

Regular Article

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New approach to analysis of railway track dynamics – Rail head vibrations

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Abstract: The article presents an original approach to the analysis of railway track dynamics. The “beam-inside-beam” concept is introduced as a dynamic generalisation of the static head-on-web effect, based on the two-layer model of rail. Rail head is a distinguished element of the rail. The system of distributed moving loads excites rail head as a beam supported by the rail web being considered as a viscoelastic layer. Rail head vibration is a kinematic excitation of the whole rail profile, including the rail head. The whole rail is also considered as a beam on viscoelastic foundation with parameters used in typical analyses of railway track dynamics. Vibrations of the whole rail obtained from the analysis of the two-layer “beam-inside-beam” model are compared with vibrations of typical one-layer track model. It is known that the static head-on-web effect is very limited. The range of significant rail head displacements covers nearly 0.6 m of area centred around the wheel position. This observation is also valid in the case of moving constant load. In other words, the dynamic effect of rail head vibrations in the case without imperfections type of track-vehicle is practically the same as in the static head-on-web case. Nevertheless, the analysis of track imperfections impact with various factors influencing the system behaviour, like the axles load and configuration, the track elastic parameters, the length of imperfection, or the train speed, show that the head-on-web effect is significant and should be analysed in more detail. For this purpose, a new model “beam-inside-beam” is proposed. The Fourier series are used to solve the two considered models. The load and unknown functions in the models’ solutions are expanded

in the arbitrarily assumed interval which practically covers non-zero track response for the analysed system of parameters (group of wheels). This solving method for equation of motion was experimentally validated in the paper by Czyczula et al. (2017). Validation of the “beam-inside-beam” model is considered as future work and the current article should be recognised as a preliminary study of the analysed effects under assumption of being purely theoretical investigation so far, although forming hierarchically organised building of more complex rail models is possible to partially verify in further steps of modelling process.

Keywords: railway track, moving load, train axles configuration, track imperfections

1 Introduction

The problem of dynamic response of the track under a moving load is the subject of many theoretical and experimental studies.

Under some assumptions, the beam on elastic foundation can be considered as a typical track model. It is worth to mention that the first study of beams on Winkler foundation subjected to a concentrated force moving with constant speed was initiated by Timoshenko [1]. The first solution to a simple stationary case of the Bernoulli–Euler beam on an elastic foundation was properly obtained by Ludwig [2]. The case of moving and oscillating force was formulated and partly solved by Mathews [3]. The case of varying moving force was studied by Fryba [4] and Bogacz and Krzyżyński [5]. The model of load moving along a sectional structure (varying along the space variable) was analysed by Bogacz [6].

Many papers are devoted to study the various effects of generalised models:

1. Analysis of the Timoshenko beam under a moving constant and varying load [7–9],
2. Analysis of beam vibration on elastic half-space [10],
3. Beam response on non-linear foundation [7,10,11] and non-linear spring contact between wheel and rail [23,24]

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4. Beam dynamic response on random foundation [12,13]
5. Dynamic response of track considered as a multilayer structure, e.g. refs. [14,15] – analytical approach and [16] – numerical approach;
6. Analysis of a set of distributed moving forces, described with Heaviside functions (e.g. ref. [8]), cosine square formula [9], or rectangular and Gauss functions [15];
7. Effect of axial force on dynamic response [15,17];
8. Analysis of a set of forces varying harmonically and associated with track imperfections – including phase of sine function for particular axles (numerically in refs. [16,18,19] and analytical approach in ref. [15]).

In all the above described generalisations of classical approach, the track response model is composed of rail (as the beam) and viscoelastic or elastic foundation. The sleepers and the ballast are modelled as additional layers.

The rail head effect in a static load case was studied analytically by Orringer *et al.* [20]. In this article, the effects of rail head vibrations on track response are studied in a steady state case. In the model, both the rail head and the whole rail profile are described as the Bernoulli–Euler beams. The moving load is modelled by a set of distributed forces moving with constant velocity.

2 Rail head as beam on elastic web – static analysis using numerical approach

In the paper by Orringer *et al.* [20], the rail head is modelled as the Bernoulli–Euler beam with the moment of inertia determined numerically (for US rail only). The stiffness of the web, treated as the head foundation, is described by the formula:

$$k_h = \frac{t}{h} \cdot E, \quad (1)$$

where t – rail web thickness; h – rail web height, and E – Young’s modulus of rail steel.

As rail web thickness is changing and its height is difficult to determine, the numerical analysis is carried out to determine both the rail head foundation and the rail head geometrical parameters. Figure 1 presents cross section of the most popular rail i.e. the 60E1 type. Rail head is selected arbitrarily and the following parameters are used:

- moment of inertia in vertical plane: $I_h = 43.84 \times 10^{-8} \text{ m}^4$;

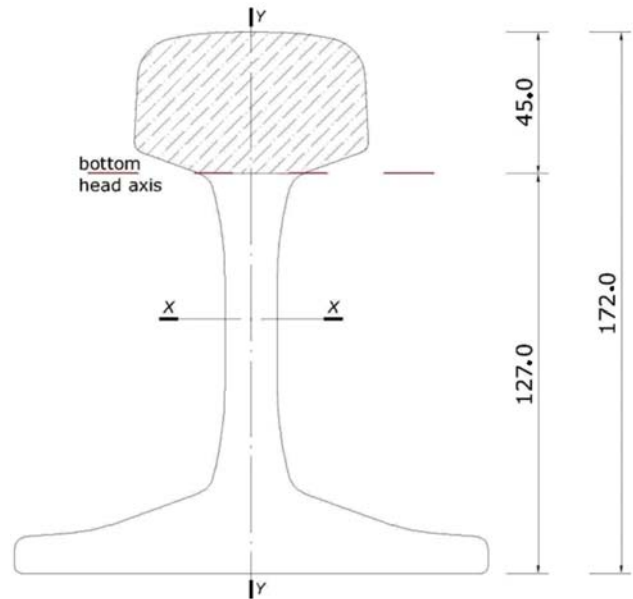


Figure 1: Rail cross section with the rail head separation.

- cross section area: $A_h = 2,990 \times 10^{-6} \text{ m}^2$;
- unit mass: $m_h = 23.3 \text{ kg/m}$.

To determine the stiffness of rail head foundation, a 3D numerical model is analysed. A 1 m long 60E1 rail considered as an elastic body is fixed at the bottom, and loaded along the top of the rail by vertical distributed forces with density $q = 78.328 \text{ kN/m}$. The rail head foundation stiffness can be calculated with the following simple formula:

$$k_h = \frac{q}{y_{av}}, \quad (2)$$

where y_{av} is the average vertical displacement of the rail head neutral axis. The rail head foundation stiffness obtained by numerical experiment, according to formula (2), is equal to $0.19 E$ which is very close to the value $0.2 E$ obtained by Orringer with the use of equation (1).

3 Steady state response in vertical direction using the two-layer “beam-inside-beam” model

The problem of rail head vibration as “beam-inside-beam” model was mentioned in the paper by Czyczula *et al.* [21] presenting non-linear formulation of the problem; however, without finalized solution. In this article,

a two-layer track model is considered, in which a rail head is distinguished, but not separated from the whole rail profile. The rail head is described as a layer, the vibrations of which are a kinematic excitation of rail longitudinal axis. It is assumed that both the rail head and the whole rail are the Bernoulli–Euler beams with the following parameters:

$E = E_h = E_r$ – Young’s modulus of rail steel (N/m²);

I_h and I_r – moments of inertia of rail head and the whole rail profile, respectively (m⁴);

m_h and m_r – unit mass of rail head and the whole rail profile, respectively (kg/m);

N_h and N_r – axial forces in rail head and the whole rail profile, respectively (should be considered as constant along the track) (N). Assuming that (k_h [N/m²], c_h [Ns/m²]) is viscoelastic constraint describing rail head foundation (rail web) properties, and (k_r [N/m²], c_r [Ns/m²]) describing viscoelastic properties of rail foundation, the “beam-inside-beam” system can be described by the following system of equations:

$$\begin{aligned} EI_h \frac{\partial^4 y_h}{\partial x^4} + N_h \frac{\partial^2 y_h}{\partial x^2} + m_h \frac{\partial^2 y_h}{\partial t^2} + c_h \left(\frac{\partial y_h}{\partial t} - \frac{\partial y_r}{\partial t} \right) + k_h (y_h - y_r) &= q(x, t), \\ EI_r \frac{\partial^4 y_r}{\partial x^4} + N_r \frac{\partial^2 y_r}{\partial x^2} + m_r \frac{\partial^2 y_r}{\partial t^2} + c_r \frac{\partial y_r}{\partial t} + k_r y_r &= c_h \left(\frac{\partial y_h}{\partial t} - \frac{\partial y_r}{\partial t} \right) + k_h (y_h - y_r), \end{aligned} \quad (3)$$

where $q(x, t)$ is a unit load acting on a rail head (N/m).

After introducing the moving coordinate system ($\eta = y$, $\chi = y_h$, $\xi = x - vt$), equation (3) can be written as follows:

$$\begin{aligned} EI_h \frac{\partial^4 y_h}{\partial \xi^4} + N_h \frac{\partial^2 y_h}{\partial \xi^2} + m_h \left(\frac{\partial^2 y_h}{\partial t^2} - 2v \frac{\partial^2 y_h}{\partial t \partial \xi} + v^2 \frac{\partial^2 y_h}{\partial \xi^2} \right) + c_h \left(\frac{\partial y_h}{\partial t} - v \frac{\partial y_h}{\partial \xi} - \frac{\partial y_r}{\partial t} + v \frac{\partial y_r}{\partial \xi} \right) + k_h (y_h - y_r) &= q(\xi, t), \\ EI_r \frac{\partial^4 y_r}{\partial \xi^4} + N_r \frac{\partial^2 y_r}{\partial \xi^2} + m_r \left(\frac{\partial^2 y_r}{\partial t^2} - 2v \frac{\partial^2 y_r}{\partial t \partial \xi} + v^2 \frac{\partial^2 y_r}{\partial \xi^2} \right) + c_r \left(\frac{\partial y_r}{\partial t} - \frac{\partial y_r}{\partial \xi} \right) + k_r y_r &= c_h \left(\frac{\partial y_h}{\partial t} - v \frac{\partial y_h}{\partial \xi} - \frac{\partial y_r}{\partial t} + v \frac{\partial y_r}{\partial \xi} \right) + k_h (y_h - y_r). \end{aligned} \quad (4)$$

For a set of moving distributed forces changing harmonically in time with circular frequency ω , the load $q(\xi, t)$ can be expressed as:

$$q(\xi, t) = q_c(\xi) \cdot \cos \omega t + q_s(\xi) \cdot \sin \omega t, \quad (5)$$

where q_c and q_s are cosine and sine parts of the load, respectively.

The steady state solution of (4) can also be described as cosine and sine parts of the rail head and the whole rail profile displacements:

$$\begin{aligned} y_h(\xi, t) &= Y_{hc}(\xi) \cdot \cos \omega t + Y_{hs}(\xi) \cdot \sin \omega t \\ y_r(\xi, t) &= Y_{rc}(\xi) \cdot \cos \omega t + Y_{rs}(\xi) \cdot \sin \omega t. \end{aligned} \quad (6)$$

By differentiating equation (6) and substituting together with equation (5), equation (4) becomes a set of ordinary equations associated with the cosine ($\cos \omega t$) and the sine ($\sin \omega t$) parts:

$$\begin{aligned} L_1(Y_{hc}, Y_{hs}, Y_{rc}, Y_{rs}, P) \cdot \cos \omega t &= q_c \cdot \cos \omega t \\ L_2(Y_{hc}, Y_{hs}, Y_{rc}, Y_{rs}, P) \cdot \sin \omega t &= q_s \cdot \sin \omega t \\ L_3(Y_{hc}, Y_{hs}, Y_{rc}, Y_{rs}, P) \cdot \cos \omega t &= 0 \cdot \cos \omega t \\ L_4(Y_{hc}, Y_{hs}, Y_{rc}, Y_{rs}, P) \cdot \sin \omega t &= 0 \cdot \sin \omega t, \end{aligned} \quad (7)$$

where $L_1, \dots, L_4$ are linear operators of functions Y_{hc} , Y_{hs} , Y_{rc} , Y_{rs} , and their derivatives, while P denotes a set of model parameters. E.g. operator L_1 has the following form:

$$\begin{aligned} L_1 = EI_h \frac{d^4 Y_{hc}}{d\xi^4} - m_h \omega^2 Y_{hc} - 2m_h v \omega \frac{dY_{hs}}{d\xi} \\ + (N_h + m_h v^2) \frac{d^2 Y_{hc}}{d\xi^2} + c_h \omega Y_{hs} - c_h v \frac{dY_{hc}}{d\xi} \\ - c_h \omega Y_{rs} + c_h v \frac{dY_{rc}}{d\xi} + k_h Y_{hc} - k_h Y_{rc}. \end{aligned} \quad (8)$$

In order to solve the considered system, a method similar to that presented in the paper by Czyczula et al. [15] can be applied. The distributed loads $q_c(\xi)$ and $q_s(\xi)$

and the unknown functions Y_{hc} , Y_{hs} , Y_{rc} and Y_{rs} are expanded by the Fourier series in the assumed interval $[0, \lambda]$, which should cover significantly different from zero response of the track for the analysed group of wheels:

$$\begin{aligned} q_c(\xi) &= \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cdot \cos \Omega_i \xi + b_i \cdot \sin \Omega_i \xi); \\ q_s(\xi) &= \frac{c_0}{2} + \sum_{i=1}^{\infty} (c_i \cdot \cos \Omega_i \xi + d_i \cdot \sin \Omega_i \xi); \\ Y_{hc}(\xi) &= \frac{Y_{h01}}{2} + \sum_{i=1}^{\infty} (A_i \cdot \cos \Omega_i \xi + B_i \cdot \sin \Omega_i \xi); \\ Y_{hs}(\xi) &= \frac{Y_{h02}}{2} + \sum_{i=1}^{\infty} (C_i \cdot \cos \Omega_i \xi + D_i \cdot \sin \Omega_i \xi); \end{aligned} \quad (9)$$

$$\begin{aligned}
Y_{rc}(\xi) &= \frac{Y_{r01}}{2} + \sum_{i=1}^{\infty} (E_i \cdot \cos \Omega_i \xi + F_i \cdot \sin \Omega_i \xi); \\
Y_{rs}(\xi) &= \frac{Y_{r02}}{2} + \sum_{i=1}^{\infty} (G_i \cdot \cos \Omega_i \xi + H_i \cdot \sin \Omega_i \xi); \\
\xi \in [0, \lambda]; \quad \Omega_i &= \frac{2\pi \cdot i}{\lambda}.
\end{aligned}$$

Differentiating functions (9) and substituting the obtained expressions to the set of equation (7) lead to the solution of the system of algebraic equations with unknown coefficients $A_i, B_i, C_i, D_i, E_i, F_i, G_i$, and H_i , which can be found by comparison of quantities with the sine and cosine i th series:

$$\begin{aligned}
&A_i \cdot P_1 + B_i \cdot (-P_2) + C_i \cdot P_3 + D_i \cdot (-P_4) + E_i \cdot P_5 \\
&\quad + F_i \cdot P_2 + G_i \cdot (-P_3) + H_i \cdot P_6 = a_i \\
&A_i \cdot P_2 + B_i \cdot P_1 + C_i \cdot P_4 + D_i \cdot P_3 + E_i \cdot (-P_2) \\
&\quad + F_i \cdot (-P_5) + G_i \cdot P_6 + H_i \cdot (-P_3) = b_i \\
&A_i \cdot (-P_3) + B_i \cdot P_4 + C_i \cdot P_1 + D_i \cdot (-P_2) + E_i \cdot P_3 \\
&\quad + F_i \cdot P_6 + G_i \cdot (-P_5) + H_i \cdot P_2 = c_i \\
&A_i \cdot (-P_4) + B_i \cdot (-P_3) + C_i \cdot P_2 + D_i \cdot P_1 + E_i \cdot P_6 \\
&\quad + F_i \cdot P_3 + G_i \cdot (-P_2) + H_i \cdot (-P_5) = d_i \\
&A_i \cdot (-P_5) + B_i \cdot P_2 + C_i \cdot (-P_3) + D_i \cdot P_6 + E_i \cdot P_7 \\
&\quad + F_i \cdot (-P_8) + G_i \cdot P_9 + H_i \cdot (-P_{10}) = 0 \\
&A_i \cdot (-P_2) + B_i \cdot (-P_5) + C_i \cdot P_6 + D_i \cdot (-P_3) + E_i \cdot P_8 \\
&\quad + F_i \cdot P_7 + G_i \cdot P_{10} + H_i \cdot P_9 = 0 \\
&A_i \cdot P_3 + B_i \cdot P_6 + C_i \cdot (-P_5) + D_i \cdot P_2 + E_i \cdot (-P_9) \\
&\quad + F_i \cdot P_{10} + G_i \cdot P_7 + H_i \cdot (-P_8) = 0 \\
&A_i \cdot P_6 + B_i \cdot P_3 + C_i \cdot (-P_2) + D_i \cdot (-P_5) + E_i \cdot (-P_{10}) \\
&\quad + F_i \cdot (-P_9) + G_i \cdot P_8 + H_i \cdot P_7 = 0.
\end{aligned} \tag{10}$$

In the system (10), the following new parameters are introduced:

$$\begin{aligned}
P_1 &= EI_h \Omega_i^4 - m_h \omega^2 - (N_h + m_h v^2) \Omega_i^2 + k_h \\
P_2 &= c_h v \Omega_i \\
P_3 &= c_h \omega \\
P_4 &= 2m_h v \omega \Omega_i \\
P_5 &= k_h \\
P_6 &= 0 \\
P_7 &= EI_r \Omega_i^4 - m_r \omega^2 - (N_r + m_r v^2) \Omega_i^2 + k_r + k_h \\
P_8 &= (c_r + c_h) v \Omega_i \\
P_9 &= (c_r + c_h) \omega \\
P_{10} &= 2m_r v \omega \Omega_i.
\end{aligned} \tag{11}$$

The constants $Y_{h01}, Y_{h02}, Y_{r01}$, and Y_{r02} (equation [9]) are calculated from the set of equations:

$$\begin{aligned}
&Y_{h01} \cdot (k_h - m_h \omega^2) + Y_{h02} (c_h \omega) + Y_{r01} (-k_h) + Y_{r02} (-c_h \omega) = a_0 \\
&Y_{h01} \cdot (-c_h \omega^2) + Y_{h02} (k_h - m_h \omega^2) + Y_{r01} (c_h \omega) + Y_{r02} (-k_h) = c_0 \\
&Y_{h01} \cdot (-k_h) + Y_{h02} (-c_h \omega) + Y_{r01} (k_r + k_h - m_r \omega^2) + Y_{r02} (c_r \omega + c_h \omega) = 0 \\
&Y_{h01} \cdot (c_h \omega) + Y_{h02} (-k_h) + Y_{r01} (-c_h \omega - c_r \omega) + Y_{r02} (k_r + k_h - m_r \omega^2) = 0.
\end{aligned} \tag{12}$$

Vibrations of the whole rail profile obtained from the analysis of the two-layer model described above can be compared to the vibrations of rail modelled by one-beam equation (cf. e.g. ref. [15]):

$$EI_r \frac{\partial^4 y_r}{\partial x^4} + N_r \frac{\partial^2 y_r}{\partial x^2} + m_r \frac{\partial^2 y_r}{\partial t^2} + c_r \frac{\partial y_r}{\partial t} + k_r y_r = q(x, t), \tag{13}$$

where all parameters are similar to those in equation (3).

4 Numerical examples

The following track and load parameters are used in computational examples:

1. Rail type of 60E1: Young's modulus $E = 2.1 \times 10^8$ kN/m²; moment of inertia in vertical plane $I_r = 3,055 \times 10^{-8}$ m⁴; unit mass $m_r = 60$ kg/m; and longitudinal force in rail $N_r \equiv 0$;
2. Rail head (60E1 rail): moment of inertia in vertical plane $I_h = 43.84 \times 10^{-8}$ m⁴; unit mass $m_h = 23.3$ kg/m (cf. Section 2), and longitudinal force in rail head $N_h \equiv 0$;
3. Rail head foundation: stiffness $k_h = 0.19 E \times 10^{10}$ kN/m² and damping coefficient $c_h = 0$; 20 Ns/m²;
4. Rail foundation: $k_r = 91.2 \times 10^6$ N/m² (reference value), also 0.3 and 0.1 of the reference value and $c_r = 3,950$ Ns/m²;
5. Track irregularities: cosine type – the depth of 10 μ m at the length of 0.3 and 0.6 m;
6. Rail head/wheel contact stiffness: $k_c = 10^6$ kN/m (cf. ref. [15]);
7. Load configuration: two wheels of Thalys train with a static load $P_{st} = 80,000$ N at the distance of 3 m one from another (cf. ref. [16]); four wheels of EMU-250 (Pendolino) train with average static load $P_{st} = 78,329$ N with distances between them: 2.7, 7.2, and 9.9 m (cf. ref. [22]), four wheels of coal wagons Falns 441 VA with static load $P_{st} = 112,500$ N; and distances: 1.8, 5.04 and 6.84 m;
8. Train speed: up to 300 km/h;
9. Distributed forces approximation: Gauss function (cf. ref. [15]); number of Fourier series coefficients: 2,000–3,000.

It is assumed that the “rail head – wheel” contact is continuous and vibration occurs with a change in the length of contact spring.

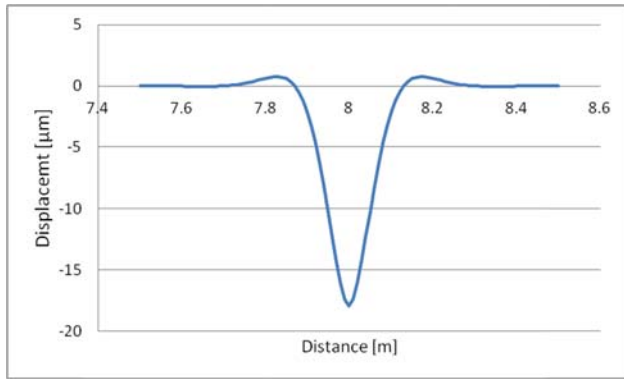


Figure 2: Difference between rail head and whole rail vertical displacements.

Figure 2 shows the difference between a rail head and a whole rail vertical displacements in the case of the Pendolino EMU-250 train in the region of the first axle and with the train speed of 300 km/h. Only forces constant in time are taken into account (corresponding to a track without imperfections). It can be seen that the effect of the head-on-web displacement is relatively small (0.018 mm) and it possesses a local nature. It can be said that displacements of the rail head and the whole rail are practically the same at a distance of more than approximately 0.3 m from the wheel. This conclusion, however, is valid for moving forces constant in time only.

Both Figure 3a and b present rail displacements obtained in the case of the two-layer “beam-inside-beam” model and the one-layer model, with the second one (Figure 3b) as a “zoomed” fragment shown for better recognition of the pattern details. The calculations are carried out for the Pendolino train moving with the speed of 300 km/h along a track with a 10 μm imperfection with the cosine wave with a length of 0.6 m and the track stiffness $k_r = 91.1 \text{ kN/m}^2$. In this case, the difference reaches 0.05 mm and it is spread over 5 m in length. Therefore, the effect of head-on-web should not be considered as local.

One should mention that an introduction of imperfection in both the considered models leads to the

necessity of the analysed distance enlargement due to the increasing range of the response significantly different from zero. Therefore, the interval $(0, \lambda)$ must be modified according to the particular cases.

Similarly prepared examples are shown in Figures 4 and 5, with second graphs presenting zoomed fragments of the response history in time.

Both Figure 4a and b present the difference between the two-layer and one-layer rail responses for the Thalys train moving with the speed of 200 km/h on a track with imperfections of 10 μm represented by the cosine wave with a length of 0.3 m and the track stiffness $k_r = 91.1 \text{ kN/m}^2$. In this case, there is also a significant difference between the maximum displacements obtained using both models.

Another example, for a coal wagons, is shown in Figure 5a and b confirming the previously described observations. Here, additionally, a phase shift can be observed which makes the necessity of the head-on-web problem analysis even stronger. However, physical reasons of this phenomena for the considered system of parameters remain an open problem.

Figure 6 shows the maximum difference of displacements between the two-layer and the one-layer cases obtained for the three considered vehicles but for the same set of track parameters and speed $v = 100 \text{ km/h}$. This is an example justifying the necessity of more detailed analysis of the analysed phenomenon, taking into account various scenarios of track loads and other track parameters. Results of such a preliminary analysis are gathered in Tables 1–3 showing that the difference in question significantly depends on the selected parameters, i.e. the train speed, the type of train (axle configuration and load characteristics), the track stiffness, and the length of track imperfection.

One can observe that the length of imperfections significantly influences the investigated characteristics when analysing the two discussed models. One should also underline that the dynamic “head-on-web” effect becomes stronger with increasing train speed and changes with rail foundation stiffness variation.

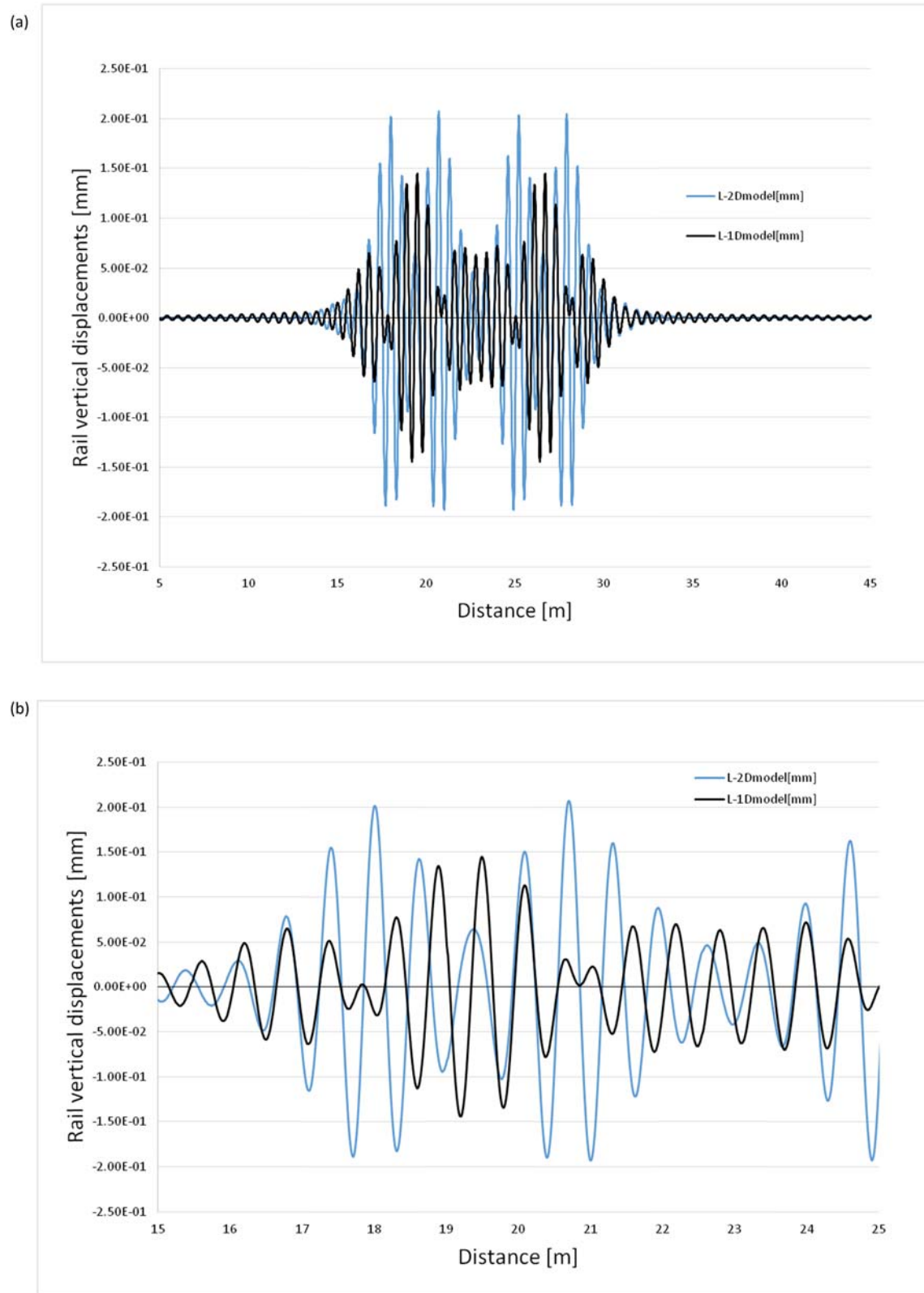


Figure 3: (a) The whole rail displacements for a two-layer (L-2D model) and a one-layer (L-1D model) models (Pendolino train, $v = 300$ km/h, $k_r = 91.1$ MPa, imperfection $s = 10$ μm , and $L_n = 0.6$ m) and (b) “Zoomed” graph of a part of (a) solution.

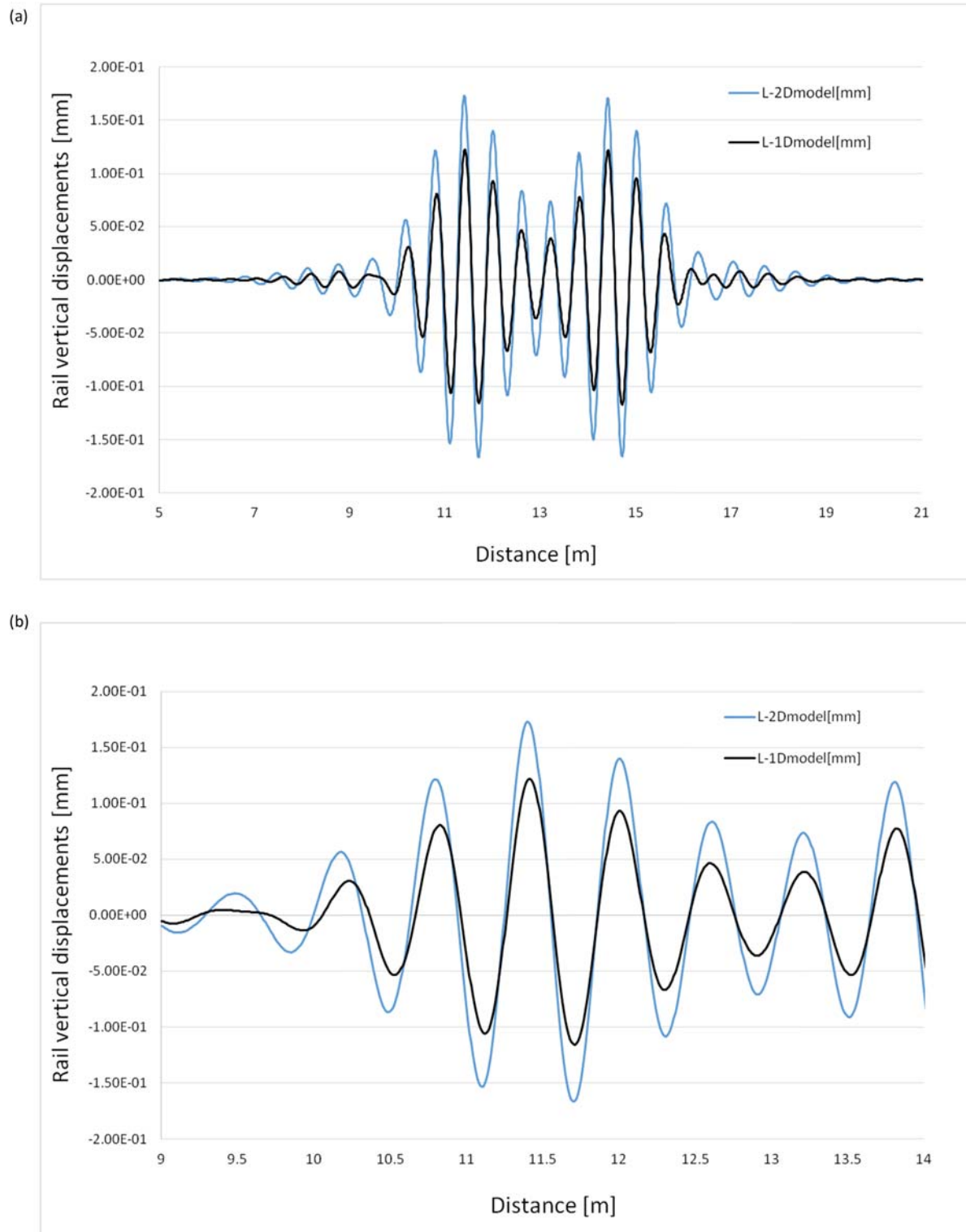


Figure 4: (a) The whole rail displacements for a two-layer (L-2D model) and a one-layer (L-1D model) models (Thalys train, $v = 200$ km/h, $k_r = 91.1$ MPa, imperfection $s = 10$ μm , and $L_n = 0.6$ m) and (b) “Zoomed” graph of a part of (a) solution.

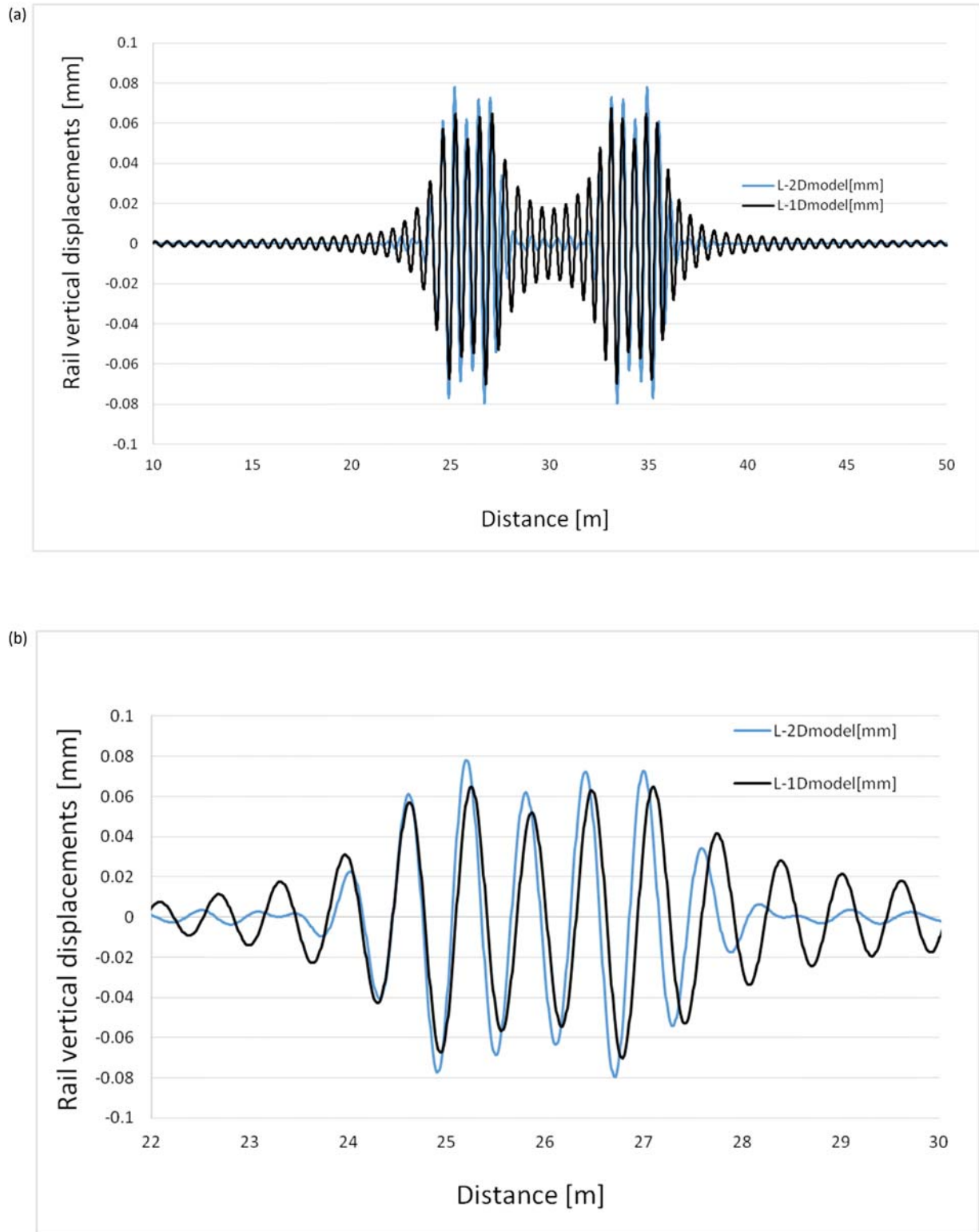


Figure 5: (a) Whole rail displacements for a two-layer (L-2D model) and a one-layer (L-1D model) models (coal wagon, $v = 100$ km/h, $k_r = 91.1$ MPa, imperfection $s = 10$ μ m, and $L_n = 0.6$ m) and (b) “Zoomed” graph of a part of (a) solution.

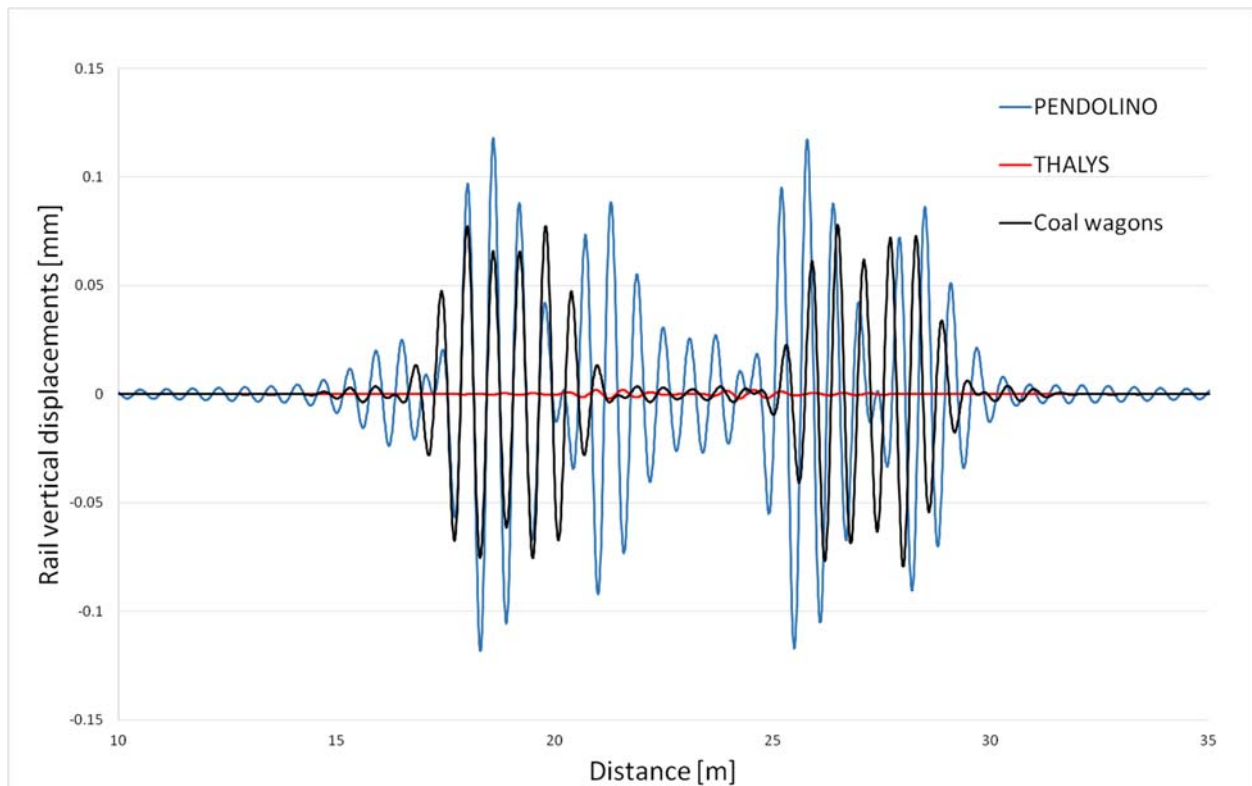


Figure 6: Maximum difference of a two-layer (2-L) and a one-layer (1-L) model displacements ($v = 100$ km/h, $k_r = 91.1$ MPa, imperfection $s = 10$ μ m, and $L_n = 0.6$ m).

Table 1: The amplitude differences for the two considered models (Pendolino)

Pendolino						
	Track stiffness					
	$k_r = 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.3 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.1 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$	
	Length of cosine wave					
	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$
$v = 100 \text{ (km/h)}$	0.164	0.002	2.209	0.019	0.215	0.745
$v = 200 \text{ (km/h)}$	0.601	0.009	0.195	1.729	0.31	0.568
$v = 300 \text{ (km/h)}$	0.295	0.054	0.117	0.312	0.256	0.384

Table 2: The amplitude differences for the two considered models (Thalys)

Thalys						
	Track stiffness					
	$k_r = 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.3 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.1 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$	
	Length of cosine wave					
	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$
	$v = 100 \text{ (km/h)}$	0.008	0.118	2.384	0.365	0.252
$v = 200 \text{ (km/h)}$	0.743	0.145	0.260	1.747	0.431	1.119
$v = 300 \text{ (km/h)}$	0.304	0.253	0.113	1.031	0.265	1.580

Table 3: The amplitude differences for the two considered models (coal wagons)

Coal wagons						
	Track stiffness					
	$k_r = 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.3 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$		$k_r = 0.1 \times 91.2 \times 10^6 \text{ (N/m}^2\text{)}$	
	Length of cosine wave					
	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$	$L_n = 0.3 \text{ (m)}$	$L_n = 0.6 \text{ (m)}$
	$\nu = 100 \text{ (km/h)}$	0.135	0.062	2.655	0.216	0.899
$\nu = 50 \text{ (km/h)}$	0.110	0.059	0.372	0.180	2.196	0.544
$\nu = 20 \text{ (km/h)}$	0.105	0.059	0.307	0.172	0.873	0.468

5 Conclusion

The rail head vibrations effect is described and studied as a new approach to the analysis of the track response under load in vertical direction moving along the track. The analysis is carried out for various train axle configurations, train speeds, lengths of track imperfections, and track foundation stiffness. The main conclusions can be formulated as follows:

1. The rail head vibrations effect is significant which is confirmed by a comparative study of the track response in the case of the two-layer “beam-inside-beam” model and the one-layer (the whole rail profile). The response difference between these two cases can reach 2 mm and occurs even at the distance of a few meters around a source of the load. It means that the rail head should be considered as a separated part of rail when one deals with more advanced discussions regarding the railway track dynamic response to moving vehicles.
2. The rail head foundation effect changes with decreasing rail foundation stiffness and with increasing train speed.
3. Train axle configurations (track load characteristics) and track imperfection parameters influence the difference between the rail response in the cases of the two-layer “beam-inside-beam” and the one-layer (the whole rail profile) models.
4. Although the new model itself still needs to be verified by experimental measurements, the method of solution was previously checked and validated in the cases of one-layer model and the two-layer track model, which is used for the “beam-inside-beam” concept introduced in the current article.
5. The article can be recognised as a preliminary study of the analysed effects under assumption of being purely theoretical investigation so far. Nevertheless, a proposed structure of this investigation follows the idea

of hierarchical extension process leading to a building of more advanced rail models available for partial verification in consecutive steps of modelling process.

Further investigations should be directed, besides the experimental verification supported by an application of other computational techniques, towards the model extensions, including an introduction of the sleeper and ballast layers and non-linear properties of track and foundation parameters. In addition, the head-on-web effect should be analysed in lateral and longitudinal directions, leading to a 3D system characteristics.

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