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Research Article

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Non-transient optimum design of nonlinear electromagnetic vibration-based energy harvester using homotopy perturbation method

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Abstract: In this paper the coupled differential equations governing the vibration of nonlinear electromagnetic energy harvesters are solved by the homotopy perturbation method. The amplitudes of odd harmonics of displacement of the magnet, coil current, and load voltage are derived up to the 5th harmonic. The frequency response of output power is plotted and it peaks at the linear mechanical resonance frequency. It should be noted that the optimum design of coil and load parameters, optimum electromagnetic coupling coefficient, and optimum vibration frequency of the magnet attached to a non-linear spring resulted in a stationary or non-transient vibration. Paying insufficient attention to this point and using typical parameters instead of optimum ones will result in transient vibration. The research aims at a rigorous semi-analytical method on a nonlinear problem which has previously solely investigated by numerical or experimental method.

Keywords: coupled differential equations; homotopy perturbation; nonlinear energy harvesting; nonlinear spring; optimum design; stationary or non-transient vibration

1 Introduction

The enormous growth has been witnessed in the realm of wireless devices in the past few decades. But, in many cases, the extent of services rendered by these devices has been dictated by the lifetime of batteries powering them. Presence of a self-sustainable power source would thus enable the exploitation of the full potential of such devices. Miniaturization is the prime motive behind the current technological revolution and as devices continue to shrink, less energy is required onboard (kamaraj, Ali, and Arockiarajan 2015). Nonlinear vibration frequently occurs in engineering problems among them the problem of nonlinear electromagnetic energy harvesters with spring non-linearity received substantial attention recently because of their high output power at the low resonance frequency. In this paper, the new perturbation method called the homotopy perturbation method is proposed, in contrast to the traditional perturbation methods, this technique does not require a small parameter in an equation. In this method, according to the homotopy technique, a homotopy with an embedding parameter p is constructed, and the embedding parameter is considered as 1, so the method is called the homotopy perturbation method, which can take full advantage of the traditional perturbation methods and homotopy techniques. To illustrate its effectiveness and its convenience, a Duffing equation with high order of nonlinearity is used (Anjum and He 2020; He 2000, 2003; Yu, He, and García 2019). Many researchers work on this subject without paying much attention to the optimum design of these harvesters so they design a harvester that vibrates with specified mechanical parameters at specified base acceleration and gives specified electrical power at generally their mechanical resonance frequency, and with the same mechanical as well as electrical parameters, these devices give different output power!. However, in this paper and the author's previous paper (Dezhara 2022a) it is proved that with the same parameters different output power can not be achieved with different mechanical configurations. With optimum parameters, the frequency response will peak at the mechanical resonance frequency for nonlinear vibration-based harvesters, but no power will be delivered to electrical loads for linear ones because efficiency is zero at the mechanical resonance frequency. Generally, it is difficult to provide an exact or closed-form solution to nonlinear dynamical equations. Therefore, researchers have attempted to implement approximate analytical solutions or numerical solutions for such equations. Approximate solutions provide important information to understand the characteristics of mechanical systems, specifically during the design procedure (Bayat et al. 2022). Li

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and Lim (Ghadimi and Kaliji 2013) applied Newton–harmonic balance (NHB) to the nonlinear oscillation systems to yield an approximate analytical expression for large amplitude problems. The accuracy of the high order of the NHB was verified, and it is not restricted to small and large positive parameters of the Duffing equation. Bayat, Pakar, and Cveticanin (2014) proposed the Hamiltonian Approach to achieve an accurate solution for nonlinear ordinary differential equations with inertia and static-type cubic nonlinearities. Their proposed solution is accurate for the whole domain. Homotopy perturbation Method (HPM) was first proposed by He (1999) and is widely used in nonlinear problems. Ganji, Tari, and Jooybari (2007) extended the HMP for nonlinear evolution equations and compared the results with the variational iteration method. Among all mentioned approaches, the Homotopy Perturbation Method (HPM) has been used widely for nonlinear systems. Compared to other approaches, it has the following advantages: (a) It is appropriate to be used for strong nonlinear oscillators, and (b) the solving procedure in higher order approximation is quite simple. In this paper, the high-order approximate Homotopy Perturbation Method (HPM) is developed to evaluate the nonlinear frequency response of an electromagnetic vibration-based energy harvester (EVEH) at a capacitive load. The governing equation of motion consists of three coupled differential equations one of them are second-order nonlinear differential equation and the other two equations are linear first-order differential equations. The coupled equations with linear and nonlinear properties are transformed into a set of differential–algebraic equations using intermediate variables. The obtained differential–algebraic equations are solved by HPM. It has been demonstrated the second order of the HPM can lead us to a highly accurate solution that is valid for the whole domain of the problem. It should be noted that in the view of mathematics, this research is novel research of its kind because the other researchers paid no or little attention to solving coupled differential equations using HPM (Figures 1 and 2).

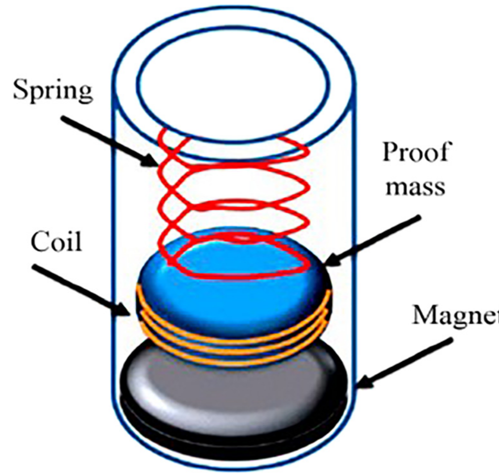

Figure 1: Typical nonlinear electromagnetic energy harvester (reprinted from reference (Kubba and Jiang 2014)).

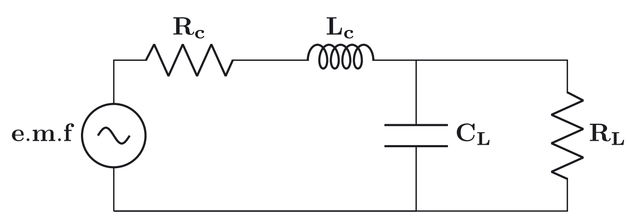

Figure 2: EVEH's electrical side for capacitive load.

2 Homotopy perturbation method

The coupled differentials equations of EVEH (Figure 1) are as follows (Aldawood, Nguyen, and Bardaweel 2019) (see Figure 2):

$$m\ddot{z} + C_m\dot{z} + k_1 z + k_3 z^3 + Ki = mA_b \sin(\omega t) \tag{1}$$

$$K\dot{z} = L_c\frac{\mathrm{d}i}{\mathrm{d}t} + R_c i + v \tag{2}$$

$$i = C_L\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{R_L} \tag{3}$$

where the z is magnet displacement and the i is coil current and the v is load voltage, C_m, K, m, A_b, k_1, k_3 are mechanical damping coefficient, electromagnetic coupling coefficient, the mass of the magnet, base acceleration, linear spring constant, and nonlinear spring constant respectively. It should be noted that in our analysis the sign of coefficients of k_1 and k_3 are not necessarily positive and they may possess negative values. In solving these coupled equations you should note that in arranging the homotopy equation of nonlinear differential equation of (1) the zero-order homotopy variable equation should be simple harmonic so that you will be able to solve higher-order homotopy variable differential equations. Consequently, the author arranges the first equation i.e. equation of (1) as follows:

$$\ddot{z} + \frac{k_1}{m}z + p\left\{\frac{C_m}{m}\dot{z} + \frac{k_3}{m}z^3 + \frac{K}{m}i - A_b\sin(\omega t)\right\} = 0 \tag{4}$$

$$\frac{k_1}{m} = \omega^2 + pa_1 + p^2 a_2 + \cdots \tag{5}$$

$$z = z_0 + pz_1 + p^2 z_2 + \cdots \quad z(0) = 0 \quad \dot{z}(0) = 0 \tag{6}$$

$$i = i_0 + pi_1 + p^2 i_2 + \cdots \quad i(0) = 0 \quad i'(0) = 0 \tag{7}$$

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \quad v(0) = 0 \quad v'(0) = 0 \tag{8}$$

where the prime and dot are derivative with respect to time. Now write the initial conditions according to homotopy theory:

$$z(0) = z_0(0) + pz_1(0) + p^2 z_2(0) = 0 = Z_0 + pZ_1 + p^2 Z_2 \tag{9}$$

$$\dot{z}(0) = \dot{z}_0(0) + p\dot{z}_1(0) + p^2\dot{z}_2(0) = 0 = V_0 + pV_1 + p^2 V_2 \tag{10}$$

$$i(0) = i_0(0) + pi_1(0) + p^2 i_2(0) = 0 = 0 + p0 + p^2 0 \tag{11}$$

$$v(0) = v_0(0) + pv_1(0) + p^2 v_2(0) = 0 = 0 + p0 + p^2 0 \tag{12}$$

Similarly the derivative of v and i at time $t = 0$ is zero. Note that we just expand the number of zero to homotopy variable p for z and not for i or v (the reason will be known later in this paper). Consequently, we have these initial conditions and relations:

$$\begin{array}{llll} z_0(0) & = Z_0 & z_1(0) = Z_1 & z_2(0) = Z_2 \\ \dot{z}_0(0) & = V_0 & \dot{z}_1(0) = V_1 & \dot{z}_2(0) = V_2 \\ i_0(0) & = 0 & i_1(0) = 0 & i_2(0) = 0 \\ v_0(0) & = 0 & v_1(0) = 0 & v_2(0) = 0 \\ 0 & = Z_0 + Z_1 + Z_2 & & \end{array} \tag{13}$$

$$0 = V_0 + V_1 + V_2 \tag{14}$$

By substituting equations of (5), (6), (7) into Equation (4):

$$\begin{aligned} &\ddot{z}_0 + p\ddot{z}_1 + p^2\ddot{z}_2 + (\omega^2 + pa_1 + p^2 a_2)(z_0 + pz_1 + p^2 z_2) \\ &+ p\frac{C_m}{m}(\dot{z}_0 + p\dot{z}_1 + p^2\dot{z}_2) \\ &+ p\frac{k_3}{m}(z_0^3 + p^3 z_1^3 + 3pz_0^2 z_1 + 3p^2 z_1^2 z_0 + \cdots) \\ &+ p\frac{K}{m}(i_0 + pi_1 + p^2 i_2) - pA_b \sin(\omega t) = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} p^0: \quad & \ddot{z}_0 + \omega^2 z_0 = 0 \quad z_0(0) = Z_0 \quad \dot{z}_0(0) = V_0 \\ p^1: \quad & \ddot{z}_1 + \omega^2 z_1 = -a_1 z_0 - \frac{C_m}{m}\dot{z}_0 - \frac{k_3}{m}z_0^3 - \\ & \frac{K}{m}i_0 + A_b \sin(\omega t) \quad z_1(0) = Z_1 \quad \dot{z}_1(0) = V_1 \\ p^2: \quad & \ddot{z}_2 + \omega^2 z_2 = -a_2 z_0 - a_1 z_1 - \\ & \frac{C_m}{m}\dot{z}_1 - 3\frac{k_3}{m}z_0^2 z_1 - \frac{K}{m}i_1 \quad z_2(0) = Z_2 \quad \dot{z}_2(0) = V_2 \end{aligned} \tag{16}$$

It should be noted that the first equation in (16) is a simple harmonic equation, second and third equations in (16) are coupled with current. To solve these coupled equations you should refer to other differential equations i.e. equations of (2) and (3). Accordingly, we proceed as follows: The solution of the first differential equation of (16) is:

$$z_0 = Z_0 \cos(\omega t) + \frac{V_0}{\omega}\sin(\omega t) \tag{17}$$

For solving the second differential equation you should first solve for i_0 and with known z_0 and i_0 one can solve for z_1. Now write the homotopy equation for equations of (2) and (3). For convenience, we can assume the prime is a symbol of the derivative with respect to time for coil current and load voltage. According to the equation of (3):

$$\begin{aligned} i_0 + pi_1 + p^2 i_2 &= C_L(v_0' + pv_1' + pv_2') \\ &+ \frac{1}{R_L}(v_0 + pv_1 + p^2 v_2) \end{aligned} \tag{18}$$

$$\begin{aligned} p^0: \quad & i_0 = C_L v_0' + \frac{v_0}{R_L} \\ p^1: \quad & i_1 = C_L v_1' + \frac{v_1}{R_L} \\ p^2: \quad & i_2 = C_L v_2' + \frac{v_2}{R_L} \end{aligned} \tag{19}$$

According to the Equation (2):

$$\begin{aligned} K(\dot{z}_0 + p\dot{z}_1 + p^2\dot{z}_2) = {} & L_c(i_0' + pi_1' + i_2') \\ & + R_c(i_0 + pi_1 + p^2 i_2) \\ & + v_0 + pv_1 + p^2 v_2 \end{aligned} \tag{20}$$

$$\begin{aligned} p^0: \quad & K\dot{z}_0 = L_c i_0' + R_c i_0' + v_0 \\ p^1: \quad & K\dot{z}_1 = L_c i_1' + R_c i_1' + v_1 \\ p^2: \quad & K\dot{z}_2 = L_c i_2' + R_c i_0' + v_2 \end{aligned} \tag{21}$$

Now consider the first equation of (19) and combine it with the first equation of (21) and eliminate i_0. As we know z_0 from the equation of (17), we can solve for v_0 and consequently i_0. we proceed as follows. First eliminate i_0:

$$L_c C_L v_0'' + \left(\frac{L_c}{R_L} + R_c C_L\right)v_0' + \left(1 + \frac{R_c}{R_L}\right)v_0 = K\dot{z}_0 \tag{22}$$

If you solve for v_0 and substitute it in the first equation of (19) the i_0 will be obtained and consequently we can solve for z_1 from the second equation of (16). The reason for not expanding the zero initial condition of v and i now became known, the differential equation of v has a damping term and the transient response of it became evanescent as time goes by, hence we assumed all initial conditions of v and i are zero. However, the z has not this condition i.e. has not had the damping term. Now turn to the differential equation of (22):

$$L_c C_L v_0'' + \left(\frac{L_c}{R_L} + R_c C_L\right)v_0' + \left(1 + \frac{R_c}{R_L}\right)v_0 = \tag{23}$$

$$-K\omega Z_0 \sin(\omega t) + KV_0 \cos(\omega t) \quad (24)$$

The forced response of the above equation is as follows:

$$v_0 = A\cos(\omega t) + B\sin(\omega t) \quad (25)$$

By substituting the above equation into the differential equation of (24) we have:

$$\begin{aligned} &-a_1\omega^2 A\cos(\omega t) - a_1\omega^2 B\sin(\omega t) \\ &-a_2 A\omega \sin(\omega t) + a_2 B\omega \cos(\omega t) + a_3 A\cos(\omega t) \\ &+a_3 B\sin(\omega t) = KV_0\cos(\omega t) - K\omega Z_0 \sin(\omega t) \end{aligned} \quad (26)$$

where a_1, a_2, and a_3 are coefficients of derivatives in the equation of (24).

$$a_1 = L_c C_L \quad (27)$$

$$a_2 = \frac{L_c}{R_L} + R_c C_L \quad (28)$$

$$a_3 = 1 + \frac{R_c}{R_L} \quad (29)$$

By arranging equation of (26) we have:

$$\begin{aligned} (-a_1\omega^2 + a_3)A + a_2\omega B &= KV_0 \\ (-a_1\omega^2 + a_3)B - a_2\omega A &= -KZ_0\omega \end{aligned}$$

In the matrix form, we have:

$$\begin{bmatrix} -a_1\omega^2 + a_3 & a_2\omega \\ -a_2\omega & -a_1\omega^2 + a_3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} KV_0 \\ -KZ_0\omega \end{bmatrix}$$

$$A = a_2 K\omega^2 Z_0 - KV_0\left(a_1\omega^2 - a_3\right) \quad (30)$$

$$B = K\omega Z_0\left(a_1\omega^2 - a_3\right) + K\omega V_0 a_2 \quad (31)$$

Now that the v_0 is known we derive i_0 using the first equation of (19).

$$i_0 = \left(\frac{A}{R_L} + C_L\omega B\right)\cos(\omega t) + \left(\frac{B}{R_L} - C_L\omega A\right)\sin(\omega t) \quad (32)$$

$$\lambda_1 = \frac{A}{R_L} + C_L\omega B \quad (33)$$

$$\lambda_2 = \frac{B}{R_L} - C_L\omega A \quad (34)$$

where λ_1 and λ_2 are arbitrary names for coefficients of i_0. Turning to second Equation (16) i.e. differential equation of z_1, we have:

$$\ddot{z}_1 + \omega^2 z_1 = -a_1 z_0 - \frac{C_m}{m} z_0 - \frac{k_3}{m} z_0^3 - \frac{K}{m} i_0 + A_b \sin(\omega t)$$

putting the i_0 into the above differential equation:

$$\begin{aligned} \ddot{z}_1 + \omega^2 z_1 &= -a_1 Z_0 \cos(\omega t) - \frac{a_1 V_0}{\omega}\sin(\omega t) \\ &+\frac{C_m}{m}\omega Z_0 \sin(\omega t) - \frac{k_3}{m} Z_0^3 \\ &\times\left(\frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t)\right) \\ &+ - \frac{K}{m}\lambda_1 \cos(\omega t) - \frac{K}{m}\lambda_2 \sin(\omega t) + A_b \sin(\omega t) \end{aligned} \quad (35)$$

To avoid secular terms the coefficients of cos(ωt) and sin(ωt) at the right side of Equation (35) should be equaled to zero.

$$-a_1 Z_0 - \frac{K}{m}\lambda_1 - \frac{3k_3 Z_0^3}{4m} = 0 \quad (36)$$

$$-\frac{a_1 V_0}{\omega} + \frac{C_m \omega Z_0}{m} - \frac{K}{m}\lambda_2 + A_b = 0 \quad (37)$$

From the equation of (36) the value of unknown parameter a_1 will be known. And from the equation of (37), the value of Z_0 will be known.

$$a_1 = -\left(\frac{K\lambda_1}{mZ_0} + \frac{3k_3 Z_0^2}{4m}\right) \quad (38)$$

$$Z_0 = \frac{K\lambda_2 - mA_b}{C_m\omega} + \frac{mV_0 a_1}{C_m\omega^2} \quad (39)$$

After omitting the secular terms the differential equation of z_1 is:

$$\ddot{z}_1 + \omega^2 z_1 = \frac{-k_3 Z_0^3}{4m}\cos(3\omega t) \quad (40)$$

Now we can solve for the z_1, assume the forced response of z_1 is:

$$z_1 = D\cos(3\omega t) \quad (41)$$

putting the equation of (41) into the Equation (40), we have:

$$-9D\omega^2 + \omega^2 D = -\frac{k_3 Z_0^3}{4m}$$

The result is:

$$D = \frac{k_3 Z_0^3}{32m\omega^2} \quad (42)$$

The general solution of z_1 is:

$$z_1 = \gamma_1 \cos(\omega t) + \gamma_2 \sin(\omega t) + D\cos(3\omega t) \quad (43)$$

$$z_1(0) = Z_1 \qquad \dot{z}_1(0) = V_1 \quad (44)$$

Hence we have:

$$\gamma_1 + D = Z_1 \qquad \gamma_2\omega = V_1 \tag{45}$$

$$z_1 = (Z_1 - D)\cos(\omega t) + \frac{V_1}{\omega}\sin(\omega t) + D\cos(3\omega t) \tag{46}$$

Now turn to the third differential equation of (16), we should first derive i_1 and then solve for z_2. Combining i_1 into the second equation of (19) and as we derived z_1 we can solve for v_1 and consequently i_1.

$$L_cC_Lv_1'' + \left(\frac{L_c}{R_L} + R_cC_L\right)v_1' + \left(1 + \frac{R_c}{R_L}\right)v_1 = Kz_1 \tag{47}$$

assume v_1 as:

$$v_1 = A_1\cos(\omega t) + B_1\sin(\omega t) + C_1\cos(3\omega t) + D_1\sin(3\omega t) \tag{48}$$

By substituting (48) into the differential equation of (47), we have:

$$\begin{aligned}
&(-\alpha_1\omega^2)B_1 + (-\alpha_2\omega)A_1 + \alpha_3B_1 = -K\omega(Z_1 - D)\\
&(-\alpha_1\omega^2)A_1 + (\alpha_2\omega)B_1 + \alpha_3A_1 = KV_1\\
&(-9\alpha_1\omega^2)D_1 + (-3\alpha_2\omega)C_1 + \alpha_3D_1 = -3K\omega D\\
&(-9\alpha_1\omega^2)C_1 + (3\alpha_2\omega)D_1 + \alpha_3C_1 = 0
\end{aligned} \tag{49}$$

$$\begin{bmatrix} -\alpha_2\omega & -\alpha_1\omega^2+\alpha_3 & 0 & 0\\ -\alpha_1\omega^2+\alpha_3 & \alpha_2\omega & 0 & 0\\ 0 & 0 & -3\alpha_2\omega & -9\alpha_1\omega^2+\alpha_3\\ 0 & 0 & -9\alpha_1\omega^2+\alpha_3 & 3\alpha_2\omega \end{bmatrix}\begin{bmatrix}A_1\\B_1\\C_1\\D_1\end{bmatrix} = \begin{bmatrix}-K\omega(Z_1-D)\\ KV_1\\ -3KD\omega\\ 0\end{bmatrix}$$

After solving the above matrix equation symbolically in MATLAB we have:

$$\begin{aligned}
A_1 &= \frac{K(\alpha_3V_1 - D\alpha_2\omega^2 - V_1\alpha_1\omega^2 + Z_1\alpha_2\omega^2)}{\delta_1}\\
B_1 &= \frac{K\omega(\alpha_3D + V_1\alpha_2 - Z_1\alpha_3 - D\alpha_1\omega^2 + Z_1\alpha_1\omega^2)}{\delta_1}\\
C_1 &= \frac{9K\alpha_2\omega^2D}{\delta_2}\\
D_1 &= \frac{-3DK\omega(-9\alpha_1\omega^2 + \alpha_3)}{\delta_2}\\
\delta_1 &= \alpha_1^2\omega^4 - 2\alpha_1\alpha_3\omega^2 + \alpha_2^2\omega^2 + \alpha_3^2\\
\delta_2 &= 81\alpha_1^2\omega^4 - 18\alpha_1\alpha_3\omega^2 + 9\alpha_2^2\omega^2 + \alpha_3^2
\end{aligned} \tag{50}$$

From the second equation of (19) we have:

$$i_1 = \lambda_3\cos(\omega t) + \lambda_4\sin(\omega t) + \lambda_5\cos(3\omega t) + \lambda_6\sin(3\omega t) \tag{51}$$

where:

$$\begin{aligned}
\lambda_3 &= \frac{A_1}{R_L} + C_L\omega B_1\\
\lambda_4 &= \frac{B_1}{R_L} - C_L\omega A_1\\
\lambda_5 &= \frac{C_1}{R_L} + 3\omega C_LD_1\\
\lambda_6 &= \frac{D_1}{R_L} - 3C_L\omega C_1
\end{aligned} \tag{52}$$

Turning to third Equation (16) i.e. differential equation for z_2, we have:

$$\ddot{z}_2 + \omega^2z_2 = -\alpha_2z_0 - \alpha_1z_1 - \frac{C_m}{m}\dot{z}_1 - \frac{3k_3}{m}z_0^2z_1 - \frac{K}{m}i_1$$

By substituting the i_1 into the above differential equation:

$$\begin{aligned}
\ddot{z}_2 + \omega^2z_2 = &-\alpha_2Z_0\cos(\omega t) - \frac{\alpha_2V_0}{\omega}\sin(\omega t) - \alpha_1(Z_1 - D)\\
&\times\cos(\omega t) - \frac{\alpha_1V_1}{\omega}\sin(\omega t) - \alpha_1D\cos(3\omega t)\\
&-\frac{C_m}{m}(Z_1 - D)\cos(\omega t) - \frac{C_mV_1}{m\omega}\sin(\omega t)\\
&-\frac{C_m}{m}D\cos(3\omega t) - \frac{3k_3}{m}\left(Z_0^2\cos^2(\omega t) + \frac{V_0^2}{\omega^2}\right.\\
&\left.\times\sin^2(\omega t) + \frac{Z_0V_0}{\omega}\sin(2\omega t)\right)\\
&\times\left((Z_1 - D)\cos(\omega t) + \frac{V_1}{\omega}\sin(\omega t)\right.\\
&\left.+D\cos(3\omega t)\right) - \frac{K}{m}\lambda_3\cos(\omega t) - \frac{K}{m}\lambda_4\sin(\omega t)\\
&-\frac{K}{m}\lambda_5\cos(3\omega t) - \frac{K}{m}\lambda_6\sin(3\omega t)
\end{aligned} \tag{53}$$

The multiplication action between the two big parentheses on the right-hand side of the differential equation of (53) can be simplified as follows:

$$\begin{aligned}
&Z_0^2(Z_1 - D)\cos^3(\omega t) + \frac{Z_0^2V_1}{\omega}\cos^2(\omega t)\sin(\omega t) + DZ_0^2\cos^2\\
&\times(\omega t)\cos(3\omega t) + \frac{V_0^2(Z_1 - D)}{\omega^2}\sin^2(\omega t)\cos(\omega t)\\
&+\frac{V_0^2V_1}{\omega^3}\sin^3(\omega t) + \frac{V_0^2D}{\omega^2}\sin^2(\omega t)\cos(3\omega t)\\
&+\frac{Z_0V_0(Z_1 - D)}{\omega}\cos(\omega t)\sin(2\omega t) + \frac{Z_0V_0V_1}{\omega^2}\sin(\omega t)\\
&\times\sin(2\omega t) + \frac{Z_0V_0D}{\omega}\sin(2\omega t)\cos(3\omega t)
\end{aligned}$$

Finally, after simplifying the above term the differential equation became:

$$\begin{aligned}
\ddot{z}_2 + \omega^2 z_2 &= \Big(-a_2 Z_0 \cos(\omega t) - \frac{a_2 V_0}{\omega}\sin(\omega t)\Big) \\
&-\Big(a_1 (Z_1 - D)\cos(\omega t) + \frac{a_1 V_1}{\omega}\sin(\omega t) \\
&+a_1 D\cos(3\omega t)\Big) - \Big(\frac{C_m}{m}(Z_1 - D)\cos(\omega t) \\
&+\frac{C_m V_1}{m\omega}\sin(\omega t) + \frac{C_m D}{m}\cos(3\omega t)\Big) \\
&-\frac{3k_3}{m}\Big(Z_0^2 (Z_1 - D)\Big(\frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t)\Big) \\
&+\frac{Z_0^2 V_1}{2\omega}\Big(\sin(\omega t) + \frac{1}{2}(\sin(3\omega t) - \sin(\omega t))\Big) \\
&+\frac{Z_0^2 D}{2}\Big(\cos(3\omega t) + \frac{1}{2}(\cos(5\omega t) + \cos(\omega t))\Big) \\
&+\frac{V_0^2 (Z_1 - D)}{2\omega^2}\Big(\cos(\omega t) - \frac{1}{2}(\cos(3\omega t) + \cos(\omega t))\Big) \\
&+\frac{V_0^2 V_1}{\omega^3}\Big(\frac{-1}{4}\sin(3\omega t) + \frac{3}{4}\sin(\omega t)\Big) \\
&+\frac{V_0^2 D}{2\omega^2}\Big(\cos(3\omega t) - \frac{1}{2}(\cos(5\omega t) + \cos(\omega t))\Big) \\
&+\frac{Z_0 V_0 (Z_1 - D)}{2\omega}(\sin(3\omega t) + \sin(\omega t)) \\
&+\frac{Z_0 V_0 V_1}{2\omega^2}(\cos(3\omega t) - \cos(\omega t)) \\
&+\frac{Z_0 V_0 D}{2\omega}(\sin(5\omega t) - \sin(\omega t))\Big) \\
&-\Big(\frac{K}{m}\lambda_3\cos(\omega t) + \frac{K}{m}\lambda_4\sin(\omega t) \\
&+\frac{K}{m}\lambda_5\cos(3\omega t) + \frac{K}{m}\lambda_6\sin(3\omega t)\Big)
\end{aligned} \tag{54}$$

To avoid secular terms the coefficients of cos(ωt) and sin(ωt) at the right side of Equation (54) should be equaled to zero.

$$\begin{aligned}
&a_2 Z_0 + a_1 (Z_1 - D) + \frac{C_m}{m}(Z_1 - D) \\
&+\frac{3k_3}{m}\Big(\frac{3Z_0^2 (Z_1 - D)}{4} + \frac{Z_0^2 D}{4} \\
&-\frac{V_0^2 (Z_1 - D)}{4\omega^2} - \frac{V_0^2 D}{4\omega^2} - \frac{Z_0 V_0 V_1}{2\omega^2}\Big) + \frac{K\lambda_3}{m} = 0
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\frac{a_2 V_0}{\omega} + \frac{a_1 V_1}{\omega} + \frac{C_m V_1}{m\omega} + \frac{3k_3}{m}\Big(\frac{Z_0^2 V_1}{4\omega} + \frac{3V_0^2 V_1}{4\omega^3} \\
&+\frac{Z_0 V_0 (Z_1 - D)}{2\omega} - \frac{Z_0 V_0 D}{2\omega}\Big) + \frac{K\lambda_4}{m} = 0
\end{aligned} \tag{56}$$

These equations are vital in the design process. We will use them directly without simplification in MALTAB to derive optimum parameters numerically.

Thus Equation (54) became:

$$\begin{aligned}
\ddot{z}_2 + \omega^2 z_2 &= \beta_1\cos(3\omega t) + \beta_2\sin(3\omega t) + \beta_3\cos(5\omega t) \\
&+\beta_4\sin(5\omega t)
\end{aligned} \tag{57}$$

where:

$$\begin{aligned}
\beta_1 &= -\Big(a_1 D + \frac{C_m D}{m} + \frac{3k_3}{m}\Big(\frac{Z_0^2 (Z_1 - D)}{4} \\
&+\frac{Z_0^2 D}{2} - \frac{V_0^2 (Z_1 - D)}{4\omega^2} \\
&+\frac{V_0^2 D}{2\omega^2} + \frac{Z_0 V_0 V_1}{2\omega^2}\Big) + \frac{K\lambda_5}{m}\Big)
\end{aligned} \tag{58}$$

$$\begin{aligned}
\beta_2 &= -\Big(\frac{3k_3}{m}\Big(\frac{Z_0^2 V_1}{4\omega} - \frac{V_0^2 V_1}{4\omega^3} + \frac{Z_0 V_0 (Z_1 - D)}{2\omega}\Big) \\
&+\frac{K\lambda_6}{m}\Big)
\end{aligned} \tag{59}$$

$$\beta_3 = -\frac{3k_3}{m}\Big(\frac{Z_0^2 D}{4} - \frac{V_0 D}{4\omega^2}\Big) \tag{60}$$

$$\beta_4 = -\frac{3k_3}{m}\Big(\frac{Z_0 V_0 D}{2\omega}\Big) \tag{61}$$

Now we can solve for the z_2, assume z_2 is:

$$\begin{aligned}
z_2 &= \gamma_3\cos(\omega t) + \gamma_4\sin(\omega t) + D_2\cos(3\omega t) \\
&+E_2\sin(3\omega t) + F_2\cos(5\omega t) + G_2\sin(5\omega t)
\end{aligned} \tag{62}$$

By substituting Equation (62) into Equation (57) and after simplifying we can solve for constants of D_2, E_2, F_2 and G_2.

$$D_2 = \frac{-\beta_1}{8\omega^2} \tag{63}$$

$$E_2 = \frac{-\beta_2}{8\omega^2} \tag{64}$$

$$F_2 = \frac{-\beta_3}{24\omega^2} \tag{65}$$

$$G_2 = \frac{-\beta_4}{24\omega^2} \tag{66}$$

Applying initial conditions we have:

$$\begin{aligned}
z_2(0) &= Z_2 \rightarrow \gamma_3 + D_2 + F_2 = Z_2 \\
\dot{z}_2(0) &= V_2 \rightarrow \omega\gamma_4 + 3\omega E_2 + 5\omega G_2 = V_2
\end{aligned}$$

Hence:

$$\gamma_3 = Z_2 + \frac{\beta_1}{8\omega^2} + \frac{\beta_3}{24\omega^2} \quad (67)$$

$$\gamma_4 = \frac{V_2}{\omega} + \frac{3\beta_2}{8\omega^2} + \frac{5\beta_4}{24\omega^2} \quad (68)$$

To derive v_2 from the third equation of (19) eliminate i_2 from this equation using the third equation of (18).

$$\alpha_1 v_2'' + \alpha_2 v_2' + \alpha_3 v_2 = K\dot{z}_2 \quad (69)$$

since z_2 has the $\cos(3\omega t)$ and $\cos(5\omega t)$ terms we can assume v_2 as follows:

$$\begin{aligned} v_2 \; &= A_2\cos(\omega t) + B_2\sin(\omega t) + C_2\cos(3\omega t) \\ &+H_2\sin(3\omega t) + J_2\cos(5\omega t) + N_2\sin(5\omega t) \end{aligned} \quad (70)$$

By putting (70) into the Equation (69) and simplifying the resulted relations we have:

$$\begin{aligned} &(-\alpha_1\omega^2)A_2 + (\alpha_2\omega)B_2 + \alpha_3 A_2 = K(\omega\gamma_4) \\ &(-\alpha_1\omega^2)B_2 + (-\alpha_2\omega)A_2 + \alpha_3 B_2 = K(-\omega\gamma_3) \\ &(-9\alpha_1\omega^2)C_2 + (3\alpha_2\omega)H_2 + \alpha_3 C_2 = K(3\omega E_2) \\ &(-9\alpha_1\omega^2)H_2 + (-3\alpha_2\omega C_2) + \alpha_3 H_2 = K(-3\omega D_2) \\ &(-25\alpha_1\omega^2)J_2 + (5\alpha_2\omega)N_2 + \alpha_3 J_2 = K(5\omega G_2) \\ &(-25\alpha_1\omega^2)N_2 + (-5\alpha_2\omega)J_2 + \alpha_3 N_2 = K(-5\omega F_2) \end{aligned} \quad (71)$$

$$\begin{bmatrix} -\omega^2\alpha_1+\alpha_3 & \omega\alpha_2 & 0 & 0 & 0 & 0 & \\ -\omega\alpha_2 & -\omega^2\alpha_1+\alpha_3 & 0 & 0 & 0 & 0 & \\ 0 & 0 & -9\omega^2\alpha_1+\alpha_3 & 3\omega\alpha_2 & 0 & 0 & \\ 0 & 0 & -3\omega\alpha_2 & -9\omega^2\alpha_1+\alpha_3 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & -25\alpha_1\omega^2+\alpha_3 & 5\omega\alpha_2 \\ & 0 & 0 & 0 & 0 & -5\omega\alpha_2 & -25\alpha_1\omega^2+\alpha_3 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ H_2 \\ J_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} K\omega\gamma_4 \\ -K\omega\gamma_3 \\ 3K\omega E_2 \\ -3K\omega D_2 \\ 5K\omega G_2 \\ -5K\omega F_2 \end{bmatrix}$$

you can solve the above matrix equation in MATLAB symbolically, the result is:

$$A_2 = \frac{K\omega(-\alpha_1\gamma_4\omega^2 + \alpha_2\gamma_3\omega + \alpha_3\gamma_4)}{\delta_3} \quad (72)$$

$$B_2 = \frac{K\omega(\alpha_1\gamma_3\omega^2 + \alpha_2\gamma_4\omega - \alpha_3\gamma_3)}{\delta_3} \quad (73)$$

$$C_2 = \frac{3K\omega(-9\alpha_1 E_2\omega^2 + 3\alpha_2 D_2\omega + \alpha_3 E_2)}{\delta_4} \quad (74)$$

$$H_2 = \frac{3K\omega(9\alpha_1 D_2\omega^2 + 3\alpha_2 E_2\omega - \alpha_3 D_2)}{\delta_4} \quad (75)$$

$$J_2 = \frac{5K\omega(-25\alpha_1 G_2\omega^2 + 5\alpha_2 F_2\omega + \alpha_3 G_2)}{\delta_5} \quad (76)$$

$$N_2 = \frac{5K\omega(25\alpha_1 F_2\omega^2 + 5\alpha_2 G_2\omega - \alpha_3 F_2)}{\delta_5} \quad (77)$$

where δ_3, δ_4 and δ_5 are:

$$\begin{aligned} \delta_3 \; &= \alpha_1^2\omega^4 - 2\alpha_1\alpha_3\omega^2 + \alpha_2^2\omega^2 + \alpha_3^2 \\ \delta_4 \; &= 81\alpha_1^2\omega^4 - 18\alpha_1\alpha_3\omega^2 + 9\alpha_2^2\omega^2 + \alpha_3^2 \\ \delta_5 \; &= 625\alpha_1^2\omega^4 - 50\alpha_1\alpha_3\omega^2 + 25\alpha_2^2\omega^2 + \alpha_3^2 \end{aligned} \quad (78)$$

Now that we know v_2 we can derive i_2 from third equation of (18):

$$\begin{aligned} i_2 \; &= \lambda_7\cos(\omega t) + \lambda_8\sin(\omega t) + \lambda_9\cos(3\omega t) \\ &+\lambda_{10}\sin(3\omega t) + \lambda_{11}\cos(5\omega t) + \lambda_{12}\sin(5\omega t) \end{aligned} \quad (79)$$

where the constants are as follows:

$$\lambda_7 = C_L\omega B_2 + \frac{A_2}{R_L} \quad (80)$$

$$\lambda_8 = -C_L\omega A_2 + \frac{B_2}{R_L} \quad (81)$$

$$\lambda_9 = 3C_L\omega H_2 + \frac{C_2}{R_L} \quad (82)$$

$$\lambda_{10} = -3C_L\omega C_2 + \frac{H_2}{R_L} \quad (83)$$

$$\lambda_{11} = 5C_L\omega N_2 + \frac{J_2}{R_L} \quad (84)$$

$$\lambda_{12} = -5C_L\omega J_2 + \frac{N_2}{R_L} \quad (85)$$

Now you can add these currents or voltages or even displacements to achieve the unknowns dependent variables in coupled differential equations. If we take the homotopy variable (p) as (1), we have:

$$\begin{aligned} z \; &= z_0 + z_1 + z_2 \\ i \; &= i_0 + i_1 + i_2 \\ v \; &= v_0 + v_1 + v_2 \\ \omega^2 \; &= \frac{k_1}{m} - a_1 - a_2 \\ Z_0 + Z_1 + Z_2 \; &= 0 \\ V_0 + V_1 + V_2 \; &= 0 \end{aligned} \quad (86)$$

where a_1 and a_2 obtained from relations of (38) and (56) or (55) respectively. The displacement of magnet is:

$$\begin{aligned} z &= (Z_0 + \gamma_1 + \gamma_3)\cos(\omega t) + \left(\frac{V_0}{\omega}\gamma_2 + \gamma_4\right)\sin(\omega t) \\ &\quad + (D + D_2)\cos(3\omega t) + E_2\sin(3\omega t) \\ &\quad + F_2\cos(5\omega t) + G_2\sin(5\omega) \end{aligned} \tag{87}$$

The coil current is:

$$\begin{aligned} i &= (\lambda_1 + \lambda_3 + \lambda_7)\cos(\omega t) + (\lambda_2 + \lambda_4 + \lambda_8)\sin(\omega t) \\ &\quad + (\lambda_5 + \lambda_9)\cos(3\omega t) + (\lambda_6 + \lambda_{10})\sin(3\omega t) \\ &\quad + \lambda_{11}\cos(5\omega t) + \lambda_{12}\sin(5\omega t) \end{aligned} \tag{88}$$

The load voltage is:

$$\begin{aligned} v &= (A + A_1 + A_2)\cos(\omega t) + (B + B_1 + B_2)\sin(\omega t) \\ &\quad + (C_1 + C_2)\cos(3\omega t) + (D_1 + H_2)\sin(3\omega t) \\ &\quad + J_2\cos(5\omega t) + N_2\sin(5\omega t) \end{aligned} \tag{89}$$

The resonance frequency is:

$$\begin{aligned} \omega^2 &= \frac{k_1}{m} + \frac{3k_3Z_0^2}{4m} + \frac{K\lambda_1}{mZ_0} - a_2 \\ \omega &= f(Z_0, Z_1, V_0, V_1) \end{aligned} \tag{90}$$

To plot frequency response after designing the optimum parameters of the system you should refer to relation (90). Note that the Equation (90) is a function of Z_1, V_0, V_1, in order to eliminate these parameters and derive frequency with respect to just Z_0 and plotting frequency response you should solve symbolically (4) equations from omitting secular terms with (6) symbolic variables i.e. ω, a_1, a_2, Z_1, V_0, V_1 and two initial homotopy relation (i.e. relations of (13) and (14)) which itself add two more parameters of Z_2 and V_2 which totally became six equation and eight parameters. The other two equation comes from the first two initial conditions i.e. $z(0) = 0$ and $z(0) = 0$ which help us to eliminate the parameters of Z_2 and V_2. After eliminating the above-mentioned parameters we have an implicit formula in terms of ω and Z_0.

The initial conditions are:

$$\begin{aligned} z(0) &= 0 \rightarrow Z_0 + \gamma_1 + \gamma_3 + D + D_2 + F_2 = 0 \\ z(0) &= 0 \rightarrow \frac{V_0}{\omega}\gamma_2 + \gamma_4 + E_2 + G_2 = 0 \\ i(0) &= 0 \rightarrow \lambda_1 + \lambda_3 + \lambda_7 + \lambda_5 + \lambda_9 + \lambda_{11} = 0 \\ i'(0) &= 0 \rightarrow \lambda_2 + \lambda_4 + \lambda_8 + \lambda_6 + \lambda_{10} + \lambda_{12} = 0 \\ v(0) &= 0 \rightarrow A + A_1 + A_2 + C_1 + C_2 + J_2 = 0 \\ v'(0) &= 0 \rightarrow B + B_1 + B_2 + D_1 + H_2 + N_2 = 0 \end{aligned} \tag{91}$$

To design optimum parameters of the system, we have (6) equations from initial conditions, (2) equations from initial homotopy relations i.e. equations of (13) and (14) plus (4) equations as a result of omitting the secular terms and one equation of frequency-amplitude i.e. relation of (90) which add up to (13) equations with (13) unknowns. Theses unknown includes R_L, C_L, R_c, L_c, ω, a_1, a_2 and (6) unknown from initial homotopy parameters. Solving these (13) nonlinear equations with (13) unknowns using fsolve command in MATLAB results in optimum parameters. Consequently, the amplitudes of sinusoidal terms of v, i, and z can be calculated.

It should be noted that the amplitudes of v, i, z are functions of (4) coil and load parameters i.e. R_L, C_L, R_c, L_c and frequency as well as initial homotopy parameters i.e. Z_0, Z_1, Z_2 and V_0, V_1, V_2.

3 Numerical example

In this example with just known values of spring constants of k_1 and k_3 and the mass of the magnet, we will design the (9) optimum parameters that lead to non-transient vibration and give us the maximum output power at this stationary situation. Consider the values of k_1, k_3 and m as follows:

According to our design procedure and based on MATLAB calculation the optimum parameters that lead to stationary or non-transient vibration are (Table 2):

Table 1: Known values ($g = 9.81\ \frac{\text{Kg m}}{\text{s}^2}$).

Known parameter	Value	Unit
k_1	700	$\frac{\text{N}}{\text{m}}$
k_3	60	$\frac{\text{N}}{\text{m}^3}$
m	14	gr
C_m	0.35	$\frac{\text{N s}}{\text{m}}$
K	3.5	$\frac{\text{N}}{\text{A}}$
A_b	1.8 g	$\frac{\text{Kg m}}{\text{s}^2}$

Table 2: Optimum parameters for known values of Table 1.

Number	Optimum parameter	Value	Unit
1	R_L	28.5	kΩ
2	C_L	5.14	nF
3	R_c	10.38	Ω
4	L_c	1.8075	H
5	w	223.6	$\frac{\text{rad}}{\text{s}}$
6	a_1	−7.64	$\frac{\text{rad}}{\text{s}}$
7	a_2	10.45	$\frac{\text{rad}}{\text{s}}$
8	Z_0	−3.2	mm
9	Z_1	4	mm
10	Z_2	−0.883	mm
11	V_0	−793.1	$\frac{\text{mm}}{\text{s}}$
12	V_1	790.3	$\frac{\text{mm}}{\text{s}}$
13	V_2	2.8	$\frac{\text{mm}}{\text{s}}$

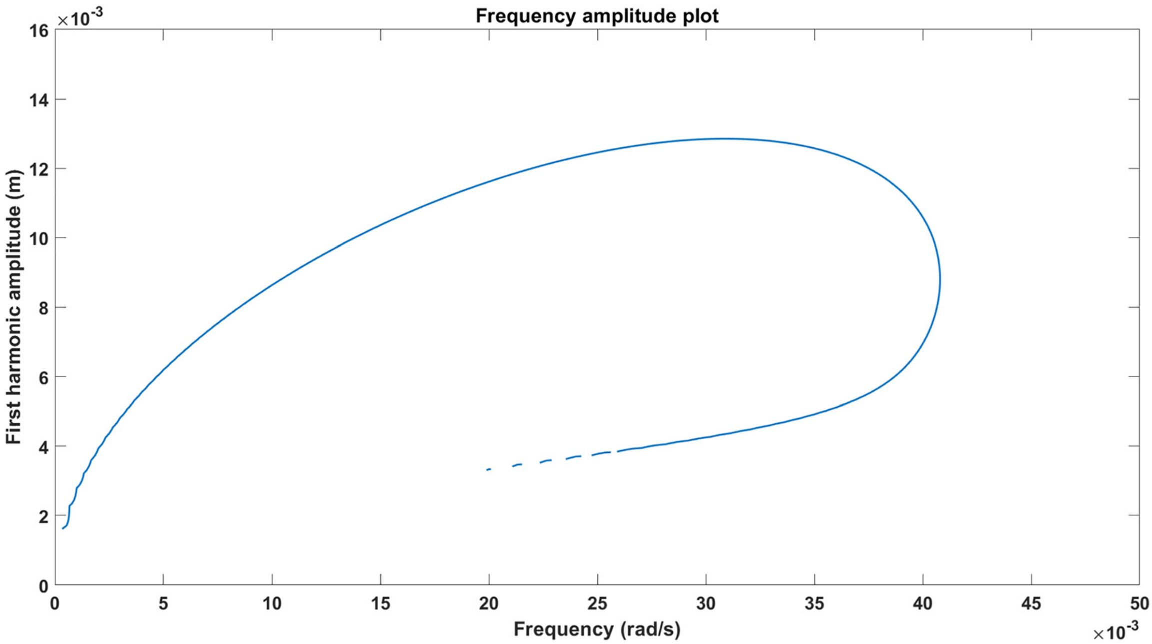

Figure 3: First harmonic displacement frequency response.

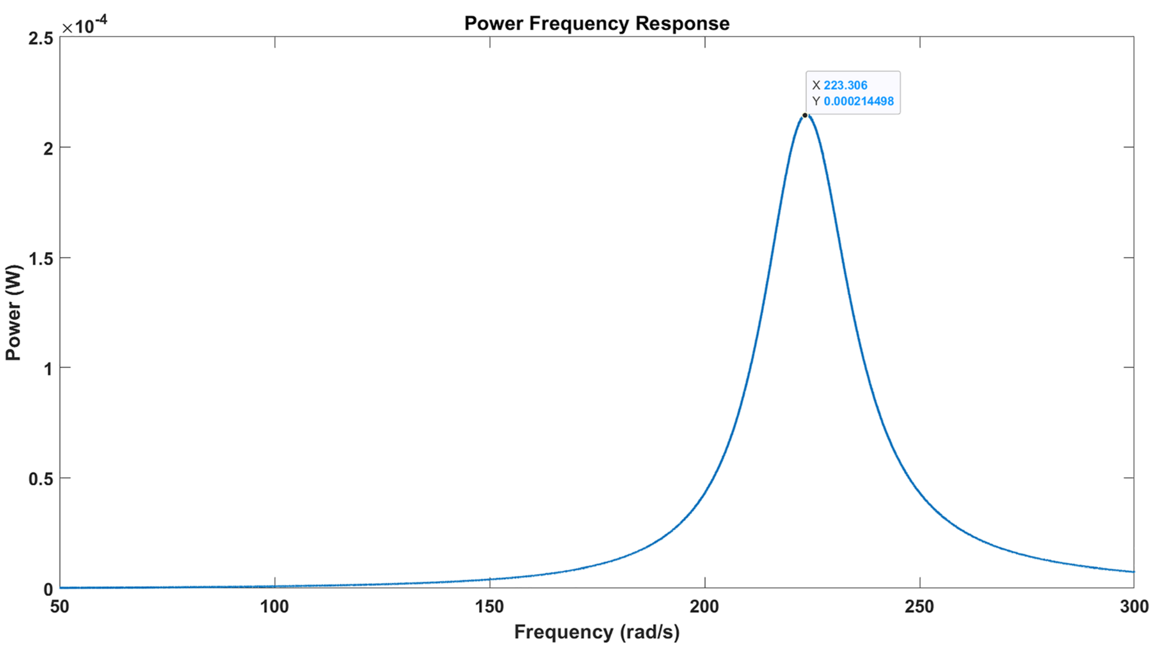

Figure 4: Power frequency response.

The frequency response based on the above parameters is as follows (Figures 3–7):

Note that in contrast to linear EVEH whose frequency response peaks at a different frequency in comparison to mechanical resonance frequency (Dezhara 2022b), the frequency response of the nonlinear one peaks at the linear mechanical resonance frequency.

4 Conclusions

We conclude that the homotopy method results in a more accurate response than other methods such as harmonic balance which assumes the response is simple harmonic and ignores higher harmonics. Plotting the frequency response after the optimum parameters design of EVEH is a good way

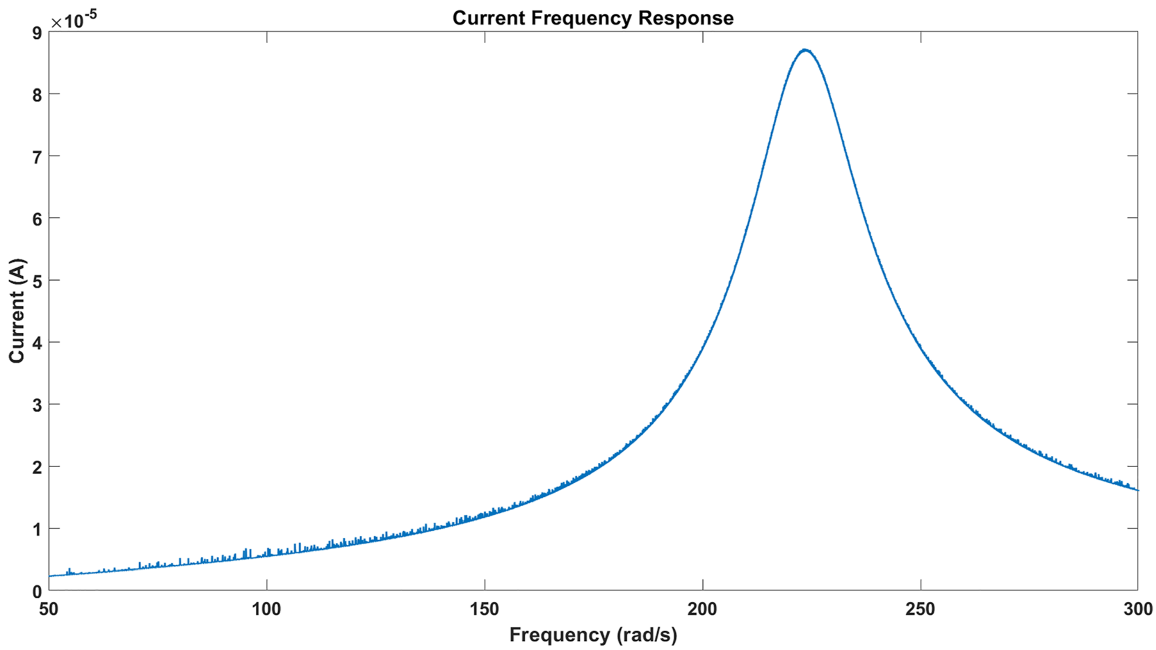

Figure 5: Current frequency response.

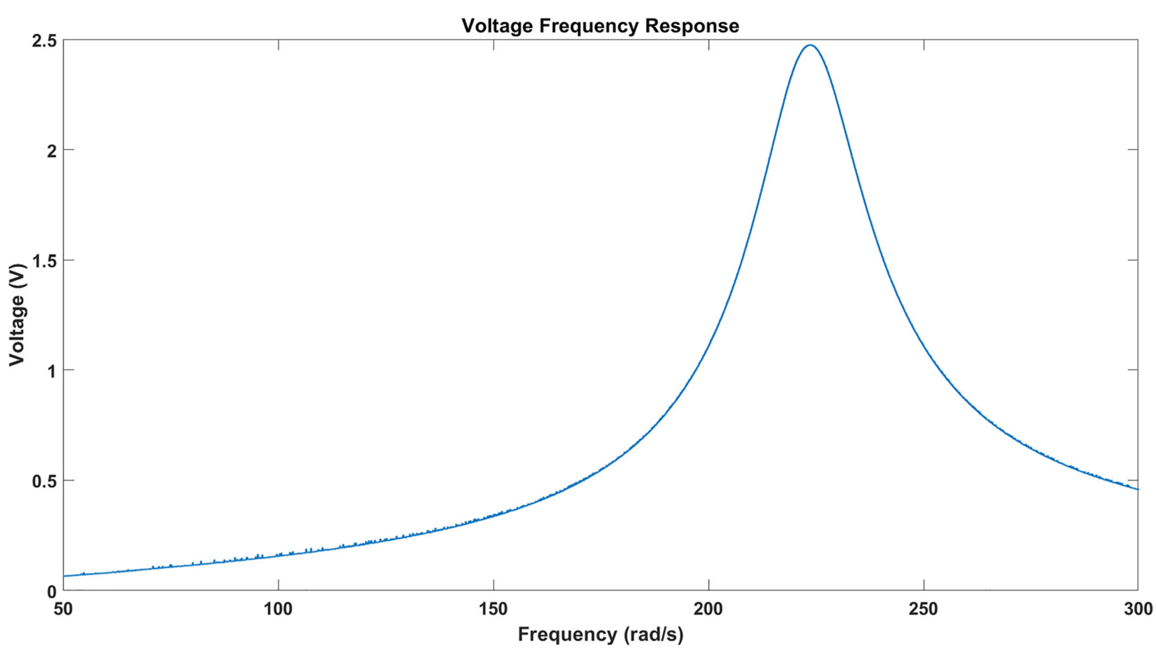

Figure 6: Voltage frequency response.

to specify the resonance frequency of EVEH, and based on the numerical example of this paper this resonance frequency occurs at the linear mechanical resonance frequency. Note that in the case of linear EVEH at resonance mechanical frequency no power will deliver to the electrical load. However, in the case of nonlinear one maximum power is delivered to the load at the linear mechanical resonance frequency.

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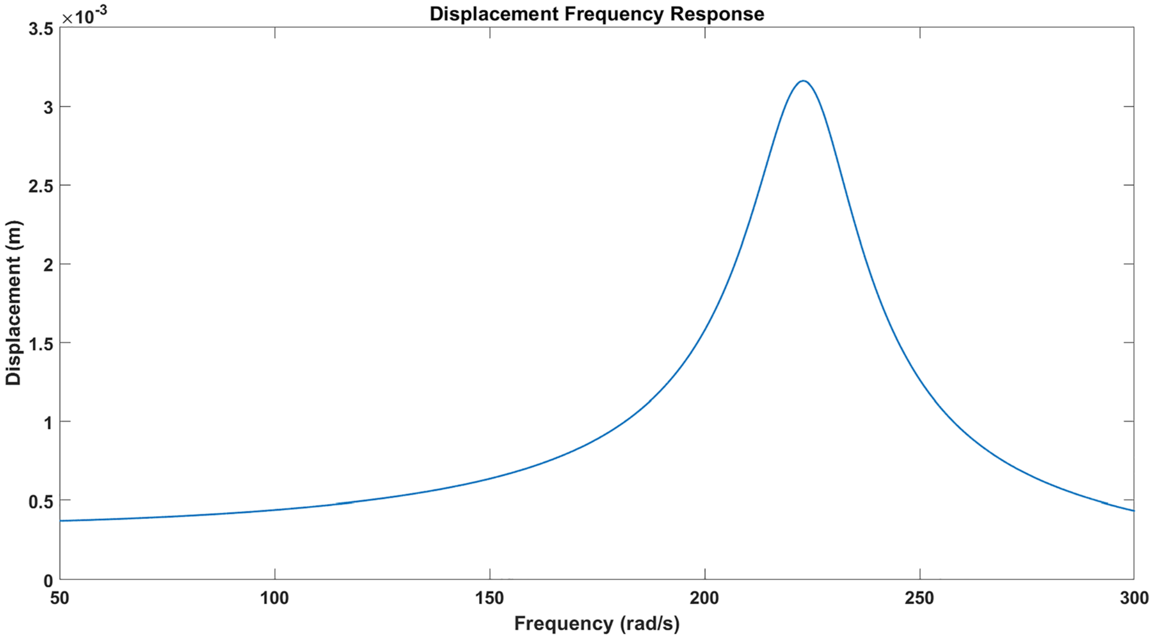

Figure 7: Displacement frequency response.

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