

Research Article

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A polynomial algorithm for packing unit squares in a hypograph of a piecewise linear function

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Abstract: We consider the problem of packing the maximal number of unit squares in a hypograph of a function. A polynomial time algorithm is described to solve this problem for a piecewise linear function.

Keywords: packing unit squares, polynomial algorithm

1 Introduction

The problems of rectangular packing and cutting are difficult problems of discrete optimization (for references see [1]). For example, the computational complexity of the problem of packing in a rectangular sheet (A, B) the maximum number of equal small rectangles (c, d) (pallet loading problem (PLP)) is unknown. This problem has number 55 in the list of open problems "The Open Problems Project". It is closely connected with another known problem, 56 in the list of open problems, about the complexity of deciding whether a given number of axis-parallel unit squares can be packed into a simple polygon (without holes). Appearance [2] conjecture the problem to be polynomially solvable. The problem is known to be NP-hard for polygons with holes [3], even if the polygon is an orthogonal polygon with all coordinates being multiples of $1/2$. Recently, this version of the problem was shown to be in NP [6], making it NP-complete. This problem arises in wood industry (cutting a log into quadratic plates), in glass industry (cutting defecting pane glass into small equal squares) etc. Applications of analogous problems can be seen in [4, 5]. In this article we consider the problem of packing of unit squares in a hypograph of a function. The

article is organized as follows. The second section contains the proof of a theorem about the structure of the optimal packing unit squares in a hypograph of a function. In the third section we propose a polynomial algorithm for the problem with piecewise linear function.

2 A theorem about optimal packing

It appears, that a two-dimensional problem about optimum packing unit squares in a hypograph of a function can be reduced to a one-dimensional problem of packing unit intervals in a one-dimensional set which have a trivial solution. If a compact set is represented as a separable union of intervals

$$M = \bigcup_{i \in I} A_i,$$

then the maximum number of single intervals which are possible to pack in this set is given by the obvious formula

$$N = \sum_{i \in I} [|A_i|].$$

Definition 1. A hypograph of a function f on the interval $[a, b]$ is the set

$$H(f, a, b) = \{(x, y) | 0 \leq y \leq f(x), a \leq x \leq b\}.$$

Note that this definition is different from the classical definition of hypograph in [7]. Cleave this set by lines parallel to the x -axis with integer y -coordinates. We designate these lines by L_i :

$$L_i = \{(x, y) | y = i \in \mathbf{N} = \{1, 2, \dots\}\}.$$

The intersection of the set L_i and the hypograph $H(f, a, b)$ is a compact set, which we designate by $H_i(f, a, b)$. Let us suppose that this set consists of finite number of intervals. Let us suppose also without loss of generality that the function f is a bounded continuous function on the interval $[a, b]$. Then, if $H_i(f, a, b) = \bigcup_{j \in J} A_{i,j}$, where $A_{i,j}$ is an interval, the maximum number N_i of unit intervals in $H_i(f, a, b)$ is given by

$$N_i = \sum_{j \in J} [|A_{i,j}|].$$

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The following theorem reduces the packing of unit squares to the packing of unit intervals.

Theorem 1. *The maximum number N of unit squares in a hypograph $H(f, a, b)$ of a function f on the interval $[a, b]$ is given by*

$$N = \sum_{i \in \mathbb{N}} N_i. \quad (1)$$

Proof. This theorem is almost obvious. We prove this claim by induction on the area of a hypograph. If the area of a hypograph is less than 1, then any unit square cannot be packed in such a hypograph. In the same way, any interval $A_{i,j} = [a_{i,j}, b_{i,j}]$ has length less than 1, otherwise, a rectangle

$$P_{i,j} = \{(x, y) | a_{i,j} \leq x \leq b_{i,j}, i-1 \leq y \leq i\},$$

lying inside a hypograph $H(f, a, b)$, contains unit square. Let for all hypographs of area less than S , theorem is valid. Let us show its justice for hypographs of area greater then or equal to S . Let us consider a set $H_1(f, a, b) = \bigcup_{j \in J} A_{1,j}$. It can cross no more than

$$N_1 = \sum_{j \in J} \lfloor |A_{1,j}| \rfloor$$

unit squares. All remaining unit squares are above the line $y = 1$. Then these squares with decreased by 1 y -coordinate are in the hypograph of a function, $f(x) - 1$. By the inductive hypothesis, in the hypograph of a function $f(x) - 1$, it can be packed $\sum_{i \in \mathbb{N} \setminus \{1\}} N_i$ unit squares. Together with squares intersecting a line $y = 1$ it can be proved easily that it is impossible to locate in the hypograph $H(f, a, b)$ more than $N = \sum_{i \in \mathbb{N}} N_i$ unit squares. But such a number of unit squares can be locate, if the first N_1 squares we locate in a hypograph between lines $y = 0$ and $y = 1$. The theorem is proved.

Let's illustrate this theorem by a problem of optimal packing of unit squares in a semicircle of radius 5 (hypograph of a function $\sqrt{25 - x^2}$). We have $N_1 = \lfloor \sqrt{96} \rfloor$, $N_2 = \lfloor \sqrt{84} \rfloor$, $N_3 = \lfloor 8 \rfloor$, $N_4 = \lfloor 6 \rfloor$. According to theorem, no more than $32=9+9+8+6$ unit squares can be packed in a semicircle of radius 5 and optimal packing is presented in Figure 1.

As another application of this theorem, we consider a problem of packing unit squares in a hypograph $H(f, a, b)$ of a function $F(x) = \frac{n}{x}$. If this problem can be solved in a polynomial time, then a known unsolved problem of factoring integers in a polynomial time has a positive answer. Let's designate $N(f, a, b)$ as the maximum number of unit squares, which can be packed into a hypograph of a function $H(f, a, b)$, then $N(\frac{n}{x}, a, b) - N(\frac{n-1}{x}, a, b)$ is equal to the number of integral points on the graph of the function $f(x) = \frac{n}{x}$, $a \leq x \leq b$. As the points $(1, n)$, $(n, 1)$ lie

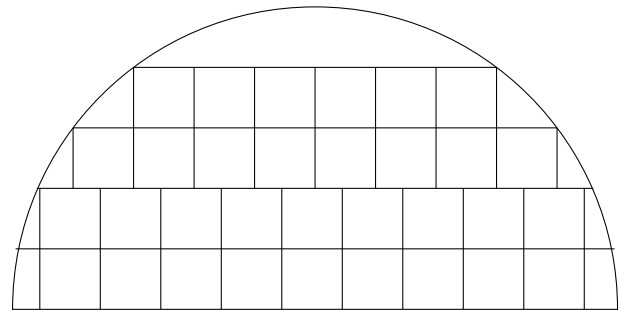


Figure 1: Packing unit squares in a semicircle

on the graph of this function, an inequality $N(\frac{n}{x}, 1, n) - N(\frac{n-1}{x}, 1, n) \geq 3$ tests the primality of n . Then, to find prime factors of n , one can apply the method of half division of the interval $[1, n]$. Thus, the complexity of a problem of optimum packing of unit squares in a hypograph of a function $H(\frac{n}{x}, a, b)$ is limited from below by the complexity of factoring integers for which the best algorithms to date do not run in a polynomial time.

3 Polynomial algorithm for a piecewise-linear function

In this section we consider a problem of optimum packing of unit squares in a hypograph of a piecewise-linear function and show the existence of a polynomial algorithm for its solution.

Let us consider in the beginning, the case of trapezoidal function:

$$f(x) = \begin{cases} c_1 + d_1 x, & a \leq x \leq a_1 \\ c, & a_1 \leq x \leq a_2 \\ c_2 + d_2 x, & a_2 \leq x \leq b. \end{cases}$$

Coefficients of this function satisfy the equalities

$$\begin{aligned} c_1 + d_1 a_1 &= c = c_2 + d_2 a_2, \\ c_1 + d_1 a &= c_2 + d_2 b = 0, \quad d_1 > 0, d_2 < 0. \end{aligned}$$

The graph of this function is presented in Figure 2.

Then, by Theorem 1, a solution to the problem of packing of unit squares in a hypograph of a function $H(f, a, b)$ is given by the formula

$$N(f, a, b) = \sum_{i=1}^{\lfloor c \rfloor} N_i,$$

where

$$N_i = \left\lfloor b - a - \frac{i(b - a - a_2 + a_1)}{c} \right\rfloor.$$

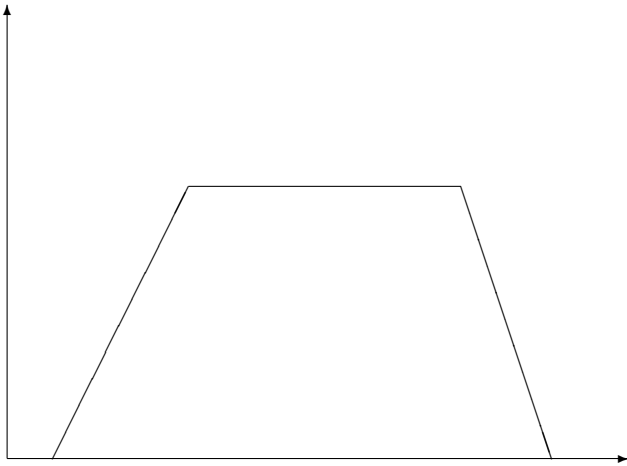


Figure 2: Trapezoidal function

This expression is equal to the number of points of a two-dimensional integral lattice without the points on axes of the abscissas in a trapezoid with vertices in points $(0, 0)$, $(0, c)$, $(a_2 - a_1, c)$, $(b - a, 0)$. The formula for the number of integral points in a trapezoid in the vertices $(0, 0)$, $(0, a)$, (b, a) , $(c, 0)$ is given by

$$N = \sum_{i=1}^{\lfloor a \rfloor} \left\lfloor c - \frac{i(c-b)}{a} \right\rfloor$$

Thus, the problem of optimum packing of unit squares in a hypograph of a trapezoidal function is reduced to the problem of counting the number of integral points in a trapezoid. This last problem in the general case of polygons is widely investigated and the existence of polynomial time algorithms is well known (see e.g. [8, 9]). So, for a solution of this reduced problem there exist effective polynomial algorithms. Thus, the problem of optimum packing of unit squares in the hypograph of trapezoidal function has effective, polynomial algorithms.

Let's consider the case of a piecewise linear function

$$F(x) = \begin{cases} c_1 + d_1x, & \text{if } a \leq x \leq a_1 \\ c_2 + d_2x, & \text{if } a_1 \leq x \leq a_2 \\ \dots & \text{if } \dots \\ c_n + d_nx, & \text{if } a_{n-1} \leq x \leq a_n = b. \end{cases}$$

Such a function can be represented as an ordered sequence of vertices of a hypograph considered as a polygon with vertices

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), a_1 < a_2 < \dots < a_n.$$

It is easy to see that $O(n)$ arithmetical operations are sufficient for transforming from one form of presentation to another. Moreover, a problem of optimum packing unit

squares in a hypograph of a piecewise linear function can be considered by virtue of the theorem 1, as a problem of counting their numbers with the help of formulas (1), i.e. as counting the sum of integer part of lengths of intervals cutting from a hypograph of a piecewise linear function by the set of lines $y = i$, $i \in \mathbf{N}$. Then a problem of counting the sum of integer part of lengths of intervals cutting from a hypograph of a piecewise linear function, considered as a sequence of vertices of a polygon

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), a_1 < a_2 < \dots < a_n.$$

by the set of lines $y = i$, $i \in \mathbf{N}$ is easily decomposed into problems with smaller numbers of vertices, and the sum of such numbers of vertices will be no more than n . The main idea consists in cutting a trapezoidal part of a hypograph of a non-unimodal piecewise linear function. Before formulating an algorithm for general case, we shall consider the case of a unimodal piecewise linear function, for which a sequence b_i is of the form:

$$b_1 \leq b_2 \leq \dots \leq b_k \geq b_{k+1} \geq \dots \geq b_{n-1} \geq b_n.$$

Such a polygon can be easily presented in a way in which

$$b_1 = b_{n-1}, b_2 = b_{n-2}, \dots, b_k = b_{n-k}, \dots$$

Then, counting the sum of the integer part of lengths of intervals cutting from a hypograph of such a function can be reduced to the formula

$$N = \sum_{i=1}^{n/2} N_i,$$

where N_i is equal to the sum of the integer part of lengths of intervals cutting from a hypograph of trapezoidal function as an appropriate trapezoid with vertices

$$(a_i, b_i), (a_{i+1}, b_{i+1}), (a_{n-i-1}, b_{n-i-1}), (a_{n-i}, b_{n-i}).$$

As we already have stated, for calculating N_i there is a polynomial algorithm based on counting the points of integral lattice in an appropriate trapezoid. Therefore, there is a polynomial algorithm for counting the sum of the integer part of lengths of intervals cutting from a hypograph of a unimodal piecewise linear function by lines $y = i$, $i \in \mathbf{N}$.

Let's consider the general case of a piecewise linear function, presented by the sorted list of vertices:

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), a_1 < a_2 < \dots < a_n.$$

If this function is not unimodal on the interval $[a_1, a_n]$, there is a local minimum of this function b_k :

$$b_{k-1} > b_k < b_{k+1}.$$

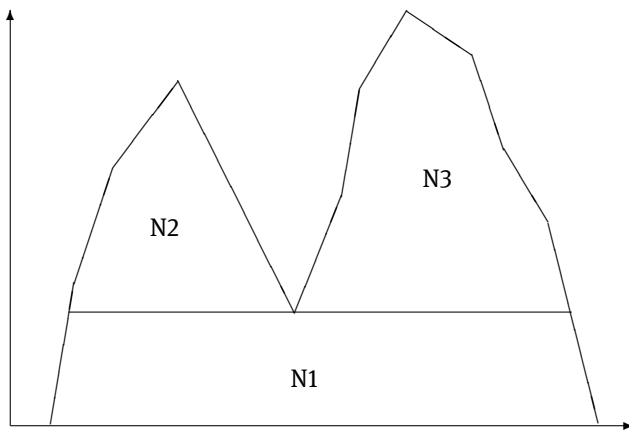


Figure 3: Piecewise linear function

Select the least among such local minima. Let this minimum has an index k . Then by a line $y = b_k$, we cut a hypograph into two parts. The bottom part corresponds to a unimodal function. The upper part corresponds to a function on a smaller interval, and this smaller interval can be divided into two intervals, on each of which it is possible to solve a problem of counting the sum of integer part of lengths of intervals cutting by lines $y = i$. Thus, the following decomposition has been derived (see figure 2):

$$N = N1 + N2 + N3,$$

where $N1$ corresponds to a solution of a problem for unimodal function, $N2$ is a solution of a problem for function on an interval $[a', a_k]$, $N3$ is a solution of a problem for function on an interval $[a_k, b']$, a' , b' are the left and right intersection of a line $Y = b_k$ with the graph of considered function.

Then the time of solution $T(n)$ of considered problem satisfies the recurrent ratio

$$T(n) = P(n) + T(k) + T(n - k),$$

where $P(n)$ is the time of the solution of a problem for trapezoidal component of a function plus elementary operations on calculation of k , a' , b' . It is easy to show that a solution of the last recurrent ratio requires an easily proved polynomial evaluation $T(n)$. Thus, taking into account that a problem of calculating the sum of integer parts of intervals cutting by lines $y = i$ in a hypograph of function is equivalent to a problem of optimum packing of unit squares in a hypograph of this function, the following theorem is proved.

Theorem 2. *For a problem of optimum packing of unit squares in a hypograph $H(f, a, b)$ of a piecewise linear function f on the interval $[a, b]$ there exists a polynomial algorithm.*

Let's remark, that the crude estimates allow to evaluate a computing time of the given algorithm as quadratic number of arithmetical operations. That is for a hypograph of function, consisting n vertices, the number of arithmetical operations required for calculating the maximum number of unit squares, which can be packed into the hypograph, is evaluated by $O(n^2)$.

4 Conclusions

The proved theorems make probable a hypothesis about an existence of polynomial algorithm for a general problem of optimum packing the unit squares in a polygon without holes. It is natural to consider at first the following weaker problems.

Problem 1. To prove the existence of a polynomial algorithm for problems of packing of unit squares in a convex polygon.

Problem 2. To prove the existence of a polynomial algorithm for the problem about packing of squares 2×2 in a rectilinear polygon without holes, whose all vertices have integer coordinates.

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