Research Article

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Performance of GRKM-method for solving classes of ordinary and partial differential equations of sixth-orders

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Abstract: A general quasilinear sixth-order ordinary differential equation (ODE) is an important class of ODEs. The primary objective of this study is to establish a numerical method for solving a general class of quasilinear sixth-order partial differential equations (PDEs) and ODEs. However, the Runge–Kutta method (RKM) approach for solving special classes of ODEs has been generalized as an effort to solve the general class of ODEs. Nonlinear algebraic order condition (OCs) equations have been obtained up to the tenth order using the Taylor-series expansion methodology which is used to derive the novel generalized Runge–Kutta method (GRKM). In this study, a GRKM integrator has been derived for solving a general class of quasilinear sixth-order ODEs and then this method is modified subsequently to solve a class of PDEs. Accordingly, the proposed GRKM is modified to solve a quasilinear sixth-order PDE by converting it to a system of sixth-ODEs using the method of lines. Nine problems have been implemented to prove the efficiency and accuracy of the proposed method. Simulation results of these problems showed that the proposed numerical GRKM is an accurate and efficient method. In contrast, by comparing the proposed GRKM numerical approach with the classical RK method, the numerical results demonstrate that the direct integrator outperforms the indirect classical RK method in terms of algorithm complexity and function evaluations, proving that the numerical GRKM is efficient.

Keyword: RK, RKM, GRKM, sixth-order, ordinary, partial, DEs, ODEs, PDEs, Taylor-series

1 Introduction

Differential equations (DEs) are essential for science and engineering mathematical models. The mathematical modeling of real-life have some applications of DEs, particularly different-orders of partially differential equations (PDEs) [1,2]. As for the overview of DEs' applications, DEs are utilized in a variety of engineering domains, including electronics, mechanics, control engineering, and quantum chemistry. However, real-world applications of fourth-order ordinary differential equations (ODEs) have been studied using various models in domains such as beam theory [3], fluid dynamics [4,5], ship dynamics [6], and neural networks [7]. The sudden movement of a flat surface, a subfield of engineering and physics, made use of a number of sixth-order (PDEs). The aim of the literature study is to identify analytical solutions or numerical approximations for the mathematical model that involves sixth-order ODEs with boundary value problems, as documented in previous studies [8,9]. Various numerical and analytical strategies have been explored in the literature for solving DEs of different orders. Twizell [10] developed a numerical technique to solve sixth-order ODEs and subsequently, employed finite-difference methods of orders 2, 4, 6, and 8 to solve these types of problems [11]. Previously, researchers employed approximation and computational methods to solve DEs. The numerical and analytical approaches for solving DEs of various orders sometimes struggle to directly or indirectly determine solutions for many types of equations. The researchers are driven to devise additional numerical methodologies to solve different categories of DEs due to the necessity of addressing diverse types of them. A group of academics developed various categories of numerical

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techniques of Runge–Kutta RK type for solving ODEs of different orders. For example, Cong [12] has devised numerical approaches for solving second-order ODEs that employ variable step-sizes, while other researchers [13,14] introduced the direct RKT method and the direct Runge–Kutta (RKD) method for finding the numerical solutions of third-order ODEs. These approaches utilize direct numerical techniques with a constant step-size. Moreover, Senu *et al.* [15] constructed three orders of non-constant step-size for direct integrators: 4(3), 5(4), and 6(5). For this purpose, many researchers have implemented one-step numerical integrators for solving IVPs of orders lower than ten with a constant step size in order to solve various classes of higher-order ODEs [13,16–32]. Finally, Ram and Davim [34] presented and demonstrated the various mathematical uses, techniques, strategies and techniques in engineering applications and the importance of practical applications of mathematics in engineering sciences. In order to derive or modify the numerical methods and then to improve the accuracy of the numerical methods, it is useful to construct more numerical methods for solving ODEs or PDEs.

The aim of this article is to achieve some goals, first, we seek to establish a direct generalized Runge–Kutta method (GRKM) integrator for solving the general class of quasilinear sixth-order ODEs. For this purpose, we have generalized the integrator of RKM which is used for solving a special class of sixth-order ODEs. Second, we have constructed the proposed method by combining GRKM with the method of lines (MOLs) to be consistent for solving classes of sixth-order PDEs.

2 Preliminary

In this section, some of concepts and definitions for a general quasilinear sixth-order ODEs and PDEs have been introduced as follows.

2.1 Quasilinear sixth-order ODEs

The general class of quasilinear sixth-order ODEs can be defined as follows:

$$\omega^{(6)}(\xi) = \phi(\xi, \omega(\xi), \omega'(\xi), \omega''(\xi), \omega'''(\xi), \omega^{(4)}(\xi), \omega^{(5)}(\xi)); \quad \xi \geq \xi_0. \tag{1}$$

The class one of general quasilinear ODEs of 6th-order with no explicit dependence of the derivatives $\omega^{(i)}(\xi)$, for $i \leq 5$ of second, third, fourth, and fifth orders has the following definition.

Definition 2.1. (Class one of quasilinear sixth-order ODEs) The formula of quasilinear sixth-order ODEs of class one is:

$$\omega^{(6)}(\xi) = \phi(\xi, \omega(\xi), \omega'(\xi)); \quad \xi \geq \xi_0, \tag{2}$$

with the initial conditions (ICs)

$$\omega^{(i)}(\xi_0) = \alpha^i, \tag{3}$$

where $\alpha^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_N^i]$ for $i = 2, 3, 4, 5$, $\phi : \Re \times \Re^N \to \Re^N$,

$$\omega(\xi) = [\omega_1(\xi), \omega_2(\xi), \ldots, \omega_N(\xi)]$$

and

$$\phi(\xi, \omega(\xi), \omega(\xi)) = [\phi_1(\xi, \omega_1(\xi), \omega_1'(\xi)), \phi_2(\xi, \omega_2(\xi), \omega_2'(\xi)), \ldots, \phi_N(\xi, \omega_N(\xi), \omega_N'(\xi))].$$

In this article, we have constructed numerical GRKM integrator for solving Equation (2) with ICs (3).

2.2 Quasilinear sixth-order PDEs

The quasilinear PDEs of sixth-order in the domain with two variables ζ, ς defined generally as follows:

$$\phi\left(\zeta, \varsigma, \omega(\zeta,\varsigma), \frac{\partial \omega(\zeta,\varsigma)}{\partial \varsigma}, \frac{\partial \omega(\zeta,\varsigma)}{\partial \varsigma}, \frac{\partial^2 \omega(\zeta,\varsigma)}{\partial \zeta^2}, \frac{\partial^2 \omega(\zeta,\varsigma)}{\partial \varsigma \partial \zeta}, \frac{\partial^2 \omega(\zeta,\varsigma)}{\partial \varsigma^2} \frac{\partial^3 \omega(\zeta,\varsigma)}{\partial \zeta^3}, \frac{\partial^3 \omega(\zeta,\varsigma)}{\partial \zeta^2 \partial \varsigma}, \frac{\partial^3 \omega(\zeta,\varsigma)}{\partial t \partial \varsigma^2}, \frac{\partial^3 \omega(\zeta,\varsigma)}{\partial \varsigma^3}, \ldots, \frac{\partial^6 \omega(\zeta,\varsigma)}{\partial \varsigma^6}\right) = 0. \tag{4}$$

The sixth-order PDE in domain with n-variables $s_1, s_2, \ldots, s_n$, is defined generally as follows.

Definition 2.2. Linear sixth-order PDE: The general linear sixth-order PDE has the following formula:

$$\sum_{i=1}^{n} f_i(\bar{\zeta}) \frac{\partial \omega(\bar{\zeta})}{\partial \varsigma_i} + \sum_{i_1 \leq i_2 = 1}^{n} g_{i_1,i_2}(\bar{\zeta}) \frac{\partial^2 \omega(\bar{\zeta})}{\partial \varsigma_{i_1} \partial \varsigma_{i_2}} + \cdots + \sum_{i_1 \leq i_2 \leq \cdots \leq i_6 = 1}^{n} h_{i_1,i_2,i_3}(\bar{\zeta}) \frac{\partial^6 \omega(\bar{\zeta})}{\partial \varsigma_{i_1} \partial \varsigma_{i_2} \partial \varsigma_{i_3}, \ldots, \partial \varsigma_{i_6}} = f(\bar{\zeta}), \tag{5}$$

where $\bar{\varsigma} = \varsigma_1, \varsigma_2, \ldots, \varsigma_n$

The general formula of quasilinear PDE of 6th-order in n-independent variables is given in the following definition:

Definition 2.3. The quasilinear sixth-order in $\boldsymbol{n}$-independent variables

$$\sum_{i=1}^{n} f_i(\bar{\eta}, \omega)\frac{\partial \omega(\bar{\eta})}{\partial \eta_i} + \sum_{i_1 \le i_2 = 1}^{n} g_{i_1,i_2}(\bar{\eta}, \omega)\frac{\partial^2 \omega(\bar{\eta})}{\partial \eta_{i_1}\partial \eta_{i_2}} + \cdots + \sum_{i_1 \le i_2 \le \cdots \le i_6 = 1}^{n} h_{i_1,i_2,i_3}(\bar{\eta}, \omega)\frac{\partial^6 \omega(\bar{\eta})}{\partial \eta_{i_1}\partial \eta_{i_2}\partial \eta_{i_3}, \ldots, \partial \eta_{i_6}} = f(\bar{\eta}). \tag{6}$$

The following categories are used to classify the quasilinear PDEs of sixth-order.

Definition 2.4. The quasilinear PDE of classes 1,2,3,4,5, and 6: The quasilinear PDE in n-independent-variables is defined in the following:

$$\begin{aligned} \omega_{\eta_{i_{i_1}}, \eta_{i_{j_2}}, \ldots, \eta_{i_{j_{m-1}}}, \eta_{i_{j_m}}} &= \sum_{i=1}^{n} f_i(\bar{\eta}, \omega)\frac{\partial \omega(\bar{\eta})}{\partial \eta_i} \\ &+ \sum_{i_1 \le i_2 = 1}^{n} g_{i_1,i_2}(\bar{\eta}, \omega)\frac{\partial^2 \omega(\bar{\eta})}{\partial \eta_{i_1}\partial \eta_{i_2}} + \cdots \\ &+ \sum_{i_1 \le i_2 \le i_3 \le i_4 \le i_5 \le i_6 = 1 \& \ne i_j}^{n} h_{i_1,i_2,i_3,i_4,i_5,i_6}(\bar{\eta}, \omega) \\ &\times \frac{\partial^6 \omega(\bar{\eta})}{\partial x_{i_1}\partial \eta_{i_2}\partial \eta_{i_3}\partial \eta_{i_4}\partial s_{i_5}\partial \eta_{i_6}} - f(\bar{\eta}, \omega), \end{aligned} \tag{7}$$

for $i_1, i_2, \ldots, i_6, i_{j_1}, i_{j_2}, \ldots, i_{j_m} = 1, 2, \ldots, n$. We say that Equation (7) of classes 1–5 for $m = 1, 2, \ldots, 6$ resp.

The general formula of PDE of sixth-order of class one in two-variables (ζ, η) is given as follows:

$$\begin{aligned} &\omega_{\zeta\zeta\zeta\zeta\zeta\zeta}(\zeta, \eta) \\ &= f(\zeta, \eta, \omega(\zeta, \eta), \omega_\eta(\zeta, \eta), \omega_\zeta(\zeta, \eta), \omega_{\eta\eta}(\zeta, \eta), \omega_{\zeta\eta} \\ &\quad (\zeta, \eta), \omega_{\zeta\zeta}(\zeta, \eta), \omega_{\zeta\zeta\eta}(\zeta, \eta), \\ &\quad \omega_{\eta\eta\zeta}(\zeta, \eta), \omega_{\eta\eta\eta}(\zeta, \eta), \ldots, \omega_{\eta\eta\eta\eta\eta}(\zeta, \eta)), \end{aligned} \tag{8}$$

or

$$\begin{aligned} &\omega_{\eta\eta\eta\eta\eta\eta}(\zeta, \eta) \\ &= f(\zeta, \eta, \omega(\zeta, \eta), \omega_\eta(\zeta, \eta), \omega_\zeta(\zeta, \eta), \omega_{\eta\eta}(\zeta, \eta), \omega_{\zeta\eta} \\ &\quad (\zeta, \eta), \omega_{\zeta\zeta}(\zeta, \eta), \omega_{\zeta\zeta\eta}(\zeta, \eta) \\ &\quad \omega_{\eta\eta\zeta}(\zeta, \eta), \omega_{\zeta\zeta\zeta}(\zeta, \eta), \ldots, \omega_{\zeta\zeta\zeta\zeta\zeta\zeta}(\zeta, \eta)). \end{aligned} \tag{9}$$

Nowadays, the various numerical methods are used to solve some types of PDEs in various-fields of applied mathematics, engineering, and physics. However, the solutions of these PDEs could approximate by these numerical integrators. However, using modified RKD method combining with MOL, Mechee *et al.* [20] solved the third-order quasilinear PDEs of class one.

In this article, the sixth-order PDE of class one is converted to a system of sixth-order ODEs and then, using the proposed GRKM integrator combining with MOL, we solved this system of ODEs. Based on the constructed GRKM, the numerical solutions of the implementations of sixth-order PDEs are compared with the exact solutions of these test problems; the comparisons show that the constructed integrator is highly accurate and efficient.

3 Analysis of constructed GRKM

The GRKM method is analyzed in this section for solving quasilinear sixth-order ODEs and PDEs.

3.1 Proposed GRKM for solving ODEs

The following is the form of the proposed GRKM integrator with s-stages for solving the class one of quasilinear sixth-order ODEs in Equation (2) with ICs (3):

$$\begin{aligned} z_{n+1} &= z_n + hz_n' + \frac{h^2}{2!}z_n'' + \frac{h^3}{3!}z_n^{(3)} + \frac{h^4}{4!}z_n^{(4)} + \frac{h^5}{5!}z_n^{(5)} \\ &+ h^6 \sum_{j=1}^{\$} b_j k_i, \end{aligned} \tag{10}$$

$$z_{n+1}' = z_n' + hz_n'' + \frac{h^2}{2!}z_n^{(3)} + \frac{h^3}{3!}z_n^{(4)} + \frac{h^4}{4!}z_n^{(5)} + h^5 \sum_{j=1}^{\$} b_j' k_i, \tag{11}$$

$$z_{n+1}'' = z_n'' + hz_n^{(3)} + \frac{h^2}{2!}z_n^{(4)} + \frac{h^3}{3!}z_n^{(5)} + h^4 \sum_{j=1}^{\$} b_j'' k_i, \tag{12}$$

$$z_{n+1}^{(3)} = z_n^{(3)} + hz_n^{(4)} + \frac{h^2}{2!}z_n^{(5)} + h^3 \sum_{j=1}^{\$} b_j^{(3)} k_i, \tag{13}$$

$$z_{n+1}^{(4)} = z_n^{(4)} + hz_n^{(5)} + h^2 \sum_{j=1}^{\$} b_i^{(4)} k_j, \tag{14}$$

$$z_{n+1}^{(5)} = z_n^{(5)} + h \sum_{j=1}^{\$} b_i^{(5)} k_j, \tag{15}$$

where

$$k_1 = f(x_n, z_n), \tag{16}$$

and

$$k_j = f\left(x_n + c_j h_1 z_n + c_j h z_n' + c_j^2 \frac{h^2}{2!} z_n'' + c_g^3 \frac{h^3}{3!} z_n^{(3)} + c_j^4 \frac{h^4}{4!} z_n^{(4)} + c_j^5 \frac{h^5}{5!} z_n^{(5)} + h^6 \sum_{m=1}^{j-1} a_{1,jm} k_m, z_n' + c_n h z_n'' + c_n^2 \frac{h^2}{2!} z_n^{(3)} + c_j^3 \frac{h^3}{3!} z_n^{(4)} + c_j^4 \frac{h^4}{4!} z_n^{(5)} + h^5 \sum_{l=1}^{j-1} a_{2,jm} k_m \right), \quad (17)$$

for $j = 2, 3, 4, \cdots, ş$. The coefficients of GRKM are listed as follows: $b_i^{(l)}$ and c_i, $a_{1,i,j}$, $a_{2,i,j}$ for $i, j \leq 8$ and $l = 0, 1, 2, \ldots, 5$. GRKM is explicit-method if $a_{1,i,j} = a_{2,i,j} = 0$, for $i \leq j$, else GRKM is implicit-method. The coefficients of GRKM are expressed as in Table 1 using Butcher notation.

c	A_1
	A_2
	b^T
	b'^T
	b''^T
	$b^{(3)T}$
	$b^{(4)T}$
	$b^{(5)T}$

3.1.1 Derivation of the order conditions (OCs) of GRKM

As shown by Equations (10)–(17), we have determined the OCs and then the coefficients of the proposed GRKM numerical integrator in this subsection. The Taylor-series expansion method is used to expand these equations. Implementing some algebraic adaptations, Taylor-expansions are then equivalent to the solutions suggested by the Taylor-serious expansion with the same local truncation error. We have generated the general OCs for the GRKM technique using the method for deriving OCs for the RK method which was described in [33]. We have obtained the OCs of the suggested GRKM technique as follows using Maple software:

OCs for y:

$$\sum_{i=1}^{ş} b_i = \frac{1}{720}, \sum_{i=1}^{ş} b_i c_i = \frac{1}{5{,}040}, \sum_{i=1}^{ş} b_i c_i^2 = \frac{1}{20{,}160}, \sum_{i=1}^{ş} b_i c_i^3 = \frac{1}{60{,}480}, \sum_{i=1}^{ş} b_i c_i^4 = \frac{1}{151{,}200}. \quad (18)$$

OCs for y′:

$$\sum_{i=1}^{ş} b_i' = \frac{1}{120}, \sum_{i=1}^{ş} b_i' c_i = \frac{1}{720}, \sum_{i=1}^{ş} b_i' c_i^2 = \frac{1}{2{,}520}, \sum_{i=1}^{ş} b_i' c_i^3 = \frac{1}{6{,}720}, \sum_{i=1}^{ş} b_i' c_i^4 = \frac{1}{15{,}120}, \sum_{i=1}^{ş} b_i' c_i^5 = \frac{1}{30{,}240}. \quad (19)$$

OCs for y″:

$$\sum_{i=1}^{ş} b_i'' = \frac{1}{2\sqrt{2}}, \sum_{i=1}^{ş} b_i'' c_i = \frac{1}{120}, \sum_{i=1}^{ş} b_i'' c_i^2 = \frac{1}{3{,}660}, \sum_{i=1}^{ş} b_i'' c_i^3 = \frac{1}{840}, \sum_{i=1}^{ş} b_i'' c_i^4 = \frac{1}{1{,}680}, \sum_{i=1}^{ş} b_i'' c_i^5 = \frac{1}{3{,}024}, \sum_{i=1}^{ş} b_i'' c_i^6 = \frac{1}{5{,}040}. \quad (20)$$

OCs for $y^{(3)}$:

$$\sum_{i=1}^{ş} b_i^{(3)} = \frac{1}{6}, \sum_{i=1}^{ş} b_i^{(3)} c_i = \frac{1}{2\pi}, \sum_{i=1}^{ş} b_i^{(3)} c_i^2 = \frac{1}{60}, \sum_{i=1}^{ş} b_i^{(3)} c_i^3 = \frac{1}{120}, \sum_{i=1}^{ş} b_i^{(3)} c_i^4 = \frac{1}{210}, \sum_{i=1}^{ş} b_i^{(3)} c_i^5 = \frac{1}{336}, \sum_{i=1}^{ş} b_i^{(3)} c_i^6 = \frac{1}{504}, \sum_{i=1}^{ş} b_i^{(3)} c_i^7 = \frac{1}{720}. \quad (21)$$

OCs for $y^{(4)}$:

$$\sum_{i=1}^{ş} b_i^{(4)} = \frac{1}{2}, \sum_{i=1}^{ş} b_i^{(4)} c_i = \frac{1}{6}, \sum_{i=1}^{ş} b_i^{(4)} c_i^2 = \frac{1}{12}, \sum_{i=1}^{ş} b_i^{(4)} c_i^3 = \frac{1}{20}, \sum_{i=1}^{ş} b_i^{(4)} c_i^4 = \frac{1}{30}, \sum_{i=1}^{ş} b_i^{(4)} c_i^5 = \frac{1}{42}, \sum_{i=1}^{ş} b_i^{(4)} c_i^6 = \frac{1}{56}, \sum_{i=1}^{ş} b_i^{(4)} c_i^7 = \frac{1}{72}, \sum_{i=1}^{ş} b_i^{(4)} c_i^8 = \frac{1}{90}. \quad (22)$$

Table 1: Butcher tableau of GRKM6 method

$\frac{1}{2} + \frac{\sqrt{15}}{10}$	0		
$\frac{1}{2} - \frac{\sqrt{15}}{10}$	$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
	$\frac{1}{2}$	0	0
	$-\frac{1}{2}$	$\frac{259}{320}$	
	$\frac{11}{17,280} + \frac{71\sqrt{15}}{432,000}$	$\frac{11}{17,280} - \frac{71\sqrt{15}}{432,000}$	$\frac{1}{8,640}$
	$\frac{31}{8,640} + \frac{\sqrt{15}}{1,080}$	$\frac{31}{172,808,640} - \frac{\sqrt{15}}{1,080}$	$\frac{1}{864}$
	$\frac{7}{432} + \frac{\sqrt{15}}{240}$	$\frac{7}{432} - \frac{\sqrt{15}}{240}$	$\frac{1}{108}$
	$\frac{1}{18} + \frac{\sqrt{15}}{72}$	$\frac{1}{18} - \frac{\sqrt{15}}{72}$	$\frac{1}{18}$
	$\frac{5}{15} - \frac{\sqrt{15}}{36}$	$\frac{5}{36} - \frac{\sqrt{15}}{36}$	$\frac{2}{9}$
	$\frac{5}{18}$	$\frac{5}{18}$	$\frac{4}{9}$

OCs for $y^{(5)}$:

$$\begin{gathered}\sum_{i=1}^{\$} b_i^{(5)} = 1, \sum_{i=1}^{\$} b_i^{(5)} c_i = \frac{1}{2}, \sum_{i=1}^{\$} b_i^{(5)} c_i^2 = \frac{1}{3},\\ \sum_{i=1}^{\$} b_i^{(5)} c_i^3 = \frac{1}{4}, \sum_{i=1}^{\$} b_i^{(5)} c_i^4 = \frac{1}{5}, \quad \sum_{i=1}^{\$} b_i^{(5)} c_i^5 = \frac{1}{6},\\ \sum_{i=1}^{\$} b_i^{(5)} c_i^6 = \frac{1}{7}, \sum_{i=1}^{\$} b_i^{(5)} c_i^7 = \frac{1}{8}, \sum_{i=1}^{\$} b_i^{(5)} c_i^8 = \frac{1}{9},\\ \sum_{i=1}^{\$} b_i^{(5)} c_i = \frac{1}{10}, \sum_{i=1}^{\$} \sum_{j<i}^{\$} b_i^{(5)} a2_{ij} = \frac{1}{720}.\end{gathered} \tag{23}$$

3.1.2 The proposed method's derivation

The coefficients of the proposed GRKM integrator, which are defined in the Equations (10)–(17) for solving ODE in Equation (1) with ICs (3), are obtained using Maple program for solving the OCs in Equations (18)–(23), Table 1.

3.2 Modified GRKM-method for solving PDEs

Consider the following form of quasilinear sixth-order PDEs of class one: ζ, η to

$$\begin{aligned}&\omega_{\eta\eta\eta\eta\eta\eta}(\zeta,\eta)\\ &= f(\zeta, \eta, \omega'(\zeta,\eta), \omega_\eta(\zeta,\eta), \omega_\zeta(\zeta,\eta), \omega_{\zeta\zeta}(\zeta,\eta), \omega_{\zeta\zeta\zeta}\\ &\quad (\zeta,\eta), \ldots, \omega_{\zeta\zeta\zeta\zeta\zeta\zeta}(\zeta,\eta)); a < \zeta < b; 0 < \eta \le T,\end{aligned} \tag{24}$$

with ICs

$$\begin{aligned}\omega_{\zeta\zeta\zeta\zeta\zeta}(\zeta,0) &= f_6(\zeta), \omega_{\zeta\zeta\zeta}(\zeta,0) = f_5(\zeta), \omega_{\zeta\zeta\zeta}(\zeta,0)\\ &= f_4(\zeta)\\ \omega_{\zeta\zeta}(\zeta,0) &= f_3(\zeta), \omega_\zeta(\zeta,0) = f_2(\zeta), \omega(\zeta,0)\\ &= f_1(\zeta), \quad a \le \zeta \le b,\end{aligned} \tag{25}$$

with the boundary conditions (BCs),

$$\omega(a,\eta) = g_1(\eta), \omega(b,\eta) = g_2(\eta); \quad \eta > 0. \tag{26}$$

We present an established approach for solving class one PDEs using the GRKM and MOL that is consistent with the following algorithm.

3.3 Algorithm of modified GRKM

Consider two intervals of the domain definition in two directions of ζ and η, which are named as $[a, b]$ and $[0, T]$ with the norms of the subintervals are $h = \frac{b-a}{n}$ and $k = \frac{T}{m}$ respectively. Here, n and m are the number of partitions of intervals in the directions of ζ, η respectively, where $zeta_i = a + ih$, and $\eta_j = jk$, for $j = 1, 2, \ldots, m$. and $i = 1, 2, \ldots, n-1$. We can combine GRKM with MOL method to solve Equation (24) with ICs in Equation (25) and BCs in Equation (26) according the following-steps:

1. Apply the steps (2)–(6) while $1 \le k \le$ mm
2. Fix $\zeta = \zeta_l$ at the point (ζ, η) of the PDE in Equation (24) in which convert to the following system of (DEs):

$$\begin{aligned}\omega_l^{(6)}(\eta) = f\Bigg(&\zeta, \eta, \omega'(\eta), \omega(\zeta,\eta), \frac{\partial \omega(\zeta,\eta)}{\partial s}, \frac{\partial^2 \omega(\zeta,\eta)}{\partial \zeta^2}.\\ &, \frac{\partial^3 \omega(\zeta,\eta)}{\partial \zeta^3}, \frac{\partial^4 \omega(\zeta,\eta)}{\partial \zeta^4},\\ &\frac{\partial^5 \omega(\zeta,\eta)}{\partial \zeta^5}, \frac{\partial^6 \omega(\zeta,\eta)}{\partial \zeta^6}\Bigg)\Bigg|_{\zeta=\zeta_l}\end{aligned} \tag{27}$$

3. Substituting the formulas of central finite-difference in the derivative function $\omega(\zeta, \eta)$ in the right-hand side of Equation (27) up to order six as follows:

$$\begin{aligned}\frac{\partial \omega(\zeta,\eta)}{\partial \zeta}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+1,j} - \omega_{l-1,j}}{2h},\\ \frac{\partial^2 \omega(\zeta,\eta)}{\partial \zeta^2}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+1,j} - 2\omega_{l,j} + \omega_{l-1,j}}{h^2},\\ \frac{\partial^3 \omega(\zeta,\eta)}{\partial \zeta^3}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+2,j} - 2\omega_{l+1,j} + 2\omega_{l-1,j} - \omega_{l-2,j}}{h^3},\\ \frac{\partial^4 \omega(\zeta,\eta)}{\partial \zeta^4}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+2,j} - 4\omega_{l+1,j} + 6\omega_{i,j} - 4\omega_{i-1,j} + \omega_{i-2,j}}{2h^3},\\ \frac{\partial^5 \omega(\zeta,\eta)}{\partial \zeta^5}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+3,j} - 4\omega_{i+2,j} + 5\omega_{l+1,j} - 5\omega_{l-1,j} + 4\omega_{l-2,j} - \omega_{l+3,j}}{2h^5},\\ \frac{\partial^6 \omega(\zeta,\eta)}{\partial \zeta^6}\Bigg|_{(\zeta,\eta)=(\zeta_i,\eta_j)} &\cong \frac{\omega_{l+3,j} - 6\omega_{l+2,j} + 15\omega_{l+1,j} - 20\omega_{l,j} + 15\omega_{l-1,j} - 6\omega_{l-2,j} + \omega_{l+3,j}}{h^6}.\end{aligned}$$

Hence, we obtain a system of sixth-order ODEs as follows:

$$\begin{aligned}&\psi_l^{(6)}(t)\\ &= f(\eta_l, \eta, \psi_{i-3}(\eta), \psi_{l-2}(\eta), \psi_{l-1}(\eta), \psi_l(\eta), \psi_l'(\eta), \psi_{l+1}\\ &\quad (\eta), \psi_{l+2}(\eta), \psi_{l+3}(\eta)),\end{aligned} \tag{28}$$

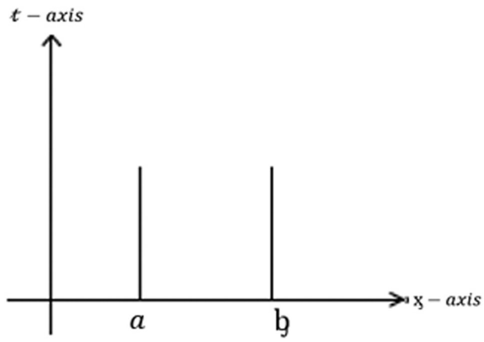

Figure 1: The domain of PDE of class one.

for $l = 1, 2, \dots, n - 1$.

4. If $j = 1$ then, ICs have the formula:

$$\omega_i^{(k)}(0) = f_{k+1}(\zeta_i). \quad (29)$$

For $2 = <j < =m$, the ICs are,

$$\omega_i^{(k)}(\eta_{j-1}) = \left. \frac{d^k \omega(\zeta, \eta_{j-1})}{d\zeta^k} \right|_{\zeta=\zeta_i} = f_k(\zeta_i), \quad (30)$$

for $k = 0, 1, 2, 3, 4, 5$.

5. Put the BCs using the following notation,

$$\omega_{0,j} = \omega(a, \eta_j) = g_1(\eta_j), \omega_{n,j} = \omega(b, \eta_j) = g_2(\eta_j). \quad (31)$$

6. Finally, the system of ODEs in Equation (27) at the line $\eta = \eta_j$ with ICs (28) and (29) and BCs (30) can be solved using the GRKM, Figure 1, The class one of quasilinear sixth-order PDE in Equation (24) with the ICs in Equation (25) and the BCs in Equation (26) in the region of definition, which are shown in Figure 1, could be solved using this algorithm.

4 Implementations

We can examine the constructed method for solving some ODEs and PDEs problems in this section.

4.1 Implementation of ODEs

This subsection investigates the proposed GRKM by studying the numerical solutions of several problems and then, their numerical results are presented in Figure 2.

Example 4.1. Homogenous in ODE

$$\omega^{(6)}(\xi) = -\omega(\xi); \quad 0 < \xi \leq b.$$

ICs

$$\omega(0) = 0, \quad \omega'(0) = 1, \quad \omega^{'}(0) = 0, \quad \omega^{(3)}(0) = -1, \quad \omega^{(4)}(0) = 0, \quad \omega^{(5)}(0) = 1.$$

The exact solution is $\omega(\xi) = \sin(\xi), \quad b = 1$.

Example 4.2. Linear ODE

$$\omega^{(6)}(\xi) = \omega'(\xi) + 2\omega(\xi), \quad 0 < \xi \leq b.$$

ICs

$$\omega^{(j)}(0) = (-1)^j; i = 1, 2, 3, 4, 5.$$

The exact solution is $\omega(\xi) = e^{-\xi}, \quad b = 1$.

Example 4.3. Non-homogenous *ODE*

$$\omega^{(6)}(\xi) = (-120 + 720\xi^2 - 480\xi^4 + 64\xi^6)\omega(\xi),$$

$0 < \xi \leq b$.

ICs

$$\omega^{(i)}(0) = 0; i = 1, 2, 3, 4, 5; \omega(0) = 1.$$

The exact solution is $\omega(\xi) = e^{-\xi^2}, \quad b = 1$.

Example 4.4. Homogenous *ODE*

$$\omega^{(6)}(\xi) = -\omega(\xi) + \omega'(\xi) - \cos(\xi);, \quad 0 < \xi \leq b.$$

ICs

$$\omega^{(2j)}(0) = 0; \omega^{(2j-1)}(0) = (-1)^j.$$

The exact solution is $\omega(\xi) = \sin(\xi), \quad b = \pi$.

Example 4.5. Nonlinear ODE

$$\omega^{(6)}(\xi) = \omega^3(\xi) - 121\omega^6(\xi), \quad 0 < \xi \leq b.$$

ICs

$$\omega^{(j)}(0) = (-1)j!; j = 0, 1, 2, 3, 4, 5.$$

The exact solution is $\omega(\xi) = \frac{1}{1+\xi}, \quad b = 1$.

Example 4.6. Linear system ODEs

$$\omega_1^{(6)}(\xi) = 666\omega_1(\xi) + 602\omega_2(\xi) + 665\omega_3(\xi),$$
$$\omega_2^{(6)}(\xi) = -665\omega_1(\xi) - 601\omega_2(\xi) - 665\omega_3(\xi),$$
$$\omega_3^{(6)}(\xi) = 728\omega_1(\xi) + 728\omega_2(\xi) + 729\omega_3(\xi).$$

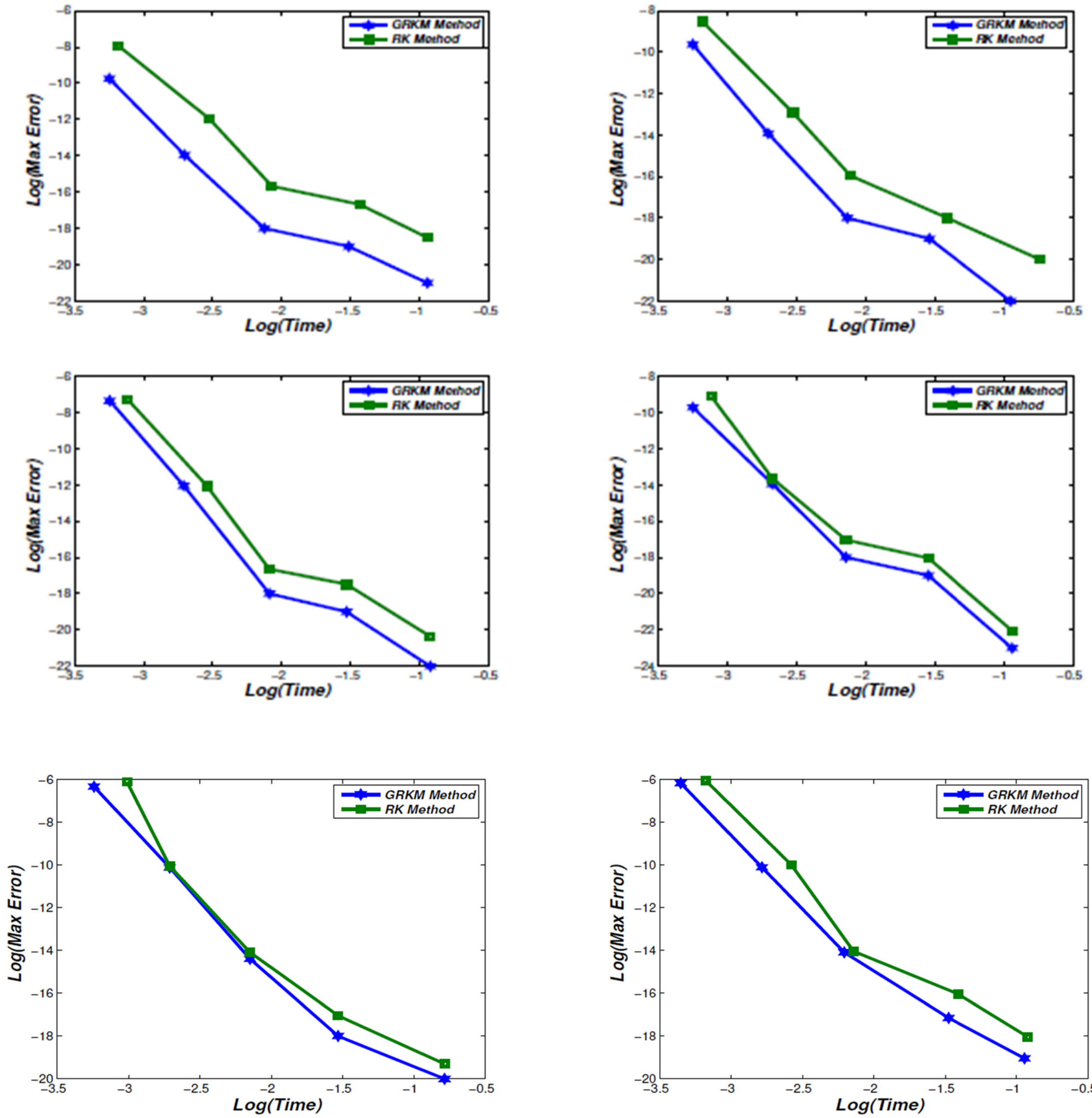

Figure 2: The efficiency curves of GRKM and their comparisons for graphs of Log10(Absolute Errors) against Log(Computational Time) of numerical solutions for Examples 4.1–4.6.

ICs

$$\omega_1(0) = 1, \quad \omega_1'(0) = -2, \quad \omega_1''(0) = 6,$$
$$\omega_1'''(0) = -20, \quad \omega_1^{(4)}(0) = 66, \quad \omega_1^{(5)}(0) = 274,$$
$$\omega_2(0) = 0, \quad \omega_2'(0) = 1, \quad \omega_2''(0) = -5,$$
$$\omega_2'''(0) = 19, \quad \omega_2^{(4)}(0) = 65, \quad \omega_2^{(5)}(0) = 211,$$
$$\omega_3(0) = 0, \quad \omega_3'(0) = -2, \quad \omega_3''(0) = 8,$$
$$\omega_3'''(0) = -26, \quad \omega_3^{(4)}(0) = 80, \quad \omega_3^{(5)}(0) = -242.$$

The exact solution for this system in [0, 1] is:

$$\omega_1(s) = e^{-3\xi} - e^{-2\xi} + e^{-\xi},$$
$$\omega_2(s) = -e^{-3\xi} + e^{-2\xi},$$
$$\omega_3(s) = e^{-3\xi} - e^{-\xi}.$$

4.2 Implementation of PDEs

In this subsection, the developed method is evaluated by using MATLAB to solve several examples of type I, quasilinear, and sixth-order PDEs. We performed a simulated comparison between the exact and numerical solutions of the implementations in Tables 3 and 4.

Example 4.7. Homogenous

Consider

$$\mathbf{f}_{tttttt}(x,t) + \mathbf{f}_{xx}(x,t) + 32\mathbf{f}_{t}(x,t) + \mathbf{f}(x,t) = 0,$$
$$a \le x \le b, \quad t > 0.$$

ICs

$$f(x,0) = \cos x,\ f_x(x,0) = -\sin x, \quad f_{xx}(x,0) = -\cos x,$$
$$f_{xxx}(x,0) = \sin x, \qquad f_{xxxx}(x,0) = \cos x,$$
$$f_{xxxxx}(x,0) = -\sin x.$$

BCs

$$f(a,t) = e^{-2t}\cos a, \quad f(b,t) = e^{-2t}\cos b.$$

The exact solution is $f(x,t) = e^{-2\pi}\cos x$, $a = 0, b = 1$, (Table 2).

Example 4.8. Homogenous

Consider

$$f_{tttttt}(x,t) + f_{xxxxx}(x,t) + f_{xxxx}(x,t) + f_{xxx}(x,t) + f_{xx}(x,t)$$
$$+f_x(x,t) + f_t(x,t) + f(x,t) = 0, \quad a \le x \le b, \quad t > 0.$$

ICs

$$f(x,0) = e^{-x}, \quad f_x(x,0) = -e^{-x}, \quad f_{xx}(x,0) = e^{-x}, \qquad f_{xxx}(x,0) = -e^{-x},$$
$$f_{xxxx}(x,0) = e^{-x}, \quad f_{xxxxx}(x,0) = -e^{-x}.$$

BCs,

$$f(a,t) = e^{-a}e^{t},\ f(b,t) = e^{-b}e^{t}.$$

Table 3: Comparison between numerical and exact solutions of GRKM6 method for example 4.8, $a = 0, b = 1$

Time (t_j)	x_1	Numerical solution	Absolute error
10^{-3}	0.2	0.810584245970172	0.000000000000015
	0.4	0.663650250136179	0.000000000000140
	0.6	0.543350869073371	0.000000000001128
	0.8	0.444858066211603	0.000000000011338
2×10^{-3}	0.2	0.802518797862978	0.000000000099500
	0.4	0.657046819346977	0.000000000468080
	0.6	0.537943574317410	0.000000863277265
	0.8	0.440428525520124	0.000003128983875
3×10^{-3}	0.2	0.794533579412687	0.000000023090647
	0.4	0.650508584521286	0.000000510202030
	0.6	0.532589799908525	0.000002001098373
	0.8	0.436037992442788	0.000011293878747
4×10^{-3}	0.2	0.786627629842670	0.000000231223884
	0.4	0.644031320063111	0.000005101020030
	0.6	0.527263950569386	0.000028473473663
	0.8	0.431598124040692	0.000112399388387

Table 2: Numerical comparison between numerical and exact solutions of GRKM6 method for example 4.7. $a = 0, b = 1$

Time (t_j)	x_i	Numerical solution	Absolute error
10^{-3}	0.2	0.960659959352262	0.000000000000011
	0.4	0.902822764356089	0.000000000000123
	0.6	0.808992874766153	0.000000000000322
	0.8	0.682910992174669	0.000000000027171
2×10^{-3}	0.2	0.941637617556523	0.000000000064710
	0.4	0.884945675385033	0.000000000563747
	0.6	0.792972879284243	0.000000637728737
	0.8	0.669385319543449	0.000001274783921
3×10^{-3}	0.2	0.922991920471520	0.000000046727811
	0.4	0.867422067217741	0.000000385885201
	0.6	0.777269809327510	0.000002000023101
	0.8	0.656122375294602	0.000046772782891
4×10^{-3}	0.2	0.904715247328577	0.000000534772782
	0.4	0.850241358563156	0.000001020101012
	0.6	0.761852323904527	0.000012948457387
	0.8	0.643028952647224	0.000846278289912

Table 4: Numerical comparison between numerical and exact solutions of GRKM6 method for example 4.9, $a = 0, b = 1$

Time (t_j)	x_i	Numerical-Solution	Absolute-Error
10^{-3}	0.2	0.931060827337037	0.000000000000022
	0.4	0.706706542681192	0.000000000000243
	0.6	0.372357587809712	0.000000000002122
	0.8	−0.019199688978460	0.000000000011454
2×10^{-3}	0.2	0.941059660596718	0.000000000083636
	0.4	0.716705375572418	0.000000000456757
	0.6	0.382355557892742	0.000000737663737
	0.8	−0.009203984591831	0.000003128983875
3×10^{-3}	0.2	0.951056471114733	0.00000003424355
	0.4	0.726701699347631	0.000000637272788
	0.6	0.392351253580796	0.000003545315661
	0.8	0.000784684022459	0.000034255616111
4×10^{-3}	0.2	0.961050096965635	0.000000534636277
	0.4	0.736690942513769	0.000002553564477
	0.6	0.402318615189645	0.000012662663778
	0.8	0.010677412496958	0.000212178237873

The exact solution is $f(x, \ell) = e^{\ell} e^{-x}$, $a = 0, b = 1$, (Table 3).

Example 4.9. Non-homogenous

Consider

$$f_{\ell\ell\ell\ell\ell\ell}(x, \ell) + f_{xxxxxx}(x, \ell) - 16 f_{xx}(x, \ell) + f_{\ell}(x, \ell) + f(x, \ell) = \cos(2x) + \cos(\ell),$$

$$a \le x \le b, \quad \ell > 0.$$

ICs

$$f(x, 0) = \cos(2x), \quad f_x(x, 0) = -2\sin(2x),$$
$$f_{xx}(x, 0) = -4\cos(2x),$$

$$\boldsymbol{f}_{xxx}(x, 0) = 8\sin(2x), \quad \boldsymbol{f}_{xxxx}(x, 0) = 16\cos(2x),$$
$$\boldsymbol{f}_{xxxxx}(x, 0) = -32\sin(2x).$$

BCs

$$\boldsymbol{f}(a, \ell) = \cos(2a) + \sin(2\ell), \quad \boldsymbol{f}(b, \ell) = \cos(2b) + \sin(2\ell).$$

The exact solution is $\boldsymbol{f}(x, \ell) = \cos(2x) + \sin(2\ell)$. $a = 0$, $b = 1$, (Table 4).

5 Discussion and conclusion

In this study, we have derived a direct numerical approach GRKM for solving the general class of quasilinear sixth-order ODEs. The aims of this article are first, to establish a direct explicit integrator for solving this general class of sixth-order ODEs and second, to combine the proposed method with the MOL method to be consistent with solving a class of quasilinear, sixth-order PDEs. Also, we have studied the efficiency of the proposed GRKM by using different examples of quasilinear, sixth-order ODEs in addition to the PDEs of the same order. The numerical results of ODEs and PDEs which are introduced in Tables 2–4 and Figure 2 proved that the solutions of the proposed method are identical to the analytical solutions. However, from the numerical results, which were obtained by the GRKM, we can conclude that GRKM is more efficient than the classical method in terms of computational time and absolute error. Numerical results obtained using the constructed GRKM have been compared with the numerical solutions of the classical RK method for solving ODEs problems in like manner, as well as they have been compared with analytical solutions of PDEs problems. As a result, the proposed method is more efficient and accurate than the indirect methods, due to the numerical results of the implementation requiring fewer function evaluations and function calls. Finally, the constructed GRKM is more cost-effective in terms of computational time, than the existing methods.

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Data availability statement: The most datasets generated and/or analysed in this study are comprised in this submitted manuscript. The other datasets are available on reasonable request from the corresponding author with the attached information.

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