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Research Article

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Generalized RKM methods for solving fifth-order quasi-linear fractional partial differential equation

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Abstract: Fractional differential equations (FDEs) are used for modeling the natural phenomena and interpretation of many life problems in the fields of applied science and engineering. The mathematical models which include different types of differential equations are used in some fields of applied sciences like biology, diffusion, electronic circuits, damping laws, fluid mechanics, and many others. The derivation of modern analytical or numerical methods for solving FDEs is a significant problem. However, in this article, we introduce a novel approach to generalize Runge Kutta Mechee (RKM) method for solving a class of fifth-order fractional partial differential equations (FPDEs) by combining numerical RKM techniques with the method of lines. We have applied the developed approach to solve some problems involving fifth-order FPDEs, and then, the numerical and analytical solutions for these problems have been compared. The comparisons in the implementations have proved the efficiency and accuracy of the developed RKM method.

Keywords: RKM method, system of fifth-order ODEs, method of lines, fifth-order PDEs

1 Introduction

We define the class of quasi-linear fractional partial differential equations (FPDEs) of fifth order as follows:

$$\begin{aligned} D_{\ell}^{5\alpha} z(x,\ell) &= f(x,\ell,z(x,\ell),z_x(x,\ell),z_{xx}(x,\ell),z_{xxx}(x,\ell),z_{xxxx}(x,\ell),z_{xxxxx}(x,\ell)),\\ &\qquad a \le x \le b,\ 0 < \ell \le T, \end{aligned} \tag{1}$$

with the initial conditions (ICs):

$$\begin{aligned} &z(x,0) = f_1(x), \quad z_x(x,0) = f_2(x),\\ &z_{xx}(x,0) = f_3(x), \quad z_{xxx}(x,0) = f_4(x),\\ &z_{xxxx}(x,0) = f_5(x), \end{aligned} \tag{2}$$

and the boundary conditions (BCs):

$$z(a,\ell) = \varphi_1(\ell), \quad z(b,\ell) = \varphi_2(\ell), \tag{3}$$

where $0 < \alpha \le 1$, f, f_i, and $\varphi_j(\ell)$ are given functions, $i = 1,\ldots,5, j = 1,2$.

The sense of fractional derivative in equation (1) can be Caputo fractional derivative or Riemann–Liouville fractional derivative.

Generally, mathematical modeling of real-life situations yields fractional differential equations (FDEs) using significant tools like special functions of mathematical physics and their expansions and generalizations in one or more variables. Many other models, such as those of fluid dynamics, quantum physics, electricity, ecological systems, and so on, rely on fractional-order PDEs to govern the vast majority of the underlying physical processes, as a consequence, it becomes essential to be familiar with all established and applications of new methods for solving PDEs of fractional order. A lot of these models and issues are unsolvable [1]. Thus, many researchers tried to study and create new numerical algorithms, for example, in 2017, Fu and Wang created finite difference technique by exploiting the scheme's underlying mathematical structure [2]; in 2018, Ara et al. and Yavuz et al. used wavelet optimization and Laplace perturbation theory to solve FPDEs, respectively [3,4]. As well as, some authors in 2019 conducted numerical research by using finite difference and Galerkin finite element techniques on three different types of FPDEs [5]. Also, Modanli [6] introduced differentiation methods for the third-order FPDE and the difference schemes have been analyzed for their stability. Zhang in 2020

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studied the numerical solution of the second problem and considered FPDE [7]. Moreover, Milici et al. studied the fractional-order dynamical systems using the Euler and fourth-order Runge–Kutta techniques [8]. As well as, Yuttanan et al. in 2021 suggested a numerical method for solving FPDEs [9]; furthermore, Burqan et al. in 2023 modified and developed the methods for solving FPDEs of different types [10–12]. Finally, Zada and Aziz in 2022 used Haar wavelet collocation technique to solve the partial differential equations with fractional order numerically [13]. In contrast, Kilbas et al. [14] and Ishteva [15] studied the theory and applications of FDEs. Finally, Mechee and Senu [16,17] solved fractional ordinary differential equations (FODEs) using least square and collocation methods, respectively, while Arshad et al. [18] investigated numerical solutions of first-order FODEs using the proposed Euler method and derived a two-stage fractional Runge–Kutta approach, whereas in 2015, Gurung et al. and Goyal et al. used numerical techniques to solve mathematical models [19,20]. This article is organized as follows: the first part contains the introductory information. In part 2, the proposed RKM method is given in specifics, while, in Section 3, the Runge Kutta Mechee (RKM) method is modified to be suitable for solving FPDEs. Finally, to prove the reliability and efficiency of the proposed method, we conduct numerical experiments.

2 Preliminaries and notations

In this section, we present the fundamental definitions and background concepts of the main issue of this study.

Definition 1. [21]

The Caputo fractional partial derivative operator with order $\alpha > 0$ is defined as

$$D_{\ell}^{\alpha} f(x,\ell) = \frac{\partial^{\alpha} f(x,\ell)}{\partial \ell^{\alpha}} = \begin{cases} \dfrac{\partial^{n} f(x,\ell)}{\partial \ell^{n}}, & \alpha = n, n \in N, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{0}^{\ell} \frac{\frac{\partial^{n} f(x,\ell)}{\partial \ell^{n}}}{(\ell-\tau)^{\alpha-n+1}} \mathrm{d}\tau, & 0 \leq n-1 < \alpha < n, \end{cases} \tag{4}$$

where n is an integer, $\ell > 0$.

Definition 2. [21]

For a function $f(x,\ell)$, the partial Riemann–Liouville fractional derivative of order $\alpha > 0$ with respect to ℓ is defined as:

$$\begin{aligned} D_{\ell}^{\alpha} f(x,\ell) &= \frac{\partial^{\alpha} f(x,\ell)}{\partial \ell^{\alpha}} \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^{n}}{\partial t^{n}} \int_{0}^{\ell} \frac{f(x,S)}{(\ell \quad S)^{n-\alpha-1}} \mathrm{d}S, \\ & n-1 < \alpha < n \in \mathbb{N}. \end{aligned} \tag{5}$$

Corollary 1. [21]

Let $m-1 < \mathcal{P} < m, m \in N, \mathcal{P} \in R$, and $f(t)$ be such that $D_{\ell}^{\mathcal{P}} f(\ell)$ exists, then the following properties for the Caputo operator hold:

a. $\lim_{\mathcal{P} \to m} D_{\ell}^{\mathcal{P}} f(\ell) = f^{(m)}(\ell)$.
b. $\lim_{\mathcal{P} \to m-1} D_{\ell}^{\mathcal{P}} f(\ell) = f^{(m-1)}(\ell) - f^{(m-1)}(0)$.
c. $D_{\ell}^{\mathcal{P}}(\alpha f(\ell) + \beta g(\ell)) = \alpha D_{\ell}^{\mathcal{P}} f(\ell) + \beta D_{\ell}^{\mathcal{P}} g(\ell)$.
d. $D_{\ell}^{\mathrm{P}} D_{\ell}^{q} f(\ell) \neq D_{\ell}^{\mathrm{P}+q} f(\ell) \neq D_{\ell}^{q} D_{\ell}^{\mathrm{P}} f(\ell), p \neq q$.
e. $D_{t}{}^{0} f(\ell) = f(\ell)$.

Definition 3. (The General Quasi-Linear FPDEs)

If all higher-order derivatives of dependent variables are linear, then the partial differential equation is said to be quasilinear. The following form describes the class of quasi-linear FPDEs:

$$\begin{aligned} D_{\ell(n-\text{times})}^{n\alpha} z(x,\ell) &= f(x,\ell,z(x,\ell),z_{x}(x,\ell),z_{xx}(x,\ell),\ldots \\ & ,z_{x(n-\text{times})}(x,\ell)), \\ & a \leq x \leq b, 0 < \ell \leq T, \end{aligned} \tag{6}$$

with ICs:

$$z_{x(i-\text{times})}(x,0) = f_{i}(x), \quad i = 0,1,2,\ldots,n-1, \tag{7}$$

and the BCs:

$$z(a,\ell) = \varphi_{1}(\ell), \quad z(b,\ell) = \varphi_{2}(\ell), \tag{8}$$

where $0 < \alpha \leq 1$.

2.1 Finite difference method [22]

The finite differences are widely used in numerical analysis to approximate derivatives of different orders in order to solve differential equations. Euler (1707–1783) established it in one dimension in 1768, while C. Runge

(1856–1927) likely developed it to two dimensions around 1908. Over the past 50 years, theoretical conclusions have been reached on the precision, convergence, and stability of the finite difference technique for PDEs. Also, FDM has been developed for solving fractional equations and the finite difference system as follows:

$$\left.\frac{\partial^n z(x,\ell)}{\partial \ell^n}\right|_{(x,\ell)=(x_i,\ell_j)} = f\left(x,\ell,z(x,\ell),\frac{\partial z(x,\ell)}{\partial x},\frac{\partial^2 z(x,\ell)}{\partial x^2},\ldots,\frac{\partial^n z(x,\ell)}{\partial x^n}\right)\Bigg|_{(x,\ell)=(x_i,\ell_j)}, \tag{9}$$

$i = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$,where N and M are the number of subintervals of domain x and ℓ, respectively.

3 Proposed RKM methods for solving fifth-order quasi-linear FPDEs

For solving fifth-order quasi-linear FPDEs, the RKM method developed by combining method of lines (MOL) with numerical RKM method.

3.1 Direct numerical RKM method [23]

Consider the following fifth-order ODE:

$$z^{(5)}(x) = f(x,z),\ x \ge x_0, \tag{10}$$

with ICs

$$z^{(i)}(x_0) = \gamma^i,\ i = 0,1,\ldots,4. \tag{11}$$

Let

$$w_n \cong z(x_n),\ n = 1,2,\ldots.$$

The general form of the RKM method with s-stage for initial value problems (10) and (11) is as follows:

$$w_{n+1} = w_n + hw'_n + \frac{h^2}{2!}w''_n + \frac{h^3}{3!}w_n^{(3)} + \frac{h^4}{4!}w_n^{(4)} + h^5\sum_{i=1}^{s} b_i \mathrm{K}_i, \tag{12}$$

$$w'_{n+1} = w'_n + hw''_n + \frac{h^2}{2!}w_n^{(3)} + \frac{h^3}{3!}w_n^{(4)} + h^4\sum_{i=1}^{s} b'_i \mathrm{K}_i, \tag{13}$$

$$w''_{n+1} = w''_n + hw_n^{(3)} + \frac{h^2}{2!}w_n^{(4)} + h^3\sum_{i=1}^{s} b''_i \mathrm{K}_i, \tag{14}$$

$$w_{n+1}^{(3)} = w_n^{(3)} + hw_n^{(4)} + h^2\sum_{i=1}^{s} b_i^{(3)} \mathrm{K}_i, \tag{15}$$

$$w_{n+1}^{(4)} = w_n^{(4)} + h\sum_{i=1}^{s} b_i^{(4)} \mathrm{K}_i, \tag{16}$$

where

$$\mathrm{K}_1 = f(\ell_n, w_n), \tag{17}$$

$$\mathrm{K}_i = f\left(\ell_n + C_i h,\ w_n + hC_i w'_n + \frac{h^2}{2!}C_i^2 w''_n + \frac{h^3}{3!}C_i^3 w_n^{(3)} + \frac{h^4}{4!}C_i^4 w_n^{(4)} + h^5\sum_{j=1}^{i-1} a_{ij}\mathrm{K}_j\right), \tag{18}$$

for I = 2, 3,…, s. The RKM method assumes the following real values for the parameters $c_i, b_i, b'_i, b_i^{(2)}, b_i^{(3)}, b_i^{(4)}$ for $i, j = 1, 2, \ldots, s$. It is an explicit method if and only if $a_{ij} = 0$ for $i \le j$, and an implicit method otherwise. Tables 1 and 2 describe the RKM method in Butcher notation.

3.2 Algorithm of the proposed method

In this subsection, we introduce a computational strategy for solving fifth-order quasi linear FPDEs (1)–(3) using a hybrid of the MOL and the RKM method.

Specifically, we presume that the x-axis interval $[a,b]$ and the t-axis interval $[0,T]$ correspond to the numerical solution, with $h = (b-a)/N$ and $k = T/M$, where N and M are the number of points in the x-direction in $[a, b]$ and t-direction in $[0, T]$, respectively. Combining the MOL and the RKM method, we can solve problem (1) with given initial and BCs (2) and (3) by the following algorithm.

Algorithm:

1. While $1 \le i \le N$, carry out steps 2–6.

Table 1: Butcher tableau for RKM technique

C	A
	b^T
	b'^T
	b''^T
	$b^{(3)T}$
	$b^{(4)T}$

Note: $A = (a_{ij})$, $C = (c_i)$, $b^T = (b_i)$, $b'^T = (b'_i)$, $b''^T = (b''_i)$, $b^{(3)T} = (b_i^{(3)})$, $b^{(4)T} = (b_i^{(4)})$.

Table 2: RKM method (three-stage, fifth-order)

0	0		
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	0	
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	1	0	$\frac{-119}{120}$
		$-\frac{1}{40}-\frac{\sqrt{6}}{360}$	
		$\frac{1}{60}+\frac{\sqrt{6}}{360}$	0
	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{6}}{48}$	$\frac{1}{18}+\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$

2. The FPDE (1) can be transformed into the following equation by fixing the value of $x = x_i$ at the point (x, ℓ), where $x_i = a + i\mathrm{h}$.

$$z_i^{(5\alpha)}(\ell) = f\left(x, \ell, z(x,\ell), \frac{\partial z(x,\ell)}{\partial x}, \frac{\partial^2 z(x,\ell)}{\partial x^2}, \frac{\partial^3 z(x,\ell)}{\partial x^3}, \frac{\partial^4 z(x,\ell)}{\partial x^4}, \frac{\partial^5 z(x,\ell)}{\partial x^5}\right)\Bigg|_{x=x_i} \tag{19}$$

where

$$z_i^{(5\alpha)}(\ell) = D_x^{5\alpha} z(x,t)|_{x=x_i}, \quad i = 1, 2, \ldots, N. \tag{20}$$

3. The derivatives on the right-hand side of ODE (19), when substituted with finite difference formulas of orders 1, ...,5, yield a system of FODEs at the fifth order.

$$\begin{aligned} z_i^{(5\alpha)}(\ell) &= f(x_i, \ell, z_{i-3}(\ell), z_{i-2}(\ell), z_{i-1}(\ell), z_i(\ell), z_{i+1}(\ell), \\ &\quad z_{i+2}(\ell), z_{i+3}(\ell)), \end{aligned} \tag{21}$$

for $i = 1, 2, \ldots, N$.

$$\frac{\partial z(x,\ell)}{\partial x}\Bigg|_{(x,\ell)=(x_i,\ell_j)} \cong \frac{z_{i+1,j} - z_{i-1,j}}{2h}, \tag{22}$$

$$\frac{\partial^2 z(x,\ell)}{\partial x^2}\Bigg|_{(x,\ell)=(x_i,\ell_j)} \cong \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2}, \tag{23}$$

$$\frac{\partial^3 z(x,\ell)}{\partial x^3}\Bigg|_{(x,\ell)=(x_i,\ell_j)} \cong \frac{z_{i+2,j} - 2z_{i+1,j} + 2z_{i-1,j} + z_{i-2,j}}{2h^3}, \tag{24}$$

$$\begin{aligned} &\frac{\partial^4 z(x,\ell)}{\partial x^4}\Bigg|_{(x,\ell)=(x_i,\ell_j)} \\ &\cong \frac{z_{i+2,j} - 4z_{i+1,j} + 6z_{i,j} - 4z_{i-1,j} + z_{i-2,j}}{h^4}, \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{\partial^5 z(x,\ell)}{\partial x^5}\Bigg|_{(x,\ell)=(x_i,\ell_j)} \\ &\cong \frac{-z_{i+3,j} + 2z_{i+2,j} - 5z_{i+1,j} + 5z_{i-1,j} - 2z_{i-2,j} + z_{i-3,j}}{2h^4}. \end{aligned} \tag{26}$$

4. Starting conditions (with $j = 1$)

$$\begin{aligned} z_i(0) &= f_1(x_i), \quad z_i'(0) = f_2(x_i), \\ z_i''(0) &= f_3(x_i), \quad z_i^{(3)}(0) = f_4(x_i), z_i^{(4)}(0) = f_5(x_i). \end{aligned} \tag{27}$$

If $2 \le j \le m$, so the ICs are

$$\begin{aligned} z_i(\ell_{j-1}) &= z(x, \ell_{j-1}), \\ z_i'(\ell_{j-1}) &= \frac{\mathrm{d}z(x,\ell_{j-1})}{\mathrm{d}x}\Bigg|_{x=x_i}, \\ z_i''(\ell_{j-1}) &= \frac{\mathrm{d}^2 z(x,\ell_{j-1})}{\mathrm{d}x^2}\Bigg|_{x=x_i}, \\ z_i^{(3)}(\ell_{j-1}) &= \frac{\mathrm{d}^3 z(x,\ell_{j-1})}{\mathrm{d}x^3}\Bigg|_{x=x_i}, \\ z_i^{(4)}(\ell_{j-1}) &= \frac{\mathrm{d}^4 z(x,\ell_{j-1})}{\mathrm{d}x^4}\Bigg|_{x=x_i}. \end{aligned} \tag{28}$$

5. Put (BCs):

$$\begin{aligned} z_{0,j} &= z(a, \ell_j) = \mathrm{g}_1(\ell_j), \\ z_{n,j} &= z(b, \ell_j) = \mathrm{g}_2(\ell_j). \end{aligned} \tag{29}$$

6. By using the RKM method, solve the system of fifth-order FODEs in equation (21), given (ICs) (27) or (28) and (BCs) (29), at $\ell = \ell_j$.

4 Implementations (numerical examples)

To test our method, we used fifth-order FPDE in the following examples:

Problem 1.

Consider the following fifth-order quasi linear FPDEs:

$$\begin{aligned} &D_\ell^{5\alpha} z(x,\ell) = z(x,\ell) + z_{xx}(x,\ell) + z_{xxxxx}(x,\ell), \\ &a \le x \le b, \quad \ell > 0, \end{aligned}$$

with ICs:

$$\begin{aligned} &z(x,0) = \mathrm{e}^{-x}, \quad z_x(x,0) = -\mathrm{e}^{-x}, \quad z_{xx}(x,0) = \mathrm{e}^{-x}, \\ &z_{xxx}(x,0) = -\mathrm{e}^{-x}, \quad z_{xxxx}(x,0) = \mathrm{e}^{-x}, \end{aligned}$$

and BCs:

$$z(a,\ell) = \mathrm{e}^{-a}\mathrm{e}^{\ell}, \quad z(b,\ell) = \mathrm{e}^{-b}\mathrm{e}^{\ell}.$$

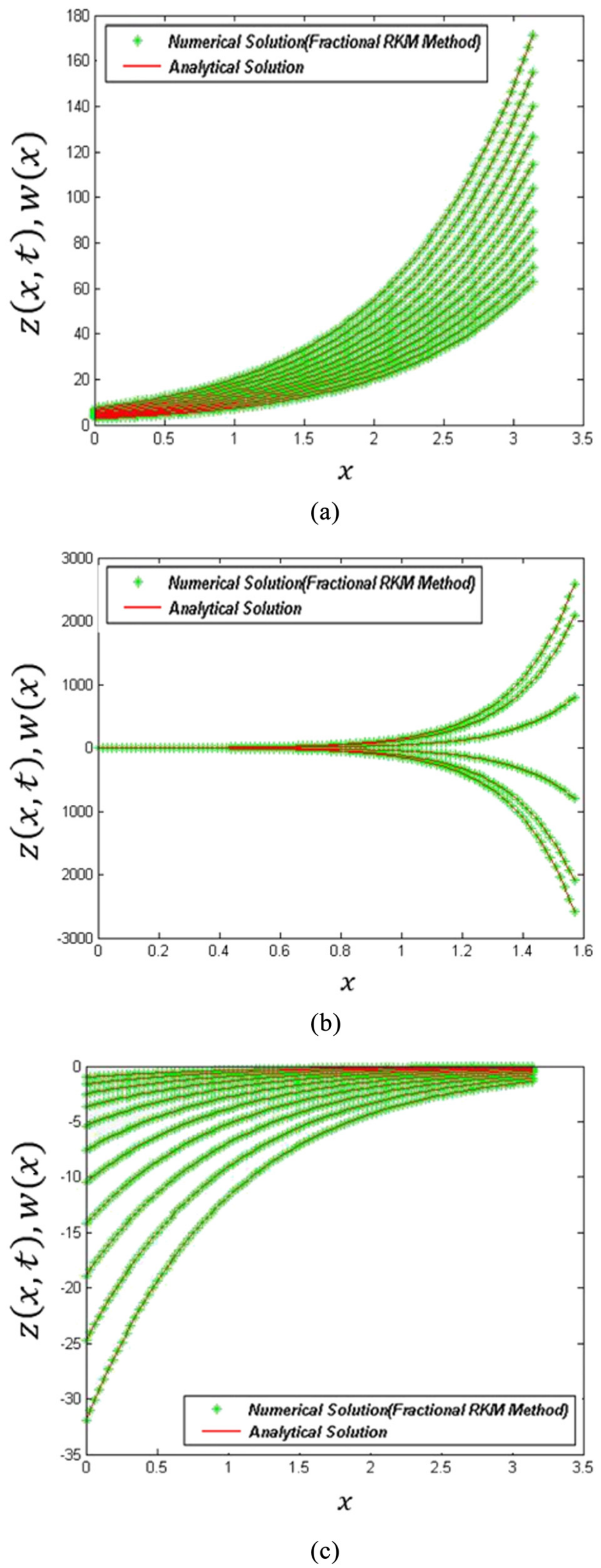

Figure 1: A comparison between the numerical solutions evaluated by generalized RKM method versus the analytical solutions for (a) Problem 1, (b) Problem 2, and (c) Problem 3 for ten lines of t in the domain with α = **0.96**.

The exact solution is $z(x,t) = e^{-x}e^{t}$, for $a = 0, b = 1$, and $\alpha = 1$.

Problem 2.

Consider the following fifth-order quasi linear FPDEs:

$$D_t^{5\alpha} z(x,t) = (5)^{5\alpha-1}(z(x,t) - z_{xx}(x,t)), \quad a \le x \le b, \quad t > 0,$$

with ICs:

$$z(x,0) = \cos(2x), \quad z_x(x,0) = -2\sin(2x),$$
$$z_{xx}(x,0) = -4\cos(2x),$$
$$z_{xxx}(x,0) = 8\sin(2x), z_{xxxx}(x,0) = 16\cos(2x),$$

and BCs:

$$z(a,t) = e^{5t}\cos(2a), \quad z(b,t) = e^{5t}\cos(2b).$$

The exact solution is $z(x,t) = e^{5t}\cos(2x)$, for $a = 0, b = \pi$, and $\alpha = 1$.

Problem 3.

Consider the following fifth-order quasi linear FPDEs:

$$D_t^{5\alpha} z(x,t) = (-1)^{5\alpha} z(x,t) + z_{xxx}(x,t) - \frac{x}{2} z_{xxxx}(x,t),$$
$$a \le x \le b, \quad t > 0,$$

with ICs:

$$z(x,0) = x^5, \quad z_x(x,0) = 5x^4, \quad z_{xx}(x,0) = 20x^3,$$
$$z_{xxx}(x,0) = 60x^2, \quad z_{xxxx}(x,0) = 120x,$$

and BCs:

$$z(a,t) = a^5 e^{-t}, \quad z(b,t) = b^5 e^{-t}.$$

The exact solution is $z(x,t) = x^5 e^{-t}$, for $a = 0, b = 1$, and $\alpha = 1$.

A comparison between the numerical solutions $w(x)$ evaluated by generalized RKM method versus the exact solutions $z(x,t)$ for the above problems and for ten lines of t and $\alpha = 0.96$ is shown in Figure 1.

5 Discussion and conclusion

In this work, we have focused on developing numerical techniques for solving fifth-order FPDEs by generalized RKM methods. This study has accomplished its purpose by demonstrating the generalizability of several efficient numerical approaches for solving FPDEs by the RKM type. In view of the comparison of the numerical solutions of the proposed method with the exact solutions of various studied cases demonstrated the efficiency and accuracy of the modified technique as in Figure 1. Furthermore, the numerical examples in the implementations of this article proved that the proposed method is a powerful numerical method for solving the class of fifth-order FPDEs.

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