

SURVEY PAPER

PROPERTIES OF THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE AND ITS DISTRIBUTIONAL SETTINGS

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Abstract

The Caputo-Fabrizio fractional derivative is analyzed in classical and distributional settings. The integral inequalities needed for application in linear viscoelasticity are presented. They are obtained from the entropy inequality in a weak form. Moreover, integration by parts, an expansion formula, approximation formula and generalized variational principles of Hamilton's type are given. Hamilton's action integral in the first principle, do not coincide with the lower bound in the fractional integral, while in the second principle the minimization is performed with respect to a function from a specified space and the order of fractional derivative.

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1. Introduction

M. Caputo and M. Fabrizio proposed in [11] a fractional derivative of order $\alpha \in (0, 1)$ of an absolutely integrable function f on $[t_0, \tilde{T}]$, for any $\tilde{T} > t_0 \geq 0$, $f \in AC_{loc}([t_0, \infty))$ (which means that the first derivative of f is integrable on $[0, \tilde{T})$ for every $\tilde{T} > 0$) in the form

$${}^{CF}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_{t_0}^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) f^{(1)}(\tau) d\tau, \quad t > t_0,$$

where $f^{(1)} = \frac{d}{d\tau}f$ and $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. We take $M(\alpha) = 1$ in the analysis which is to follow. Thus, we consider (see [21], p. 89)

$$\begin{aligned} {}^{CF}_{t_0}D_t^\alpha f(t) &= \frac{1}{1-\alpha} \int_{t_0}^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) f^{(1)}(\tau) d\tau \\ &= \frac{1}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}t\right) * f^{(1)}(t), \quad t \in (t_0, \infty), \end{aligned} \quad (1.1)$$

where $*$ denotes the convolution.

In this paper we reconsider ${}^{CF}_{t_0}D_t^\alpha, \alpha \in (0, 1)$, in the distributional setting, even introduce a Riemann-Liouville version of such fractional derivative and examine several properties of ${}^{CF}_{t_0}D_t^\alpha f(t)$ that are important in applications. One can simply derive the similar properties of the Riemann-Liouville version of this notion. By the use of the Caputo-Fabrizio fractional derivative (CFFD), we shall formulate the Hamilton principle of analytical mechanics and its generalization corresponding to the case when a memory of a system and a time of a process do not coincide. The results presented here follow the corresponding ones for “standard” Caputo fractional derivative ${}^C_{t_0}D_t^\alpha f(t)$, cf. [8, 14, 17, 22] and references given therein.

We comment now a class of fractional derivatives to which (1.1) belongs (cf. [20]). In general, fractional derivative of Riemann-Liouville and Caputo type can be defined as

$${}^{RL}D_t^\alpha y(t) = \frac{d}{dt} \int_{t_0}^t k(t-\tau) y(\tau) d\tau, \quad {}^CD_t^\alpha y(t) = \int_{t_0}^t k(t-\tau) \frac{dy(\tau)}{d\tau} d\tau, \quad (1.2)$$

where k is a function with specified properties. For example, in [20] it is required that the Laplace transform of k satisfies conditions that guarantee certain useful properties of (1.2). Relation (1.2) may also be used to define variable order fractional derivatives as it is proposed in [26]. However, in this case k satisfies additional conditions. Many choices of k , in forms of special functions, with integrals not necessarily of convolution type, for example with k being Meijer G -function, and more generally, Fox H -function of the specific form, lead to generalized fractional calculus presented in [18] and [19]. The classical cases of Riemann-Liouville and Caputo fractional derivatives correspond to k given as $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $t > 0$, $0 < \alpha < 1$. The analysis presented here corresponds to the case when $k(t) = \frac{1}{(1-\alpha)T^\alpha} \exp\left(-\frac{\alpha}{1-\alpha}\frac{t}{T}\right)$, $t > 0$, $0 < \alpha < 1$, $T > 0$, called CF kernel. We refer to Section 2 for the explanation of the introduction of parameter T . Note that in papers [10],[13] in the definition of (CFFD), it was assumed that $T = 1$.

The paper is organized as follows. We present in Section 2 distributional framework for (CFFD) as well as some generalizations including symmetrized complex order (CFFD). Moreover, we introduce the Riemann-Liouville type fractional derivative with CF kernel. Several properties of (CFFD) are derived in Section 3, while Section 4 is related to two variational problems for (CFFD).

2. An intrinsic approach to new fractional derivatives

We assume that $f \in AC_{loc}([t_0, \infty))$, $t_0 \geq 0$, and that $f^{(1)}$ is supported by $[t_0, \infty)$. The independent variable $t \in [t_0, \infty)$ in (1.1) is dimensionless. In cases when t has a dimension (time for example), we introduce a parameter T having the dimension of t and replace t in kernel of ${}^{CF}_{t_0}D_t^\alpha f(t)$, by t/T . The modified definition of (CFFD) proposed in this paper is

$${}^{CF}_{t_0}D_t^\alpha f(t) = \frac{1}{T^\alpha(1-\alpha)} \int_{t_0}^t \exp\left(-\frac{\alpha}{1-\alpha} \frac{t-\tau}{T}\right) f^{(1)}(\tau) d\tau, \quad t \in (t_0, \infty). \quad (2.1)$$

In this definition one has two parameters, $\alpha \in (0, 1)$ and $T \in (0, \infty)$. If we take $T = 1$, then (2.1) becomes (1.1).

REMARK 2.1. The kernel, initially proposed in [11] was modified in [12] so that the following definition was proposed (see [12], eq. (1))

$$D_t^\alpha f(t) = \frac{1}{(1-\alpha)} \int_{t_0}^t \exp\left(-\frac{\alpha}{1-\alpha} \frac{t-\tau}{T_0}\right) f^{(1)}(\tau) d\tau, \quad t > t_0, \quad (2.2)$$

where T_0 has the same meaning as our T . However, the expression (2.2) does not reduce to $f^{(1)}$ in the limiting case $\alpha \rightarrow 1^-$ since

$$\lim_{\alpha \rightarrow 1^-} D_t^\alpha f(t) = T_0 f^{(1)}(t), \quad t > t_0.$$

Therefore, the modification of the kernel, in order to make the argument of the exponential function dimensionless, requires that the normalization function (in the terminology of [12]) has to be modified with $\frac{1}{T^\alpha(1-\alpha)}$ instead of $\frac{1}{1-\alpha}$.

2.1. Distributional framework. In order to simplify the notation, we assume $T = 1$ and $t_0 = 0$. Let H denote the Heaviside function so that it is right continuous at 0, that is $H(0) = 1$. We will consider $f \in AC_{loc}([0, \infty))$ as well as functions f in $AC_{loc}((-\infty, \infty))$ so that their restrictions on $[0, \infty)$ denoted by fH belong to $AC_{loc}([0, \infty))$.

Let $\alpha \in (0, 1)$ and

$$\varphi_\alpha(t) = \frac{1}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}t\right), \quad t \in \mathbb{R} = (-\infty, \infty).$$

By the use of distribution theory, one has, for any $\theta \in C_0^\infty(\mathbb{R})$,

$$\langle H\phi_\alpha, \theta \rangle = \frac{1}{1-\alpha} \int_0^\infty e^{\frac{-\alpha\tau}{1-\alpha}} \theta(\tau) d\tau = \int_0^\infty e^{-\alpha s} \theta((1-\alpha)s) ds,$$

which implies

$$H\phi_\alpha \rightarrow \delta \text{ in } \mathcal{S}'(\mathbb{R}) \text{ as } \alpha \rightarrow 1^-.$$

Then, we define the generalization of (1.1) corresponding to Riemann-Liouville of fractional derivative, (RLCFFD), with CF kernel

$${}^{RLCF}_0 D_t^\alpha f(t) = ((H\varphi_\alpha)' * (Hf))(t), \quad t \in (-\infty, \infty),$$

so that

$${}^{CF}_0 D_t^\alpha f(t) = {}^{RLCF}_0 D_t^\alpha f(t) - f(0) \varphi_\alpha(t), \quad t \in (0, \infty).$$

The properties are stated in the next proposition. The proofs are simple (in the distribution framework) and they are omitted. In the sequel, the assumption $f \in AC([0, \infty))$ enable us to consider f on $(-\infty, \infty)$ being equal zero on $(-\infty, 0)$.

PROPOSITION 2.1. *Suppose that $f \in AC_{loc}([0, \infty))$. Then the following holds*

$${}^{RLCF}_0 D_t^\alpha f(t) = \frac{1}{1-\alpha} f(t) - \frac{\alpha}{1-\alpha} {}^{CF}_0 I_t^\alpha f(t), \quad t \in (0, \infty),$$

where the Caputo-Fabrizio fractional integral is defined as

$${}^{CF}_0 I_t^\alpha f(t) = (H\varphi_\alpha) * (Hf)(t), \quad t \in (0, \infty).$$

Moreover,

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} {}^{CF}_0 I_t^{1-\alpha} f(t) &= f(t), \quad t \geq 0, \\ \lim_{\alpha \rightarrow 1^-} {}^{CF}_0 I_t^{1-\alpha} f(t) &= \begin{cases} \int_0^t f(u) du, & t > 0, \\ 0, & t < 0, \end{cases} \\ \lim_{\alpha \rightarrow 1^-} {}^{RLCF}_0 D_t^\alpha f(t) &= (Hf)'(t) = (Hf')(t) + f(0) \delta(t), \quad t \in (-\infty, \infty), \\ \lim_{\alpha \rightarrow 1^-} {}^{CF}_0 D_t^\alpha f(t) &= (Hf')(t) = f'(t), \quad t > 0, \\ &\quad (f' \text{ is locally integrable on } [0, \infty)), \\ \lim_{\alpha \rightarrow 0^+} {}^{RLCF}_0 D_t^\alpha f(t) &= f(t), \quad t > 0, \\ \lim_{\alpha \rightarrow 0^+} {}^{CF}_0 D_t^\alpha f(t) &= f(t) - f(0), \quad t > 0. \end{aligned}$$

Next, we consider the distributional Laplace transform (see [27], p. 127). Let f be of exponential growth, that is, f is locally integrable on $[0, \infty)$ and $|f(t)| \leq Me^{\omega t}$, $t > 0$ ($\omega \in \mathbb{R}$). Then, the Laplace transform is defined as

$$\mathcal{L}f(s) = \langle f(t), e^{-st} \rangle = \int_0^\infty e^{-st} f(t) dt, \quad s \in \mathbb{C}, \Re s > \omega.$$

It is a matter of simple calculations to show that

$$\mathcal{L}({}^{CF}_0 D_t^\alpha f(t))(s) = \frac{1}{\alpha + s(1 - \alpha)} [s\mathcal{L}(f)(s) - f(0)], \quad \Re s > \omega,$$

$$\mathcal{L}({}^{RLCF}_0 D_t^\alpha f(t))(s) = \frac{1}{\alpha + s(1 - \alpha)} s\mathcal{L}(f)(s), \quad \Re s > \omega.$$

Recall that the Fourier transform is defined by

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^\infty e^{-i\omega t} f(t) dt, \quad \omega \in \mathbb{R}, f \in L^1(\mathbb{R}).$$

Changing s with $i\omega$, $\omega \in \mathbb{R}$, we obtain the Fourier transforms of both types of fractional derivatives (having in mind that $\text{supp } f \subset [0, \infty)$). Let $\Phi(t) = \frac{1}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}|t|\right)$, $t \in \mathbb{R}$. For the later use, we note that the same arguments give, for

$$\mathcal{F}(\Phi)(\omega) = \frac{2\alpha}{\alpha^2 + (1 - \alpha)^2 \omega^2}, \quad \omega \in \mathbb{R}. \quad (2.3)$$

Let $\alpha = A + iB$ be a complex number so that $A = \Re \alpha \in (0, 1)$. We can define ${}^{CF}_{t_0} D_t^\alpha f(t)$ by (1.1). Since we prefer to work with real valued functions after fractional differentiation, we follow our approach presented in [9] and define combination of (CFD) of complex order as:

$${}^{CF}_0 \bar{D}_t^{A,B} f(t) = \frac{1}{2} \left[\mathbb{T}^{iB} {}^{CF}_0 D_t^{A+iB} f(t) + \mathbb{T}^{-iB} {}^{CF}_0 D_t^{A-iB} f(t) \right], \quad t > 0,$$

where $\mathbb{T}$ is a constant having the dimension of time and can be interpreted as relaxation time.

3. Some properties of ${}^{CF}_{t_0} D_t^\alpha f$

Since the previous exposition of Section 2 is the same if we consider $t_0 > 0$ instead of $t_0 = 0$, for the sake of simplicity, we continue with the assumption $t_0 = 0$.

3.1. A consistency result. In viscoelasticity and heat conduction problems with fractional derivatives, in proving consistency of a model with the Second law of Thermodynamics, the estimate of the following functional is needed

$$I(f) = \int_0^{\tilde{T}} f^{(1)}(t) {}^C D_t^\alpha f(t) dt = \frac{1}{1-\alpha} \int_0^{\tilde{T}} f^{(1)}(t) [\varphi_\alpha * f^{(1)}](t) dt, \quad (3.1)$$

for arbitrary $\tilde{T} > 0$. In order to estimate I , we shall examine the properties of the kernel in (1.1).

PROPOSITION 3.1. *Suppose that $f \in AC_{loc}([0, \infty))$ is real valued. Then, $I(f) \geq 0$ for every $\tilde{T} > 0$.*

P r o o f. By (2.3), $\mathcal{F}(e^{-\lambda|\tau|})(\omega) > 0, \omega \in \mathbb{R}$. The Bochner-Schwartz theorem ([23], p. 331) implies that $e^{-\lambda|\tau|}$ is a function of positive type (positive definite). Recall that a function f is of positive type if for any test function $\theta \in \mathcal{D}(\mathbb{R})$ there holds $\langle f(t), \theta * \theta^* \rangle \geq 0$, where $\theta^*(\tau) = \overline{\theta(-\tau)}$, $\theta * \theta^*(t) = \int_{\mathbb{R}} \theta(\tau) \theta^*(t - \tau) d\tau$. This result holds in a more general case for positive definite tempered distributions which is not needed here. The derivative of f in $[0, \tilde{T}]$ is an integrable function so we consider $f^{(1)}$ as a locally integrable function in $[0, \infty)$ equal zero outside $[0, \tilde{T}]$. Let us denote by $\phi_n, n \in \mathbb{N}$, a sequence of smooth functions, supported by $[0, \tilde{T}]$, which converges to $f^{(1)}$ in $L^1([0, \tilde{T}])$. Since f is real valued, we take $\phi_n, n \in \mathbb{N}$ to be real valued. By the Bochner-Schwartz theorem, for every n ,

$$\int_0^{\tilde{T}} \int_0^{\tilde{T}} \exp(-\lambda|t - \tau|) \phi_n(t) \phi_n(\tau) d\tau dt \geq 0.$$

Letting $n \rightarrow \infty$, by the Lebesgue theorem, we obtain

$$\int_0^{\tilde{T}} \int_0^{\tilde{T}} \exp(-\lambda|t - \tau|) f'(t) \overline{f'}(\tau) d\tau dt \geq 0.$$

Since

$$\begin{aligned} & \int_0^{\tilde{T}} \int_0^{\tilde{T}} \exp(-\lambda|t - \tau|) \phi_n(t) \phi_n(\tau) d\tau dt \\ &= \int_0^{\tilde{T}} \phi_n(t) \left[\int_0^t \exp(-\lambda(t - \tau)) \phi_n(\tau) d\tau \right] dt \\ &+ \int_0^{\tilde{T}} \phi_n(\tau) \left[\int_t^{\tilde{T}} \exp(-\lambda(\tau - t)) \phi_n(t) dt \right] d\tau \end{aligned}$$

$$= 2 \int_0^{\tilde{T}} \phi_n(t) \left[\int_0^t \exp(-\lambda(t-\tau)) \phi_n(\tau) d\tau \right] dt,$$

we obtain $I(f) \geq 0$. $\square$

REMARK 3.1. The result of Proposition 3.1, for the case of Riemann-Liouville fractional derivative, was proved in [25], [15] and [3]. In [24] the problem of estimating an integral of type (3.1) was treated for the case when the lower bound in the integral is $-\infty$ and when the support of the functions involved in the integration is not $[0, \tilde{T}]$ but $[-\infty, \tilde{T}]$. Our result presented here can be simply transferred to the similar assertion with the (RLCFFD).

The result of Proposition 3.1 can be generalized in order to derive dynamically consistent models for viscoelastic bodies of Kelvin-Voigt type. We formulate this result as follows.

PROPOSITION 3.2. Suppose that $f \in AC_{loc}([0, \infty))$ and that it is real-valued. Then, for $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $a, b \in \mathbb{R}$, $\tilde{T} > 0$,

$$I(f, a, b) = \int_0^{\tilde{T}} f^{(1)}(t) \left[a {}^{CF}D_t^\alpha f(t) + b {}^{CF}D_t^\beta f(t) \right] dt,$$

if the following condition is satisfied:

$$a \alpha (1 - \beta)^2 + b (1 - \alpha)^2 \beta > 0. \quad (3.2)$$

P r o o f. Without loss of generality we assume that $\beta \geq \alpha$. Let

$$F_{\alpha, \beta}(\tau) = \frac{a}{1 - \alpha} \exp\left(-\frac{\alpha}{1 - \alpha} |\tau|\right) + \frac{b}{1 - \beta} \exp\left(-\frac{\beta}{1 - \beta} |\tau|\right), \quad \tau \in \mathbb{R}.$$

Then, by (2.3),

$$\mathcal{F}(F_{\alpha, \beta})(\omega) = 2 \left[a \frac{\alpha}{\alpha^2 + (1 - \alpha)^2 \omega^2} + b \frac{\beta}{\beta^2 + (1 - \beta)^2 \omega^2} \right], \quad \omega \in \mathbb{R}. \quad (3.3)$$

It follows from (3.3) that $\mathcal{F}(F_{\alpha, \beta})(\omega) > 0, \omega \in \mathbb{R}$, if

$$a \alpha \frac{\beta^2 + \omega^2 (1 - \beta)^2}{\alpha^2 + \omega^2 (1 - \alpha)^2} + b \beta > 0. \quad (3.4)$$

Since

$$\min\left\{\frac{\beta^2 + \omega^2 (1 - \beta)^2}{\alpha^2 + \omega^2 (1 - \alpha)^2}; \omega \in \mathbb{R}\right\} = \frac{(1 - \beta)^2}{(1 - \alpha)^2} \leq 1,$$

(3.4) leads to (3.2). Thus, as in the previous assertion, we conclude that F is of positive type. Now we proceed in the same way as in the proof of Proposition 3.1 and conclude that the assertion holds true. $\square$

3.2. Partial integration. We present integration by parts formula for ${}^C_0D_t^\beta f(t)$. First, we define the right (CFFD) of real order $0 < \alpha < 1$ for $f \in AC_{loc}([0, \infty))$ as

$${}^C_tD_b^\alpha f(t) = -\frac{1}{1-\alpha} \int_t^b \exp\left(-\frac{\alpha}{1-\alpha}(\tau-t)\right) f^{(1)}(\tau) d\tau,$$

where $0 < a < t < b$.

PROPOSITION 3.3. *Suppose that $f \in AC_{loc}([a, b])$, $0 < \alpha < 1$, $0 < a < b$. Then*

$$\begin{aligned} \int_a^b f(t) {}^C_aD_t^\alpha g(t) dt &= \int_a^b g(t) {}^C_tD_b^\alpha f(t) dt \\ &+ f(b) \int_a^b g(\tau) \varphi_\alpha(b-\tau) d\tau - g(a) \int_a^b f(\tau) \varphi_\alpha(\tau) d\tau. \end{aligned} \quad (3.5)$$

The proof is easy and we omit it. Similarly, one can prove the corresponding assertion for the (RLCFFD).

3.3. Approximation formula. Next we derive an approximation formula needed for the development of an appropriate numerical procedure for solving differential equations with fractional derivatives. We follow our approach given in [4, 5] for ${}_0^C D_t^\alpha f$ and continue to consider functions in $AC_{loc}([0, \infty))$.

PROPOSITION 3.4. *Suppose that $f \in AC_{loc}([0, \infty))$ and $0 < \alpha < 1$. Then, for $\tilde{T} > 0$,*

$${}_0^C D_t^\alpha f(t) = \varphi_\alpha(t) \sum_{k=0}^{\infty} \frac{\alpha^k V_k(f^{(1)})(t)}{(1-\alpha)^k k!}, \quad t \in (0, \tilde{T}],$$

where the convergence is uniform on $[0, \tilde{T}]$ and

$$V_k(f^{(1)})(t) = \int_0^t \tau^k f^{(1)}(\tau) d\tau, \quad t \in [0, \tilde{T}], \quad (3.6)$$

are moments of $f^{(1)}$. In particular, the approximation of ${}_0^C D_t^\alpha f(t)$ may be written as

$${}_0^C \overline{D}_t^\alpha f(t) \approx \varphi_\alpha(t) \sum_{k=0}^N \frac{\alpha^k V_k(f^{(1)})(t)}{(1-\alpha)^k k!},$$

with

$${}^{CF}_0 \overline{D}_t^\alpha f(t) = {}^{CF}_0 D_t^\alpha f(t) - Q_{N+1}(t), \quad t > t_0,$$

$$\text{and } \lim_{N \rightarrow \infty} \|Q_{N+1}(t)\|_{L^\infty([0, \tilde{T}])} = 0.$$

P r o o f. We have

$$\begin{aligned} {}^{CF}_0 D_t^\alpha f(t) &= \frac{1}{1-\alpha} \exp\left(\frac{-\alpha t}{1-\alpha}\right) \int_0^t \sum_{k=0}^{\infty} \frac{\alpha^k \tau^k}{(1-\alpha)^k k!} f^{(1)}(\tau) d\tau \\ &= \frac{1}{1-\alpha} \exp\left(\frac{-\alpha t}{1-\alpha}\right) \sum_{k=0}^{\infty} \frac{\alpha^k V_k(f^{(1)})(t)}{(1-\alpha)^k k!}, \quad t \in [0, \tilde{T}]. \end{aligned} \quad (3.7)$$

Decompose (3.7) as

$${}^{CF}_0 D_t^\alpha f(t) = \frac{1}{1-\alpha} \exp\left(\frac{-\alpha t}{1-\alpha}\right) \sum_{k=0}^N \frac{\alpha^k V_k(f^{(1)})(t)}{(1-\alpha)^k k!} + Q_{N+1}(t), \quad t \in [0, \tilde{T}],$$

where $Q_{N+1}(t)$ is the remainder. We have the estimate for $Q_{N+1}(t), t \in [0, \tilde{T}]$

$$\begin{aligned} Q_{N+1}(t) &= \frac{1}{1-\alpha} \exp\left(\frac{-\alpha t}{1-\alpha}\right) \sum_{k=N+1}^{\infty} \frac{\alpha^k \int_0^t \tau^k f^{(1)}(\tau) d\tau}{(1-\alpha)^k k!} \\ &\leq \frac{1}{1-\alpha} \exp\left(\frac{-\alpha t}{1-\alpha}\right) \|f^{(1)}\|_{L^1([0, \tilde{T}])} \sum_{k=N+1}^{\infty} \frac{\alpha^k t^k}{(1-\alpha)^k (k)!}. \end{aligned}$$

Therefore, the statement of the proposition follows, with $V_k(f^{(1)})(t)$ given by (3.6). $\square$

REMARK 3.2. Using the same procedure, we obtain the approximation formula for ${}^{CF}_t D_{\tilde{T}}^\alpha f(t)$ as

$${}^{CF}_t D_{\tilde{T}}^\alpha f(t) = -\varphi_\alpha(t) \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k \tilde{V}_k(f^{(1)})(t)}{(1-\alpha)^k k!}, \quad t \in (0, \tilde{T}],$$

where the series converges in the uniform sense and

$$\tilde{V}_k(f^{(1)})(t) = \int_t^{\tilde{T}} \tau^k f^{(1)}(\tau) d\tau, \quad t \in [0, \tilde{T}].$$

We derive now the expression for derivative of ${}^{CF}_0 D_t^\alpha f(t)$ with respect to α . We state this as:

PROPOSITION 3.5. Suppose that $f \in AC_{loc}([0, \infty))$, $0 < \alpha < 1$. Then

$$\frac{\partial {}^{CF}_0 D_t^\alpha f(t)}{\partial \alpha} = \frac{1}{1-\alpha} \left[{}^{CF}_0 D_t^\alpha f(t) - \frac{1}{(1-\alpha)} \left([t(\phi_\alpha)(t)] * f^{(1)}(t) \right) \right], \quad t \in (0, \tilde{T}]. \quad (3.8)$$

P r o o f. Definition gives (1.1) we have

$$\begin{aligned} \frac{\partial {}^{CF}_0 D_t^\alpha f(t)}{\partial \alpha} &= \frac{1}{(1-\alpha)^2} {}^{CF}_0 D_t^\alpha f(t) \\ &\quad - \frac{1}{(1-\alpha)^2} \int_0^t (t-\tau) \phi_\alpha(t-\tau) f^{(1)}(\tau) d\tau, \quad t \in (0, \tilde{T}]. \end{aligned}$$

Therefore, (3.8) follows. $\square$

Note that the result equivalent to (3.8) for Riemann-Liouville fractional derivative was obtained in [7] and can be, as above, transferred for the (RLCFFD).

4. Variational principles of Hamilton type with ${}^{CF}_0 D_t^\alpha y(t)$

In this section we present the necessary conditions for an extremum in the case when a Lagrangian density contains ${}^{CF}_0 D_t^\alpha y(t)$. We start with the generalization of the classical Hamilton principle that we state as the following two problems.

4.1. Problem 1. Find necessary conditions for the existence of minimum of a functional

$$I(y) = \int_A^B L(t, y(t), {}^{CF}_a D_t^\alpha y(t)) dt, \quad \alpha \in (0, 1),$$

where y belongs to a prescribed set $\mathcal{U} \subset AC_{loc}([0, \infty))$ described below, $[A, B] \subset (a, b) \subset [0, \infty)$. In this exposition we follow our paper [6].

Let $y^* \in \mathcal{U}$ exist so that

$$\min_{y \in \mathcal{U}} I(y) = I(y^*). \quad (4.1)$$

Suppose that

$$L \in C^1((a, b) \times \mathbb{R} \times \mathbb{R}), \quad (4.2)$$

$$\begin{aligned} t \mapsto \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} &\text{ is integrable on } (a, b), \\ t \mapsto \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} &\in AC(a, b) \text{ for every } y \in AC(a, b). \end{aligned} \quad (4.3)$$

PROPOSITION 4.1. *Additionally to (4.2) and (4.3), assume that the set of admissible functions $\mathcal{U}$ is*

$$\mathcal{U} = \left\{ y : y \in AC(a, b), y(a) = y_0, \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \Big|_{t=A} = 0, \right. \\ \left. \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \Big|_{t=B} = 0 \right\}. \quad (4.4)$$

Then, for $t \in (a, B)$, y^* in (4.1) has to satisfy

$$\frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} + {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} = 0, \\ t \in (A, B) \\ {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} - {}^{CF}_t D_A^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} = 0. \\ t \in (a, A) \quad (4.5)$$

P r o o f. Suppose that (4.1) holds and let $y = y^* + \varepsilon f$. From $y(a) = y_0$, we conclude that $f(a) = 0$. Then, by the standard procedure, the condition $\frac{dI(y^* + \varepsilon f)}{d\varepsilon} \Big|_{\varepsilon=0} = 0$, gives

$$\int_A^B \left[\frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} f(t) + \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} {}^{CF}_a D_t^\alpha f(t) \right] \\ \times dt = 0. \quad (4.6)$$

Integration by parts formula (3.5), applied to the interval $[a, B]$, leads to

$$\int_a^B \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} {}^{CF}_a D_t^\alpha f(t) dt \\ = \int_a^B {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} f(t) dt \\ + \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \Big|_{t=B} \int_a^B f(\tau) \varphi_\alpha(B - \tau) d\tau \\ - f(a) \int_a^B \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \varphi_\alpha(t) dt.$$

After the use of boundary conditions defined by (4.4), we obtain

$$\begin{aligned}
& \int_A^B \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} {}^{CF}_a D_t^\alpha f(t) dt \\
&= \int_A^B {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} f(t) dt \\
&+ \int_a^A {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} f(t) dt \\
&- \int_a^A \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} {}^{CF}_a D_t^\alpha f(t) dt.
\end{aligned}$$

Integration by parts formula for the interval $[a, A]$ leads to

$$\begin{aligned}
& \int_a^A \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} {}^{CF}_a D_t^\alpha f(t) dt \\
&= \int_a^A {}^{CF}_t D_A^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} f(t) dt \\
&+ \left. \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \right|_{t=A} \int_a^A f(\tau) \varphi_\alpha(B - \tau) d\tau \\
&- f(a) \int_a^A \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \varphi_\alpha(t) dt.
\end{aligned}$$

Again, the boundary terms vanish so that

$$\begin{aligned}
& \int_A^B \left[\frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} + {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \right] f(t) dt \\
&+ \int_a^A \left[{}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} - {}^{CF}_t D_A^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \right] \\
&\times f(t) dt = 0.
\end{aligned}$$

Since f is arbitrary, the fundamental lemma of variational calculus (see [1], Lemma 3.31 and [16]) leads to (4.5). $\square$

REMARK 4.1. Let $a = A$. Then (4.5) reduces to the known equations (see [2]) and the elements of the set of admissible functions satisfy the natural boundary conditions. The interpretation of (4.5)₂ subjected to $y(a) = 0$, is that $y(t)$ represents the history of the process for $t \in (a, A)$ which contributes to the extremum of the action integral I in the interval (A, B) .

4.2. Problem 2. Here we consider the minimization problem of a functional when both y and α are independent variables in the functional. Thus, we consider

$$\min_{y \in \mathcal{U}, \alpha \in A} I(y, \alpha) = \int_0^1 L(t, y(t), {}^{CF}_0 D_t^\alpha y(t)) dt, \quad (4.7)$$

where $\mathcal{U}$ is given by (4.4) and $A = [\alpha_0, \alpha_1]$, with $0 < \alpha_0 < \alpha_1 < 1$. We assume that

$$L(t, y, {}^{CF}_0 D_t^\alpha y, \alpha) \in C^1((a, b) \times \mathbb{R} \times \mathbb{R} \times A), \quad (4.8)$$

$$\begin{aligned} t \mapsto \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} & \text{ is integrable on } (a, b), y \in AC(a, b), \alpha \in A, \\ t \mapsto \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} & \in AC(a, b), y \in AC(a, b), \alpha \in A. \end{aligned} \quad (4.9)$$

PROPOSITION 4.2. Suppose that (4.8) and (4.9) hold. If $\min_{y \in \mathcal{U}, \alpha \in A} I(y, \alpha) = I(y^*, \alpha^*)$, then y^* and α^* satisfy

$$\begin{aligned} \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} + {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} &= 0, \quad t \in (0, 1), \\ \int_0^1 \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} & \times \left[{}^{CF}_0 D_t^\alpha y(t) - \frac{1}{(1-\alpha)^2} \left(t \exp\left(\frac{-\alpha}{1-\alpha} t\right) * y^{(1)}(t) \right) \right] dt = 0. \end{aligned}$$

P r o o f. Let $y = y^* + \varepsilon_1 f$, $\alpha = \alpha^* + \varepsilon_1 \Delta\alpha$, where f and $\Delta\alpha$ are fixed. Substituting this in (4.7), we obtain

$$I(y^* + \varepsilon_1 f, \alpha^* + \varepsilon_1 \Delta\alpha) = \int_0^1 L(t, y^*(t) + \varepsilon_1 f(t), {}^{CF}_0 D_t^{\alpha^* + \varepsilon_1 \Delta\alpha} y(t)) dt.$$

Conditions $\frac{\partial I}{\partial \varepsilon_1} \Big|_{\varepsilon_1 = \varepsilon_2 = 0} = \frac{\partial I}{\partial \varepsilon_2} \Big|_{\varepsilon_1 = \varepsilon_2 = 0} = 0$ imply

$$\begin{aligned} \int_0^1 \left[\frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial y} + {}^{CF}_t D_B^\alpha \frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \right] f(t) dt &= 0, \\ \Delta\alpha \int_0^1 \left[\frac{\partial L(t, y(t), {}^{CF}_a D_t^\alpha y(t))}{\partial {}^{CF}_a D_t^\alpha y(t)} \frac{\partial {}^{CF}_0 D_t^\alpha f(t)}{\partial \alpha} \right] dt &= 0. \end{aligned} \quad (4.10)$$

By (3.8) in (4.10)₂ and the fact that f is arbitrary, we obtain the necessary conditions of the proposition. $\square$

REMARK 4.2. The application of ${}_0^C D_t^\alpha f(t)$ in Viscoelasticity for the Stress relaxation and the Creep in viscoelastic body of Hooke-Newton type, Kelvin-Voigt and Generalized Zener model will be given in our paper in preparation.

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