcal{L}_q f(x) \Longrightarrow \mathcal{L}_q f(x) = \lambda w(x) f(x).$$

Therefore, we conclude that on the $C_B[a,b]$ space the equivalence of the differential and integral FSLP (3.14) is valid. □

Next, we extend the T-operator to the $L^2_w(a,b)$ - space and note that if order α fulfills $1 \geq \alpha > 1/2$, then

$$u \in L^2_w(a,b) \Longrightarrow T_w u \in C[a,b]. \tag{3.20}$$

The following lemma is a straightforward corollary from Lemma 3.2.

LEMMA 3.4. *Let $1 \geq \alpha > 1/2$ and $\xi < 1$. Then, for any function $u \in L^2_w(a,b)$ the image $Tu \in C[a,b]$ and it obeys the boundary conditions:*

$$Tu(a) = 0 \quad D^\alpha_{a+}Tu(b) = 0, \tag{3.21}$$

i.e. $Tu \in C_B[a,b]$.

P r o o f. Under assumptions of the lemma, we have $T_w u \in C[a,b]$ and series $\sum_{k=0}^{\infty}(-1)^k (T_q)^k T_w f(x)$ is uniformly convergent on interval $[a,b]$, thus its sum is also a continuous function, i.e.

$$Tu \in C[a,b], \quad Tu(a) = 0. \tag{3.22}$$

Similar to the proof of Lemma 3.2, we obtain the expression for the left Riemann-Liouville derivative

$$D^\alpha_{a+}Tu(x) \tag{3.23}$$

$$= \sum_{k=1}^{\infty}(-1)^k \frac{1}{p(x)} I^\alpha_{b-}\left(q(x)(T_q)^{k-1}T_w u(x)\right) + \frac{1}{p(x)} I^\alpha_{b-} w(x)u(x)$$

and the following useful inequality is valid for any function $u \in L^2_w(a,b)$ on interval $[a,b]$

$$|I^\alpha_{b-}u(x)w(x)| \leq \frac{1}{\Gamma(\alpha)}\left(\int_x^b (s-x)^{2\alpha-2}ds\right)^{1/2}\left(\int_x^b (u(s)w(s))^2 ds\right)^{1/2}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{(b-x)^{2\alpha-1}}{2\alpha-1}\right)^{1/2} \cdot \sqrt{||w||} \cdot ||u||_{L^2_w}.$$

Thence, for $k \in \mathbb{N}$ we have

$$I^\alpha_{b-}w(x)u(x)|_{x=b} = 0, \qquad I^\alpha_{b-}\left(q(x)(T_q)^{k-1}T_w u(x)\right)|_{x=b} = 0, \tag{3.24}$$

and we recover the second boundary condition

$$D^\alpha_{a+}Tu(b) = {}^cD^\alpha_{a+}Tu(b) = 0.$$

□

In order to consider the spectral properties of the integral operator T, we shall convert it to the integral Hilbert-Schmidt operator. To this aim we explicitly calculate its kernel. First, we express operators T_q and T_w as integral operators with the corresponding kernels. We obtain the following formula for the kernel K_q of the T_q operator defined by Eq. (3.6)

$$T_q u(x) = \int_a^b K_q(x,s)u(s)ds \tag{3.25}$$

and

$$K_q(x,s) = q(s)K_1(x,s) \tag{3.26}$$

with symmetric part

$$K_1(x,s) = \int_a^{\min\{x,s\}} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{p(t)} dt\,. \tag{3.27}$$

For the T_w operator defined by Eq. (3.4):

$$T_w u(x) = \int_a^b K_w(x,s)u(s)ds, \tag{3.28}$$

we calculate kernel K_w:

$$K_w(x,s) = w(s)K_1(x,s). \tag{3.29}$$

We now estimate the norms and values of the above kernels.

LEMMA 3.5. *Let* $1 \geq \alpha > 1/2$, $q, \dfrac{1}{p} \in C[a,b]$. *Then, kernel* $K_q \in L^2_{1\otimes 1}([a,b]\times[a,b])$ *and the following inequalities are valid:*

$$|K_q(x,s)| \leq \xi/(b-a), \tag{3.30}$$

$$||K_q||_{L^2_{1\otimes 1}} \leq \xi/(b-a)^{1/2}. \tag{3.31}$$

P r o o f. Let us start with the estimation of the absolute values of the K_q-kernel on square $[a,b]\times[a,b]$:

$$|K_q(x,s)| = \left| \int_a^{\min\{x,s\}} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{q(s)}{p(t)} dt \right| \tag{3.32}$$

$$\leq |q(s)| \int_a^{\min\{x,s\}} \frac{(\min\{x,s\}-t)^{2\alpha-2}}{(\Gamma(\alpha))^2 |p(t)|}\, dt$$

$$\leq ||\frac{1}{p}|| \cdot |q(s)| \cdot \frac{(-1)(\min\{x,s\}-t)^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)} \Big|_{t=a}^{\min\{x,s\}}$$

$$\leq ||\frac{1}{p}|| \cdot |q(s)| \cdot \frac{(b-a)^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)} \leq \xi/(b-a).$$

From the above inequality we infer that estimation (3.31) is also valid. □

From the above lemma we obtain $|K_w(x,s)| \leq \xi_w/(b-a)$ and $K_w \in L^2_{1\otimes 1}([a,b]\times[a,b])$ by assuming $q = w$ and denoting:

$$\xi_w = ||w|| \cdot ||\frac{1}{p}|| \cdot (b-a)^{2\alpha}/(2\alpha-1)(\Gamma(\alpha))^2.$$

Now, we reformulate the T operator given by Eq. (3.11) in the form of Hilbert-Schmidt operator with kernel K:

$$Tu(x) = \int_a^b K(x,s)u(s)ds. \tag{3.33}$$

The next step is the explicit calculation of the kernel K and examine its properties. It appears that it can be expressed as a series of convolutions described in the lemma below.

LEMMA 3.6. *Let $1 \geq \alpha > 1/2$, $q, \frac{1}{p} \in C[a,b]$, kernels K_q, K_w be defined by formulas (3.31) and (3.30) respectively and $\xi < 1$, where parameter ξ is defined in (3.9). Then, kernel K is given by the formula*

$$K := K_w + \sum_{n=1}^{\infty} (-1)^n (K_q)^{*n} * K_w \tag{3.34}$$

and it fulfills condition $K \in L^2_{w\otimes w}([a,b]\times[a,b])$.

P r o o f. First, by using the convolution properties we prove formulas for operators $(T_q)^n T_w$. For $n = 1$ we obtain

$$T_q T_w u(x) = \int_a^b K_q(x,s) T_w u(s) ds \tag{3.35}$$

$$= \int_a^b K_q(x,s) \left(\int_a^b K_w(s,t)u(t)dt \right) ds$$

$$= \int_a^b u(t) \left(\int_a^b K_q(x,s) K_w(s,t) ds \right) dt$$

$$= \int_a^b K_q * K_w(x,t) u(t) dt,$$

thus it is an integral operator with the kernel given as the convolution of the K_q and K_w kernels. Terms $(T_q)^n T_w u$, $n > 1$ can be expressed analogously:

$$(T_q)^n T_w u(x) = \int_a^b K_q(x,s) (T_q)^{n-1} T_w u(s) ds \tag{3.36}$$

$$= \int_a^b K_q(x,s) \left(\int_a^b [(K_q)^{*(n-1)} * K_w](s,t) u(t) dt \right) ds$$

$$= \int_a^b u(t) \left(\int_a^b K_q(x,s) [(K_q)^{*(n-1)} * K_w](s,t) ds \right) dt$$

$$= \int_a^b (K_q)^{*n} * K_w(x,t)u(t)dt,$$

where we applied the mathematical induction principle with its assumption $(T_q)^{n-1}T_w u(x) = \int_a^b (K_q)^{*(n-1)} * K_w(x,s)u(s)ds$. From the above calculations we infer that operator T defined by formula (3.33) can be rewritten as the Hilbert-Schmidt operator with the integral kernel K (3.34) provided this series is convergent on square $[a,b]\times[a,b]$. We apply Lemma 3.5 and estimate the absolute value of the series terms

$$|K_q * K_w(x,s)| \leq \left(\int_a^b (K_q(x,t))^2 dt\right)^{1/2} \left(\int_a^b (K_w(x,t'))^2 dt'\right)^{1/2} \leq \frac{\xi\cdot\xi_w}{b-a},$$

and by means of the mathematical induction we have

$$|(K_q)^{*n} * K_w(x,s)| \leq \left(\int_a^b ((K_q)^{*n}(x,t))^2 dt\right)^{1/2} \left(\int_a^b (K_w(x,t'))^2 dt'\right)^{1/2}$$
$$\leq \frac{\xi^n\cdot\xi_w}{b-a}.$$

We observe that at any point $(x,s)\in[a,b]\times[a,b]$ we have the following estimation valid

$$|K_w(x,s) + \sum_{n=1}^{\infty}(-1)^n (K_q)^{*n} * K_w(x,s)|$$
$$\leq |K_w(x,s)| + \sum_{n=1}^{\infty}|(K_q)^{*n} * K_w(x,s)|$$
$$\leq \frac{\xi_w}{b-a} + \sum_{n=1}^{\infty}\frac{\xi^n\cdot\xi_w}{b-a} = \frac{\xi_w}{(b-a)(1-\xi)}.$$

Thence, we infer that series K is absolutely and uniformly convergent on square $[a,b]\times[a,b]$. This fact implies that it also is convergent in the $L^2_{w\otimes w}([a,b]\times[a,b])$ space and $K\in L^2_{w\otimes w}([a,b]\times[a,b])$. □

Now, we are ready to formulate the main theorem on the integral operator T.

THEOREM 3.1. *Let $1\geq\alpha>\frac{1}{2}$, $q,\frac{1}{p}\in C[a,b]$ and $\xi<1$. Then, the Hilbert-Schmidt operator $T:\ L^2_w(a,b)\to L^2_w(a,b)$ defined by formulas (3.33), (3.34) is a compact and self-adjoint operator.*

P r o o f. The compactness of the operator T results from Lemma 3.6, i.e. the fact that $K\in L^2_{w\otimes w}([a,b]\times[a,b])$. Next, the following equalities are valid for any pair of functions $f,g\in L^2_w(a,b)$:

$$\langle f, T_w g\rangle_w = \langle T_w f, g\rangle_w \qquad \langle f, T_q g\rangle_w = \langle \frac{q}{w}T_w f, g\rangle_w. \tag{3.37}$$

It is easy to check that

$$\langle f, T_q T_w g\rangle_w = \langle T_q T_w f, g\rangle_w.$$

Let us now assume that $\langle f, (T_q)^{n-1} T_w g\rangle_w = \langle (T_q)^{n-1} T_w f, g\rangle_w$. We obtain for $n \in \mathbb{N}$, by applying the mathematical induction assumption and principle, that the following equality is fulfilled

$$\langle f, (T_q)^n T_w g\rangle_w = \langle f, T_q (T_q)^{n-1} g\rangle_w = \langle \frac{q}{w} T_w f, (T_q)^{n-1} T_w g\rangle_w$$

$$= \langle (T_q)^{n-1} T_w \frac{q}{w} T_w f, g\rangle_w = \langle (T_q)^n T_w f, g\rangle_w.$$

As the series K is uniformly convergent on square $[a,b]\times[a,b]$ we infer that the partial sums sequence $\sum_{n=0}^{m}(-1)^n (T_q)^n T_w f$ is uniformly convergent on interval $[a,b]$ for any function $f \in L^2_w(a,b)$. Thus, from the limit theorem for integrals we get

$$\langle g, Tf\rangle_w = \lim_{m\to\infty} \sum_{n=0}^{m} (-1)^n \langle g, (T_q)^n T_w f\rangle_w$$

$$= \lim_{m\to\infty} \sum_{n=0}^{m} (-1)^n \langle (T_q)^n T_w g, f\rangle_w$$

$$= \langle \lim_{m\to\infty} \sum_{n=0}^{m} (-1)^n (T_q)^n T_w g, f\rangle_w = \langle Tg, f\rangle_w.$$

We note that for any functions $f, g \in L^2_w(a,b)$ we have $\langle g, Tf\rangle_w = \langle Tg, f\rangle_w$ which means that operator T is a self-adjoint operator on $L^2_w(a,b)$. □

The above theorem on the Hilbert-Schmidt operator T implies the following result on its spectrum and eigenfunctions.

Corollary 3.1. *Let $1 \geq \alpha > \frac{1}{2}$, $q, \dfrac{1}{p} \in C[a,b]$ and $\xi < 1$. Then, the operator T has a purely discrete (countable spectrum) enclosed in interval $(-1,1)$ with 0 being its only limit point. Eigenfunctions y_n corresponding to the respective eigenvalues are continuous, obey the boundary conditions (3.2), i.e. belong to the $C_B[a,b]$-space and form a basis in the $L^2_w(a,b)$-space.*

As the eigenfunctions of the operator T belong to the $C_B[a,b]$-space, we can apply Lemma 3.3 on the equivalence of the differential and integral eigenvalue problem to obtain a principal result on the discrete spectrum of the studied FSLP.

Theorem 3.2. *Let $1 \geq \alpha > \frac{1}{2}$, $q, \dfrac{1}{p} \in C[a,b]$ and $\xi < 1$. Then, operator $\mathcal{L}_q$ has a purely discrete (countable spectrum) with $|\lambda_n| \to \infty$.*

Eigenfunctions y_n corresponding to the respective eigenvalues are continuous, obey the boundary conditions (3.2), i.e. belong to the $C_B[a,b]$-space and form a basis in the $L^2_w(a,b)$-space. Moreover, the following number series is convergent:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\lambda_n)^2} < \infty. \tag{3.38}$$

P r o o f. We can assume that eigenfunctions' basis is orthonormal in the $L^2_w(a,b)$ space, then functions $y_k \otimes \overline{y_j}$ produce an orthonormal basis in the $L^2_{w\otimes w}([a,b]\times[a,b])$ function space. Thus, the kernel K can de expressed as follows

$$K(x,s) = w(s) \sum_{k,j=-\infty}^{\infty} a_{kj} y_k(x) \overline{y_j(s)}.$$

Assuming y_m is an eigenfunction corresponding to eigenvalue λ_m and applying the orthonormality property of eigenfunctions, we get the relation below

$$\frac{1}{\lambda_m} y_m(x) = T y_m(x) = \int_a^b K(x,s) y_m(s) ds = \sum_{k=-\infty}^{\infty} a_{km} y_k(x).$$

We use the orthonormality of eigenfunctions once again and obtain the explicit formula for coefficients a_{lm}:

$$\delta_{lm} \frac{1}{\lambda_m} = a_{lm}.$$

Now, we can write the kernel K as the series

$$K(x,s) = w(s) \sum_{k=-\infty}^{\infty} \frac{1}{\lambda_k} y_k(x) \overline{y_k(s)}.$$

We know that $K \in L^2_{w\otimes w}([a,b]\times[a,b])$, obtain the inequality

$$||K||^2_{L^2_{w\otimes w}}$$

$$= \int_a^b \int_a^b w(x)|K(x,s)|^2 w(s) dx ds \geq \left(\min_{s'\in[a,b]} w(s') \right)^2 \cdot \sum_{k=-\infty}^{\infty} \frac{1}{(\lambda_k)^2} > 0,$$

and we infer that (3.38) is valid. □

The assumption $\frac{1}{p} \in C[a,b]$ leads to two possible cases, namely p can be a positive or negative function. When function p is positive the spectrum is bounded from below which means that the fractional Sturm-Liouville operator has at least a finite number of negative and an infinite number of positive eigenvalues.

COROLLARY 3.2. *Let* $1 \geq \alpha > \frac{1}{2}$, $q, \dfrac{1}{p} \in C[a,b]$, $p > 0$ *and* $\xi < 1$. *Then, the operator* $\mathcal{L}_q$ *has a purely discrete (countable spectrum) with* $\lambda_n \to \infty$. *The eigenfunctions* y_n *corresponding to the respective eigenvalues are continuous, obey the homogeneous mixed boundary conditions, i.e. belong to the* $C_B[a,b]$*-space and form an orthogonal basis in the* $L^2_w(a,b)$*-space. Moreover, the following estimation is valid:*

$$\lambda_n > \min_{x\in[a,b]} \frac{q(x)}{w(x)} \tag{3.39}$$

and the series below is convergent

$$\sum_{n=-n_0}^{\infty} \frac{1}{(\lambda_n)^2} < \infty. \tag{3.40}$$

P r o o f. We again can assume that the basis of eigenfunctions is orthonormal in $L^2_w(a,b)$ and let eigenfunction y_n correspond to eigenvalue λ_n, i.e. it fulfills the equation:

$$\mathcal{L}_q y_n(x) = \lambda_n w(x) y_n(x).$$

Multiplying both sides by $\overline{y_n}$ and integrating over interval $[a,b]$ we get the relation below

$$\int_a^b \overline{y_n(x)} \mathcal{L}_q y_n(x) dx = \lambda_n \langle y_n, y_n \rangle_w$$

which leads to the following formula for eigenvalue λ_n

$$\lambda_n = \langle y_n, {}^cD^{\alpha}_{b-} p\, {}^cD^{\alpha}_{a+} y_n \rangle + \langle y_n, q y_n \rangle.$$

Now, we use the fact that each eigenfunction fulfills the homogeneous mixed boundary conditions and by applying the fractional integration by parts rule we arrive at the equality

$$\lambda_n = ||\sqrt{p}\ {}^cD^{\alpha}_{a+} y_n||^2_{L^2} + \langle y_n, q y_n \rangle.$$

Clearly, the first term on the right-hand side is positive, therefore the following inequality is valid:

$$\lambda_n > \langle y_n, q y_n \rangle \geq \min_{x\in[a,b]} \frac{q(x)}{w(x)} \cdot \langle y_n, y_n \rangle_w$$

which means

$$\lambda_n > \langle y_n, q y_n \rangle \geq \min_{x\in[a,b]} \frac{q(x)}{w(x)},$$

because we have assumed $\langle y_n, y_n \rangle_w = 1$.

The proven inequality means that in the case when function p is positive we have at least a finite number of negative eigenvalues and an infinite number of positive ones tending to infinity. Numbering the negative eigenvalues by $n = -n_0, ..., -1$, we get the convergence in Eq. (3.40) from the general formula (3.38). $\square$

The proof of the analogous result for p negative is similar so we just formulate the corollary as follows noting that here we have at least a finite number of positive eigenvalues and an infinite number of the negative ones tending to $-\infty$.

COROLLARY 3.3. *Let $1 \geq \alpha > \frac{1}{2}$, $q, \dfrac{1}{p} \in C[a,b]$, $p < 0$ and $\xi < 1$. Then, the operator $\mathcal{L}_q$ has a purely discrete (countable spectrum) with $\lambda_n \to -\infty$. Eigenfunctions y_n corresponding to the respective eigenvalues are continuous, obey the homogeneous mixed boundary conditions i.e. belong to the $C_B[a,b]$-space and form an orthogonal basis in the $L^2_w(a,b)$-space. Moreover, the following estimation is valid:*

$$\lambda_n < \max_{x\in[a,b]} \frac{q(x)}{w(x)} \tag{3.41}$$

and the series below is convergent

$$\sum_{n=n_0}^{-\infty} \frac{1}{(\lambda_n)^2} < \infty. \tag{3.42}$$

Finally, let us note that the above statements on the regular FSLP with the fractional Sturm-Liouville operator defined in Eq. (2.11) are also valid for its reflected version with FSLO in the form of

$$\widetilde{\mathcal{L}_q} := {}^cD^{\alpha}_{a+}p(x)\ \ D^{\alpha}_{b-} + q(x) \tag{3.43}$$

and boundary conditions

$$f(b) = 0, \quad D^{\alpha}_{b-}f(a) = {}^cD^{\alpha}_{b-}f(a) = 0, \quad f \in C[a,b]. \tag{3.44}$$

As the proof of the result on spectrum and eigenfunctions' system of the reflected FSLP is very simple and based only on the properties of the reflection operator in interval $[a,b]$ we formulate the theorem omitting the proof.

THEOREM 3.3. *Let $1 \geq \alpha > \frac{1}{2}$, $q, \dfrac{1}{p} \in C[a,b]$ and $\xi < 1$. Then, the operator $\widetilde{\mathcal{L}_q}$ given in Eq. (3.43) has a purely discrete (countable spectrum) with $|\lambda_n| \to \infty$. Eigenfunctions y_n corresponding to the respective eigenvalues are continuous, obey the reflected boundary conditions (3.44) and form a basis in the $L^2_w(a,b)$-space. Moreover, the following number series is convergent:*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\lambda_n)^2} < \infty. \tag{3.45}$$

4. **Numerical solution**

We start the construction of numerical scheme by dividing the considered interval $[a, b]$ into N equidistant subintervals of length $\Delta x = (b-a)/N$ with the central points $x_i = a + (i-0.5)\Delta x$ for $i = 1, ..., N$. A value of function y at node x_i we denote as $y_i = y(x_i)$. In addition, we introduce the notation $x_{i\pm 0.5} = x_i \pm 0.5\Delta x$.

The intermediate integral equation Eq. (3.7) can be written as follows

$$\begin{aligned} y(x) + \sum_{k=1}^{N} \int_{x_{k-0.5}}^{x_{k+0.5}} K_1(x,s)\, q(s)\; y(s)\, ds \\ = \lambda \sum_{k=1}^{N} \int_{x_{k-0.5}}^{x_{k+0.5}} K_1(x,s)\, w(s)\; y(s)\, ds \end{aligned} \tag{4.1}$$

with kernel K_1 given by Eq. (3.27). Next, we approximate the above integrals by the quadrature rule

$$y(x) + \sum_{k=1}^{N} u_k\, K_1(x, x_k)\; q(x_k)\; y(x_k) = \lambda \sum_{k=1}^{N} u_k\, K_1(x, x_k)\; w(x_k)\; y(x_k), \tag{4.2}$$

where u_k are the weights of the quadrature rule (for the midpoint rectangular rule: $u_k = \Delta x$).

If we evaluate the equation at every node x_i, $i = 1, ..., N$, then we obtain the following system of N linear algebraic equations

$$y(x_i) + \sum_{k=1}^{N} u_k\, K_1(x_i, x_k)\; q(x_k)\; y(x_k) = \lambda \sum_{k=1}^{N} u_k\, K_1(x_i, x_k)\; w(x_k)\; y(x_k) \tag{4.3}$$

which can be written in the short notation for node values of eigenfunctions looks as follows:

$$y_i + \sum_{k=1}^{N} u_k\, (K_1)_{i,k}\, q_k\, y_k = \lambda \sum_{k=1}^{N} u_k\, (K_1)_{i,k}\, w_k\, y_k, \tag{4.4}$$

where $(K_1)_{i,k} = K_1(x_i, x_k)$.

Now, the above system of equations can be rewritten in the matrix form [25]. We introduce:

- two diagonal matrices $\mathbf{Q}$ and $\mathbf{W}$:

$$\begin{aligned} \mathbf{Q} &= \mathrm{diag}(q_1, q_2, ..., q_N), \\ \mathbf{W} &= \mathrm{diag}(w_1, w_2, ..., w_N), \end{aligned} \tag{4.5}$$

where

$$q_i = \frac{1}{\Delta x} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} q(x)\, dx, \quad w_i = \frac{1}{\Delta x} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} w(x)\, dx, \tag{4.6}$$

- the matrix $\mathbf{M}$:

$$\mathbf{M} = \{M_{i,k}\} = \left\{ u_k \left(K_1\right)_{i,k} \right\}, \tag{4.7}$$

for $i = 1, ..., N$ and $k = 1, ..., N$,
- the vector $\mathbf{Y}$:

$$\mathbf{Y} = \left[y_1, y_2, ..., y_N\right]^T. \tag{4.8}$$

By using these notations, the system of equations (4.4) takes the following matrix form

$$\mathbf{Y} + \mathbf{MQY} = \lambda \mathbf{MWY}, \tag{4.9}$$

and after transformations it is a matrix eigenvalue problem:

$$\mathbf{AY} = \lambda \mathbf{Y}, \tag{4.10}$$

where

$$\mathbf{A} = \left(\mathbf{MW}\right)^{-1} \left(\mathbf{I} + \mathbf{MQ}\right). \tag{4.11}$$

In order to compute the eigenvalues and eigenvectors of Eq. (4.10) one can use mathematical software. Let us note that the eigenvectors corresponding to distinct eigenvalues are orthogonal in a N-dimensional weighted Hilbert space with the scalar product defined as follows:

$$\langle \mathbf{Y}, \mathbf{X} \rangle_W = \sum_{k=1}^{N} \left(\overline{\mathbf{Y}}\right)_k w_k \left(\mathbf{X}\right)_k. \tag{4.12}$$

In calculations demonstrating the orthogonality of eigenvectors, we apply the fact that matrix $\mathbf{M}$ is symmetric. Let eigenvector $\mathbf{Y}_\lambda$ correspond to eigenvalue λ and $\mathbf{Y}_\rho$ correspond to eigenvalue ρ. First, we have the following relation valid:

$$\begin{aligned} &\langle \mathbf{Y}_\lambda, \mathbf{AY}_\rho \rangle_W \\ &= \langle \mathbf{Y}_\lambda, \left(\mathbf{W}^{-1}\mathbf{M}^{-1} + \mathbf{W}^{-1}\mathbf{Q}\right) \mathbf{Y}_\rho \rangle_W \\ &= \sum_{k=1} \overline{(\mathbf{Y}_\lambda)}_k \left[\mathbf{M}_{kj}(\mathbf{Y}_\rho)_j + q_k(\mathbf{Y}_\rho)_k\right] \\ &= \overline{\sum_{k=1} \overline{(\mathbf{Y}_\rho)}_k \left[\mathbf{M}_{kj}(\mathbf{Y}_\lambda)_j + q_k(\mathbf{Y}_\lambda)_k\right]} \\ &= \overline{\langle \mathbf{Y}_\rho, \mathbf{AY}_\lambda \rangle_W}. \end{aligned}$$

Now, we calculate the scalar products on the left- and right-hand side of the above equality remembering that $\mathbf{Y}_\lambda$ and $\mathbf{Y}_\rho$ are eigenvectors of matrix $\mathbf{A}$:

$$\langle \mathbf{Y}_\lambda, \mathbf{AY}_\rho \rangle_W = \rho \langle \mathbf{Y}_\lambda, \mathbf{Y}_\rho \rangle_W$$

$$\overline{\langle \mathbf{Y}_\rho, \mathbf{A}\mathbf{Y}_\lambda \rangle_W} = \lambda \overline{\langle \mathbf{Y}_\rho, \mathbf{Y}_\lambda \rangle_W} = \lambda \langle \mathbf{Y}_\lambda, \mathbf{Y}_\rho \rangle_W.$$

Subtracting the above equalities we arrive at the result

$$(\rho - \lambda)\langle \mathbf{Y}_\lambda, \mathbf{Y}_\rho \rangle_W = 0$$

which implies that when eigenvalues are distinct, the corresponding eigenvectors are orthogonal:

$$\langle \mathbf{Y}_\lambda, \mathbf{Y}_\rho \rangle_W = 0 \quad \rho \neq \lambda. \tag{4.13}$$

In the next step, we construct the approximate eigenfunctions by applying the eigenvectors obtained via the numerical scheme corresponding to the equidistant partition of interval $[a, b]$ into N subintervals:

$$y_\lambda^{ap}(x) = \sum_{k=1}^{N} (\mathbf{Y}_\lambda)_k \, \chi_{[x_{k-0.5}, x_{k+0.5})}(x). \tag{4.14}$$

Let us note that from the orthogonality of eigenvectors the lemma on approximate eigenfunctions results.

Lemma 4.1. *Approximate eigenfunctions corresponding to distinct eigenvalues are orthogonal in the* $L^2_w(a, b)$ *- space for each* $N \in \mathbb{N}$

$$\langle y_\lambda^{ap}, y_\rho^{ap} \rangle_w = 0 \quad \lambda \neq \rho. \tag{4.15}$$

P r o o f. The orthogonality of approximate eigenfunctions is a straightforward result of the orthogonality of eigenvectors (4.13):

$$\langle y_\lambda^{ap}, y_\rho^{ap} \rangle_w = \Delta x \langle \mathbf{Y}_\lambda, \mathbf{Y}_\rho \rangle_W = 0 \quad \rho \neq \lambda.$$

□

If we assume in the considered FSLP the coefficient and weight functions constant: $p = w = 1$ and $q = 0$, then matrix $\mathbf{A}$ is determined by simple formula

$$\mathbf{A} = \mathbf{M}^{-1}, \tag{4.16}$$

where $\mathbf{M} = \{M_{i,k}\} = \{u_k (K_0)_{i,k}\} = \{u_k K_0 (x_i, x_k)\}$, for $i = 1, ..., N$, $k = 1, ..., N$. Let us point out that in this case kernel K_0 and therefore matrix $\mathbf{M}^{-1}$ can be calculated explicitly.

4.1. **The special case for functions:** $p(x) = w(x) = 1$ **and** $q(x) = 0$**.** Now, we consider the mentioned special case which extends the classical harmonic oscillator equation and eigenvalue problem. Namely, we choose $q = 0$ and we assume that the coefficient and weight functions are constant: $p = w = 1$. We shall calculate exact values of the kernel and apply them further in calculating the numerical solutions. In this case, Eq. (3.7) reduces to the following form

$$y(x) = \lambda I^\alpha_{a^+} I^\alpha_{b^-} y(x) \tag{4.17}$$

and the composition of integral operators can be written as

$$I^{\alpha}_{a^+} I^{\alpha}_{b^-} y\,(x) = \int_a^b K_0\,(x,s)\; y\,(s)\,ds\,, \tag{4.18}$$

where the kernel K_0 (defined by Eq. (3.27) for $p = 1$) for order $1 \geq \alpha > 1/2$ is of the following explicit form

$$K_0\,(x,s) = \frac{(x-a)^{\alpha}\,(s-a)^{\alpha}}{\Gamma\,(\alpha)\,\Gamma\,(\alpha+1)} \times \begin{cases} (x-a)^{-1}\;{}_2F_1\left(1-\alpha,1;1+\alpha;\dfrac{s-a}{x-a}\right), & \text{if } s \leqslant x\,, \\ (s-a)^{-1}\;{}_2F_1\left(1-\alpha,1;1+\alpha;\dfrac{x-a}{s-a}\right), & \text{if } s > x\,. \end{cases} \tag{4.19}$$

For $x = s$ (by using properties of the hypergeometric function ${}_2F_1$) one has

$$K_0\,(x,x) = \frac{1}{\Gamma^2\,(\alpha)} \frac{(x-a)^{2\alpha-1}}{2\alpha-1}. \tag{4.20}$$

One can see that kernel K_0 is a symmetric function on square $[a,b] \times [a,b]$

$$K_0\,(x,s) = K_0\,(s,x)\;. \tag{4.21}$$

For order $\alpha = 1$ we obtain

$$K_0\,(x,s) = \begin{cases} s-a & \text{if } s \leqslant x \\ x-a & \text{if } s > x \end{cases}\;. \tag{4.22}$$

5. **Example of calculations**

In order to verify the proposed numerical method, we present two examples of numerical calculations of eigenvalues and eigenfunctions. As the first example we consider the generalization of the classical harmonic oscillator problem with $p = w = 1$ and $q = 0$ (this corresponds to the case studied in Subsection 4.1), and in the second example, we assumed the functions to be $p(x) = x^2 + \exp(x)$, $q(x) = \frac{1}{4}\sin(4\pi x)$, and $w(x) = x^2 + 2$. In both examples, we consider the interval $[0, 1]$.

In Tables 1 and 2 we present the numerical values of the first ten eigenvalues for orders $\alpha \in \{1, 0.8, 0.6\}$ and different values of $N \in \{250, 500, 1000, 2000, 4000\}$. While, in Figures 1 and 2 we show graphs of the approximate eigenfunctions corresponding to the first four eigenvalues for the considered cases, respectively, and $N = 4000$. The approximate eigenfunctions were normalized by

$$\int_a^b w(x)\, y^{ap}_{\lambda}(x)\, y^{ap}_{\lambda}(x)\, dx = 1. \tag{5.1}$$

Also, in Tables 1 and 2, the experimental rate of convergence (erc_λ) of numerical calculations of the k-th eigenvalue is presented. The values of erc_λ for fixed parameters α and variable values of N we determined from the following formula ([12])

$$erc_\lambda\left(N,\alpha,k\right)=\log_2\frac{\lambda_{(k)}^{(N,\alpha)}-\lambda_{(k)}^{(N/2,\alpha)}}{\lambda_{(k)}^{(2N,\alpha)}-\lambda_{(k)}^{(N,\alpha)}}. \tag{5.2}$$

Analyzing the values in Tables 1 and 2, one can observe that the values of erc_λ depends on the fractional order α and does not depend on the various types of functions p, q, and w and it is close to the value of 2α.

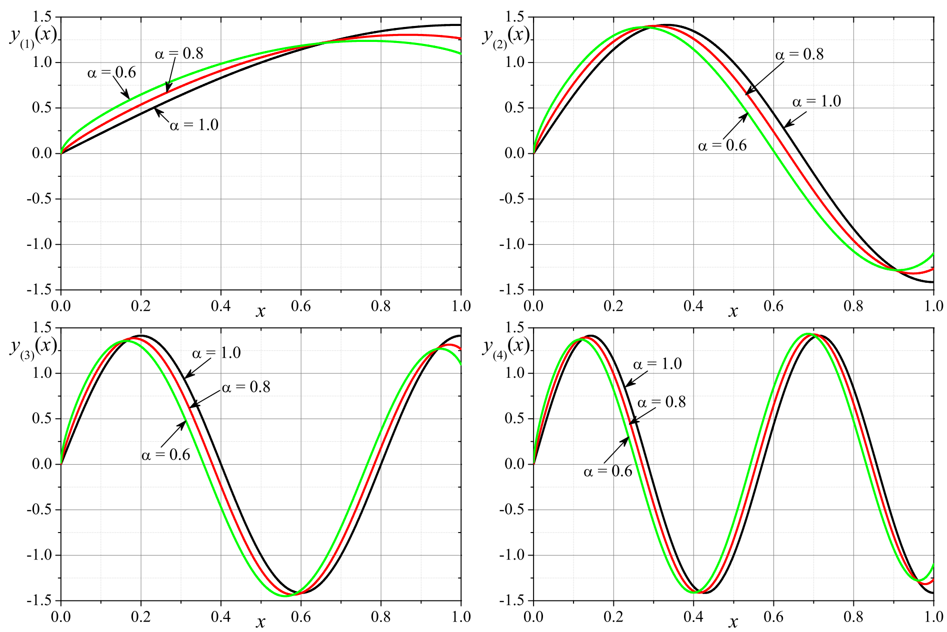

FIGURE 1. Eigenfunctions for the first 4 eigenvalues for $p(x) = 1$, $q(x) = 0$, $w(x) = 1$ and $\alpha \in \{1, 0.8, 0.6\}$ ($a = 0$, $b = 1$) (classical fractional oscillator)

6. **Conclusions**

In the paper, we studied the regular FSLP in a bounded domain, subjected to the homogeneous mixed boundary conditions. We transformed the analyzed problem into an integral one. Then, we proved the existence of a purely discrete, countable spectrum and the orthogonal system of eigenfunctions by utilizing the Hilbert-Schmidt operators theory. In the numerical approach to the FSLP, we made the discretization of the integral

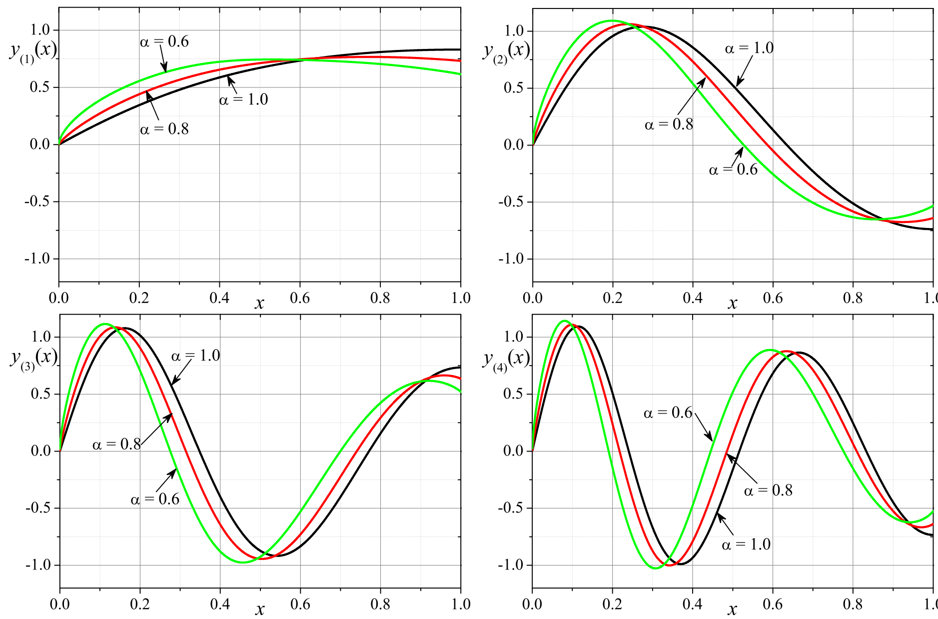

FIGURE 2. Eigenfunctions for the first 4 eigenvalues for $p(x) = x^2 + \exp(x)$, $q(x) = \frac{1}{4}\sin(4\pi x)$, and $w(x) = x^2 + 2$ and $\alpha \in \{1, 0.8, 0.6\}$ ($a = 0$, $b = 1$)

equation by using the numerical quadrature rule to the approximation of the integral. The obtained system of algebraic equations was rewritten as the matrix equation, from which the eigenvalues and eigenvectors are determined by using standard methods. The presented procedure allows us to calculate the approximate eigenvalues and eigenfunctions. We presented the obtained eigenvalues and eigenfunctions, for the selected functions p, q, w and the order α, to show the influence of these parameters on the solution to the considered FSLP. We also determined the experimental rate of convergence of the proposed method. The experimental rate of convergence of the numerical scheme, in all considered examples, is close to 2α. Moreover, orthogonality of the approximate eigenfunctions is kept at each step of the procedure. It should be pointed out that the presented numerical method can be treated as an extension of method used for the solution of the Sturm-Liouville problem of the integer order and, of course, for $\alpha = 1$ we obtain the approximate eigenvalues and eigenfunctions for the classical Sturm-Liouivlle problem.

		$\alpha = 1$		$\alpha = 0.8$		$\alpha = 0.6$	
k	N	$\lambda_{(k)}$	erc_λ	$\lambda_{(k)}$	erc_λ	$\lambda_{(k)}$	erc_λ
1	250	2.4673930	-	1.9580409	-	1.5966035	-
	500	2.4673991	2.000	1.9581331	1.607	1.5989630	1.198
	1000	2.4674006	1.995	1.9581633	1.605	1.5999915	1.199
	2000	2.4674010	1.660	1.9581733	1.604	1.6004394	1.200
	4000	2.4674011	-	1.9581766	-	1.6006344	-
2	250	22.205952	-	11.994294	-	6.4004171	-
	500	22.206446	2.000	11.997725	1.603	6.4384921	1.189
	1000	22.206569	2.000	11.998854	1.603	6.4551977	1.195
	2000	22.206600	1.997	11.999225	1.602	6.4624940	1.198
	4000	22.206607	-	11.999348	-	6.4656744	-
3	250	61.679954	-	26.986045	-	11.608982	-
	500	61.683759	2.000	27.003363	1.602	11.734829	1.178
	1000	61.684710	2.000	27.009070	1.602	11.790437	1.191
	2000	61.684948	1.999	27.010950	1.601	11.814799	1.196
	4000	61.685008	-	27.011570	-	11.825433	-
4	250	120.88317	-	46.289229	-	17.275740	-
	500	120.89778	2.000	46.340141	1.600	17.555896	1.167
	1000	120.90144	2.000	46.356931	1.601	17.680642	1.186
	2000	120.90235	2.000	46.362466	1.601	17.735481	1.194
	4000	120.90258	-	46.364290	-	17.759454	-
5	250	199.80624	-	69.085904	-	23.092775	-
	500	199.84617	2.000	69.199264	1.599	23.596078	1.156
	1000	199.85616	2.000	69.236685	1.601	23.821973	1.181
	2000	199.85866	2.000	69.249025	1.601	23.921631	1.192
	4000	199.85928	-	69.253093	-	23.965264	-
6	250	298.43670	-	95.198549	-	29.105802	-
	500	298.52582	2.000	95.413812	1.598	29.909873	1.144
	1000	298.54811	2.000	95.484937	1.600	30.273754	1.175
	2000	298.55368	2.000	95.508401	1.601	30.434883	1.189
	4000	298.55507	-	95.516137	-	30.505545	-
7	250	416.75900	-	124.20624	-	35.176421	-
	500	416.93283	2.000	124.57277	1.596	36.357661	1.132
	1000	416.97630	2.000	124.69399	1.599	36.896741	1.170
	2000	416.98716	2.000	124.73400	1.600	37.136358	1.187
	4000	416.98988	-	124.74719	-	37.241618	-
8	250	554.75442	-	156.01634	-	41.335872	-
	500	555.06252	2.000	156.59492	1.595	42.976638	1.119
	1000	555.13956	2.000	156.78647	1.599	43.731895	1.164
	2000	555.15883	2.000	156.84971	1.600	44.068916	1.184
	4000	555.16364	-	156.87057	-	44.217221	-

TABLE 1. Numerical values of the first 8 eigenvalues and the experimental rates of convergence erc_λ for $\alpha \in \{1, 0.8, 0.6\}$, $p(x) = 1$, $q(x) = 0$, and $w(x) = 1$

		$\alpha = 1$		$\alpha = 0.8$		$\alpha = 0.6$	
k	N	$\lambda_{(k)}$	erc_λ	$\lambda_{(k)}$	erc_λ	$\lambda_{(k)}$	erc_λ
1	250	1.3584974	-	1.1387715	-	0.9893191	-
	500	1.3584988	1.999	1.1388057	1.606	0.9903662	1.199
	1000	1.3584992	2.007	1.1388169	1.604	0.9908223	1.200
	2000	1.3584993	1.775	1.1388206	1.603	0.9910208	1.200
	4000	1.3584993	-	1.1388218	-	0.9911072	-
2	250	17.415682	-	9.5114080	-	5.0943056	-
	500	17.416073	2.000	9.5142567	1.602	5.1267401	1.188
	1000	17.416170	2.000	9.5151948	1.602	5.1409749	1.195
	2000	17.416195	1.998	9.5155039	1.601	5.1471929	1.198
	4000	17.416201	-	9.5156058	-	5.1499036	-
3	250	49.362322	-	21.578465	-	9.2190735	-
	500	49.365493	2.000	21.592998	1.601	9.3256384	1.178
	1000	49.366286	2.000	21.597789	1.601	9.3727246	1.191
	2000	49.366485	2.000	21.599368	1.601	9.3933541	1.196
	4000	49.366534	-	21.599888	-	9.4023590	-
4	250	97.153196	-	37.129314	-	13.762621	-
	500	97.165512	2.000	37.172412	1.600	14.001140	1.167
	1000	97.168591	2.000	37.186630	1.601	14.107352	1.186
	2000	97.169361	2.000	37.191318	1.601	14.154046	1.194
	4000	97.169554	-	37.192864	-	14.174459	-
5	250	160.88338	-	55.433830	-	18.367541	-
	500	160.91720	2.000	55.529924	1.599	18.795232	1.156
	1000	160.92565	2.000	55.561652	1.600	18.987191	1.181
	2000	160.92777	2.000	55.572118	1.600	19.071880	1.192
	4000	160.92830	-	55.575569	-	19.108960	-
6	250	240.52999	-	76.439512	-	23.143959	-
	500	240.60564	2.000	76.622406	1.597	23.828700	1.144
	1000	240.62455	2.000	76.682845	1.600	24.138585	1.175
	2000	240.62928	2.000	76.702787	1.600	24.275809	1.189
	4000	240.63046	-	76.709365	-	24.335990	-
7	250	336.07766	-	99.73969	-	27.942888	-
	500	336.22542	2.000	100.05127	1.596	28.948387	1.132
	1000	336.26237	2.000	100.15432	1.599	29.407255	1.170
	2000	336.27161	2.000	100.18834	1.600	29.611222	1.187
	4000	336.27391	-	100.19956	-	29.700823	-
8	250	447.51002	-	125.31117	-	32.816924	-
	500	447.77215	2.000	125.80346	1.595	34.214801	1.119
	1000	447.83769	2.000	125.96645	1.599	34.858243	1.164
	2000	447.85408	2.000	126.02027	1.600	35.145375	1.184
	4000	447.85818	-	126.03803	-	35.271729	-

TABLE 2. Numerical values of the first 8 eigenvalues and the experimental rates of convergence erc_λ for $\alpha \in \{1, 0.8, 0.6\}$, $p(x) = x^2 + \exp(x)$, $q(x) = \frac{1}{4}\sin(4\pi x)$ and $w(x) = x^2 + 2$

References

[1] Q.M. Al-Mdallal, An efficient method for solving fractional Sturm-Liouville problems. *Chaos Solitons Fractals* **40**, No 1 (2009), 183–189; DOI: 10.1016/j.chaos.2007.07.041.

[2] O.P. Agrawal, M. M. Hasan, X. W. Tangpong, A numerical scheme for a class of parametric problem of fractional variational calculus. *J. Comput. Nonlinear Dyn.* **7** (2012), # 021005-1–021005-6; DOI: 10.1115/1.4005464.

[3] R. Almeida, A.B. Malinowska, M.L. Morgado, T. Odzijewicz, Variational methods for the solution of fractional discrete/continuous Sturm-Liouville problems. *J. of Mechanics of Materials and Structures* **12**, No 1 (2017), 3–21; DOI: 10.2140/jomms.2017.12.3.

[4] F.V. Atkinson, *Discrete and Continuous Boundary Value Problems.* Academic Press, New York, London (1964).

[5] D. Baleanu, J.H. Asad, I. Petras, Fractional Bateman-Feshbach Tikochinsky oscillator. *Commun. Theor. Phys.* **61** (2014), 221–225; DOI: 10.1088/0253-6102/61/2/13.

[6] T. Blaszczyk, M. Ciesielski, M. Klimek, J. Leszczynski, Numerical solution of fractional oscillator equation. *Appl. Math. Comput.* **218** (2011), 2480–2488; DOI: 10.1016/j.amc.2011.07.062.

[7] T. Blaszczyk, M. Ciesielski, Numerical solution of fractional Sturm-Liouville equation in integral form. *Fract. Calc. Appl. Anal.*, **17**, No 2 (2014), 307–320; DOI: 10.2478/s13540-014-0170-8; https://www.degruyter.com/view/j/fca.2014.17.issue-2/issue-files/fca.2014.17.issue-2.xml.

[8] T. Blaszczyk, A numerical solution of a fractional oscillator equation in a non-resisting medium with natural boundary conditions. *Romanian Reports in Phys.* **67**, No 2 (2015), 350–358.

[9] T. Blaszczyk, M. Ciesielski, Fractional oscillator equation - transformation into integral equation and numerical solution. *Appl. Math. Comput.* **257** (2015), 428–435; DOI: 10.1016/j.amc.2014.12.122.

[10] L. Bourdin, J. Cresson, I. Greff, P. Inizan, Variational integrator for fractional Euler-Lagrange equations. *Appl. Numer. Math.* **71** (2013), 14–23; DOI: 10.1016/j.apnum.2013.03.003.

[11] Ciesielski M., Blaszczyk T., Numerical solution of non-homogenous fractional oscillator equation in integral form. *J. of Theoretical and Applied Mechanics* **53**, No 4 (2015), 959–968.

[12] M. Ciesielski, M. Klimek, T. Blaszczyk, The fractional Sturm-Liouville Problem – Numerical approximation and application in fractional diffusion. *J. of Computational and Applied Math.* **317**, (2017), 573–588; DOI: 10.1016/j.cam.2016.12.014.

[13] K. Diethelm, *The Analysis of Fractional Differential Equations*. Lecture Notes in Mathematics Vol. 2004, Springer-Verlag (2010).

[14] M.A. Hajji, Q.M. Al-Mdallal, F.M. Allan, An efficient algorithm for solving higher-order fractional Sturm-Liouville eigenvalue problems. *J. Comput. Phys.* **272**, (2014), 550–558; DOI: 10.1016/j.jcp.2014.04.048.

[15] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

[16] M. Klimek, O.P. Agrawal, On a regular fractional Sturm–Liouville problem with derivatives of order in $(0, 1)$. In: *Proc. of the 13th International Carpathian Control Conf.*, Vysoke Tatry (2012); DOI: 10.1109/CarpathianCC.2012.6228655.

[17] M. Klimek, O.P. Agrawal, Fractional Sturm-Liouville problem. *Comput. Math. Appl.* **66** (2013), 795–812;
DOI: 10.1016/j.camwa.2012.12.011.

[18] M. Klimek, T. Odzijewicz, A.B. Malinowska, Variational methods for the fractional Sturm-Liouville problem. *J. Math. Anal. Appl.* **416**, No 1 (2014), 402–426; DOI: 10.1016/j.jmaa.2014.02.009.

[19] M. Klimek, Fractional Sturm-Liouville problem and 1D space-time fractional diffusion problem with mixed boundary conditiond. In: *Proc. of the ASME 2015 Internat. Design Engineering Technical Conf. (IDETC) and Computers and Information in Engineering Conf. (CIE)*, Boston (2015); DOI: 10.1115/DETC2015-46808.

[20] M. Klimek, A.B. Malinowska, T. Odzijewicz, Applications of the fractional Sturm-Liouville problem to the space-time fractional diffusion in a finite domain. *Fract. Calc. Appl. Anal.* **19**, No 2 (2016), 516–550; DOI: 10.1515/fca-2016-0027; https://www.degruyter.com/view/j/fca.2016.19.issue-2/issue-files/fca.2016.19.issue-2.xml.

[21] A.B. Malinowska, D.F.M. Torres, *Introduction to the Fractional Calculus of Variations*. Imperial College Press, London (2012).

[22] A.B. Malinowska, T. Odzijewicz, D.F.M. Torres, *Advanced Methods in the Fractional Calculus of Variations*. Springer Internat. Publ., London (2015).

[23] M. d'Ovidio, From Sturm-Liouville problems to to fractional and anomalous diffusions. *Stochastic Process. Appl.* **122** (2012), 3513–3544; DOI: 10.1016/j.spa.2012.06.002.

[24] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[25] I. Podlubny, Matrix approach to discrete fractional calculus. *Fract. Calc. Appl. Anal.* **3**, No 4 (2000), 359–386.

[26] J.D. Pryce, *Numerical Solution of Sturm-Liouville Problems*. Oxford Univ. Press, London (1993).

[27] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E* **53** (1996), 1890–1899; DOI: 10.1103/PhysRevE.53.1890.

[28] M. Rivero, J.J. Trujillo, M.P. Velasco, A fractional approach to the Sturm-Liouville problem. *Cent. Eur. J. Phys.* **11**, No 10 (2013), 1246–1254; DOI: 10.2478/s11534-013-0216-2.

[29] J. Siedlecki, M. Ciesielski, T. Blaszczyk, The Sturm-Liouville eigenvalue problem - a numerical solution using the control volume method. *J. Appl. Math. Comput. Mech.* **15**, No 2 (2016), 127–136; DOI: 10.17512/jamcm.2016.2.14.

[30] Y. Xu, O.P. Agrawal, Models and numerical solutions of generalized oscillator equations. *J. Vib. Acoust.* **136**, No 5 (2014), Article ID 050903; DOI: 10.1115/1.4027241.

[31] M. Zayernouri, G.E. Karniadakis, Fractional Sturm–Liouville eigenproblems: theory and numerical approximation. *J. Comput. Phys.* **252** (2013), 495–517; DOI: 10.1016/j.jcp.2013.06.031.

[32] M. Zayernouri, M. Ainsworth, G.E. Karniadakis, Tempered fractional Sturm–Liouville eigenproblems. *SIAM J. Sci. Computing* **37**, No 4 (2015), A1777–A1800; DOI: 10.1137/140985536.

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