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Research Article

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Neutrosophic Inventory Management: A Cost-Effective Approach

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Abstract: Classical inventory models (IM) serve as quantitative tools for determining the optimal order quantities, timing of orders, and safety stock levels for specific inventory items or item groups. Zadeh (1965. Fuzzy sets. Information and Control, *8*, 338–353) introduced fuzzy theory and Dubois and Parade (1988. Fuzzy logic in expert systems: The role of uncertainty management. Fuzzy Sets and Systems, *28*, 3–17) presented the study of fuzzy inventory model, which, however, exhibits limitations in effectively handling uncertainty, inaccuracies, and imprecise data. In 1999, Smarandache presented the idea of neutrosophic set theory to handle uncertainty. Using trapezoidal neutrosophic numbers, this study extends the idea of neutrosophic sets to inventory management, concentrating on resolving the uncertainty associated with holding costs, ordering costs, and shortage costs. First time within the literature of the neutrosophic set, our new method not only addresses existing problems but can also tackle other issues that no other authors have successfully resolved so far. Additionally, we conduct a comparative analysis of our proposed model against existing models in this article. Based on this comparative study, our findings assert the superior performance of our proposed model in relation to some of the existing models. In conclusion, we wrap up our research by presenting graphical, logical, and tabular comparisons with the existing methods.

Keywords: neutrosophic, inventory, fuzzy, holding cost, ordering cost

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1 Introduction

Operations research, often abbreviated as OR, is a multidisciplinary field at the intersection of mathematics, statistics, and decision science. The field of OR is incredibly vast and fascinating. It encompasses numerous applications, making it challenging to fully comprehend. As researchers, we have observed that the most significant issue across all these applications is uncertainty, as highlighted in Table 1.

Now, as we know, our article primarily focuses on IM. Therefore, we have proceeded to discuss IM further. Operations research is intricately connected to inventory management by providing a systematic approach to optimize key inventory decisions. Through mathematical models, simulations, and analysis, operations research enables businesses to determine optimal reorder points, order quantities, and inventory policies, accounting for factors like demand variability, lead times, and cost structures. This connection empowers organizations to minimize holding and ordering costs while maintaining desired service levels, enhancing overall supply chain efficiency and profitability through data-driven decision-making in the realm of inventory management.

The last few years have witnessed a growing body of research focusing on various dimensions of inventory control and management. Numerous studies have explored complex facets with an emphasis on dealing with demand that changes depending on pricing and partial backlog, Das et al. (2020) integrated preservation technology into an inventory control system. In 2020, Mashud contributed by introducing an EOQ framework for a failing IM that considers various forms of demands and full backlog. In the year 2021, Khan and Sarkar investigated the landscape of risk transfer in the supply chain, integrating price and inventory decisions and handling shortages with great care. To tackle difficult inventory control situations, Setiawan et al. (2021) provided helpful information on how to handle exponential and quadratic demand in the context of Weibull deterioration. In the following year, Sharma et al. (2022) published their work, which developed a model that considers demand-driven production and accounts for time and stock-related demand for items

Table 1: OR at work: exploring the diverse applications across various fields

S. no.	Authors and year	Uncertain environments	Applications	Significance
1	Akram et al. (2016)	Bipolar fuzzy	Computing	Bipolar fuzzy digraphs for decision support systems
2	Butt and Akram (2016b)	Intuitionistic fuzzy	Decision making	A novel decision-making system based on intuitionistic fuzzy rule for an operating system process scheduler
3	Butt and Akram (2016a)	Fuzzy	CPU scheduling	Enhancing CPU scheduling algorithms using a novel fuzzy decision-making system
4	Habib et al. (2017)	Fuzzy	Climate decision	Developing fuzzy climate decision support systems for tomato cultivation in high tunnels
5	Farnam and Darehmiraki (2021)	Hesitant fuzzy	Multi-objective problem	Procedure for solving a multi-objective fractional programming issue in an environment of hesitant fuzzy decision-making
6	Mohanta and Sharanappa (2024)	Neutrosophic	Data analysis	A thorough review and current trends in neutrosophic data envelopment analysis
7	Iqbal et al. (2023)	Fermatean probabilistic hesitant fuzzy	Disaster	Enhancing earthquake response using Fermatean probabilistic hesitant fuzzy sets
8	Edalatpanah (2023)	Fuzzy	LPP	Solving fuzzy LPP in multiple dimensions
9	Alburaikan et al. (2023)	Neutrosophic	Goal programming	The goal programming approach to solving linear fractional programming problems with multiple objectives is based on a Neutrosophic scenario
10	Masoomi et al. (2023)	Neutrosophic	Supply chain	An improved best-worst method using Neutrosophic logic for evaluating performance indicators in the renewable energy supply chain

with declining inventory. An adaptive IM that is specifically designed for pharmaceutical distribution, with dynamic discreteness and the ability to account for both deterministic and stochastic demand, was presented by Antic et al. (2022). Duary et al. (2022) extended the scope by considering payment timing and inventory discounts in a model for deteriorating items, thoughtfully incorporating capacity constraints and partially backlogged shortages. The most recent addition, by Jani et al. (2023), offers a decision support system tailored for retailer's deterioration control, factoring in trade credit dynamics and the presence of shortages. Farahbakhsh and Kheirkhah (2023) presented a useful genetic algorithm-Taguchi-based method for solving the multi-period inventory route problem. In the same year, Dash et al. (2023) explored the coordination of a single-manufacturer multi-retailer supply chain, taking into account price and green-sensitive demand under stochastic lead time. Miriam et al. (2023) focused on decision-making processes in customer-centric IM. Moving into (2024), Das and Samanta presented an EOQ model for a two-warehouse system during lockdown, considering linear time-dependent demand. Lastly, Nazabadi et al. (2024) using agent-based and reinforcement learning models came up with a joint policy for production, maintenance, and product quality that works for a multi-machine production system. Together, these studies form a comprehensive mosaic of inventory management research, contributing valuable perspectives to this intricate and dynamic field. To avoid stockouts or excess inventory, classical IM must accurately predict future demand to strike a balance between service levels, costs, and reorder points, it must optimize safety stock levels and reorder points etc. In the sun case for a better alternative to classical inventory models (IM), fuzzy logic is useful in situations, where there are numerous complicated factors. In 1965, fuzzy theory was introduced by Lotfi A. Zadeh, which significantly improved its ability to aid in better decision-making.

The fuzzy inventory model (FIM), first proposed by Dubois and Parade (1988), achieves a flexible method of inventory management compared to classical IM. In 2011, Leopoldo Eduardo Cárdenas-Barron used analytical geometry and algebra to create EPQ/EOQ IMs including dual backorder costs. Based on this, Sulak (2015) proposed an EOQ model under a fuzzy set that handled defective shortages and products, offering a more realistic approach to inventory management. In 2019, Gani and colleagues used the AGM inequality approach to determine EOQ/EPQ in a fuzzy situation. This helped with inventory estimations. With a focus on variable holding costs, Alfares and Ghaithan (2019) investigated EOQ and EPQ production-IM, an essential part of contemporary supply chain dynamics. Additional research by Thinakaran et al. (2019) examined partial backorders in

EOQ and EPQ IMs. A typical IM issue that may be rather difficult for corporations was shed light on here. To enhance EOQ/EPQ IM formulations with two backorder prices, Lin (2019) utilized algebra and analytic geometry. Gani and Rafi (2020), enhanced EOQ/EPQ calculation using algebraic and AGM inequality approaches to streamline decision-making in a fuzzy context.

Later, Das (2020) introduced the multi-item IM with lead time in his research. In a fuzzy setting, demand influences both production cost and setup cost. In 2022, 2 years after this idea was first proposed, Das further expanded the proposed multi-objective IM in a fuzzy context. In 2022, a new approach was launched to handle production faults in EOQ/EPQ IM with shortages utilizing fuzzy methodologies. This method significantly improved the accuracy and resilience of inventory management. As a group, these academics have tackled many of the challenges and unknowns that contemporary companies confront by making great theoretical and practical advances in the field of inventory management. So far, we have discussed the FIM, but some challenges in FIM. Challenges in FIM involve handling fluctuating demand patterns, addressing imprecise or incomplete data, optimizing inventory allocation and distribution across complex supply chains, and adapting to unforeseen disruptions, whereas extended fuzzy inventory methodologies refine traditional approaches by integrating higher-order fuzzy reasoning, multiple granularities, and dynamic adjustments, enabling more adaptive, resilient, and agile inventory control strategies in volatile and uncertain environments.

However, in 1999, F. Smarandache outlined the neutrosophic set and its unique features that distinguish it apart from the classical and fuzzy models. In the context of this progression, we will now proceed to discuss the neutrosophic inventory model (NIM).

1.1 Motivation and Novelties

Neutrosophic set theory is a technique used for uncertainty in IM. It deals with situations where we're not entirely sure about something. NIM with Neutrosophic number etc. are described by few researchers. This manuscript's primary contribution is as follows:

- The proposed model can address the problems that have already been resolved by existing models.
- The proposed model has the capability to address novel problem sets that have not been explored in any existing research article to date.
- It reduces the time and space complexity.

1.2 Objective

After reading many research books and articles, it has become clear that having a lot of knowledge about management is very important. Also, it is very important to understand that uncertainty is a big part of real-life problems. Because of these things, the present study aims to explore NIM as a cost-effective approach.

The study aims to explore NIM as a cost-effective approach. To achieve this, the following objectives have been set:

(1) Conducting a comprehensive literature survey on inventory management systems to enhance understanding of the overall management framework.
(2) Proposing a new method to effectively address and manage the neutrosophic within the IM context.
(3) Comparing the proposed method with existing approaches to establish its superiority.
(4) Conducting a thorough comparison, employing logical, graphical, and tabular analyses, with existing methods in the field.
(5) Discussing the practical applications of the proposed method within the inventory management system.
(6) Introducing a new method designed not only to address existing numerical challenges but also to tackle novel problem types.

This research endeavours to contribute to the existing body of knowledge in the field by addressing the identified gaps and offering innovative solutions to enhance the efficacy of IM systems in the face of uncertainty.

The article consists of five sections and Appendix. The first section is the introduction, which provides an overview of the study. The second section presents the proposed NIM Model. The third section is dedicated to numerical analysis with results and discussion. The fourth section presents the sensitive analysis. Finally, the fifth section concludes the article. The appendix contains at first Neutrosophic numbers and their arithmetic and logical operators. at second EOQ model in Neutrosophic environment.

2 Proposed Neutrosophic Inventory Management Model

2.1 Notations

Define the following parameters used in IM:

τ **stands for** "Total cost";	i **stands for** "Holding cost for one unit per day".
ℓ **stands for** "Length of the cycle";	$\tilde{n}^{\text{neu}}$ **stands for** "Neutrosophic ordering cost per cycle".
v^* **stands for** "Optimal order quantity";	$(v^*)^{\text{neu}}$ **stands for** "Neutrosophic optimal order quantity".
v **stands for** "Order quantity per cycle";	k **stands for** "Shortage (backorder) cost per unit per day".
n **stands for** "Ordering cost per cycle";	$(v_s^*)^{\text{neu}}$ **stands for** "Neutrosophic optimal shortage quantity".
$\tilde{\tau}^{\text{neu}}$ **stands for** "Neutrosophic total cost";	$\tilde{i}^{\text{neu}}$ **stands for** "Neutrosophic holding cost for one unit per day".
v_s **stands for** "Shortage quantity per cycle";	
a **stands for** "Annual demand in period $[0, \ell]$";	$\tilde{k}^{\text{neu}}$ **stands for** "Neutrosophic shortage (backorder) cost per unit per day".

2.2 Assumptions

This IM allows inventory shortages and constant demand and plan time.

2.3 Mathematical Formation and Solution of Model

2.3.1 EOQ Model in Classical Environment

For a crisp IM with shortage quantity, then IM in the classical sense is,

Total cost = Ordering Cost + Holding Cost + Shortage Cost

$$\tau = i\frac{(v - v_s)^2}{2v}\ell + k\frac{v_s^2}{2v}\ell + n\frac{a}{v}. \tag{1}$$

where $\tau(v)$ is minimal to which by optimizing total cost we use $\frac{d\tau(v)}{dv} = 0$, $\frac{d^2\tau(v)}{dv^2} > 0$.

Now, differentiating equation (1) with respect to v, we have, the optimal order quantity is

$$v^* = \sqrt{\frac{2(i + k)na}{ki\ell}}. \tag{2}$$

And optimal shortage quantity is

$$v_s^* = \sqrt{\frac{2 \cdot ina}{k(i + k)\ell}}. \tag{3}$$

From equation (1), the minimal total cost is

$$\tau^* = \sqrt{\frac{2 \cdot ikna\ell}{i + k}}. \tag{4}$$

2.3.2 EOQ Model in Neutrosophic Environment

In an environment characterized by its clarity and precision, we can determine the total cost using equation (1). However, in real-world scenarios, this cost may exhibit slight fluctuations, thereby impacting the quantity ordering (n), shortage (v_s), and holding (i). To address this variability, we adopt a model that accounts for permissible shortages within a Neutrosophic framework. The neutrosophic methodology necessitates carrying cost (known as holding cost) per unit, ordering cost per order, and shortage cost per unit quantity. Consequently, we transform the values of n, v_s, and i into Neutrosophic numbers using a trapezoidal Neutrosophic representation as follows:

$$\tilde{n}^{\text{neu}} = \langle (n_1^{\text{neu}}, n_2^{\text{neu}}, n_3^{\text{neu}}, n_4^{\text{neu}}); (T_{\tilde{n}^{\text{neu}}}, I_{\tilde{n}^{\text{neu}}}, F_{\tilde{n}^{\text{neu}}}) \rangle,$$
$$\tilde{k}^{\text{neu}} = \langle (\tilde{k}_1^{\text{neu}}, \tilde{k}_2^{\text{neu}}, \tilde{k}_3^{\text{neu}}, \tilde{k}_4^{\text{neu}}); (T_{\tilde{k}^{\text{neu}}}, I_{\tilde{k}^{\text{neu}}}, F_{\tilde{k}^{\text{neu}}}) \rangle, \text{ and}$$
$$\tilde{i}^{\text{neu}} = \langle (\tilde{i}_1^{\text{neu}}, \tilde{i}_2^{\text{neu}}, \tilde{i}_3^{\text{neu}}, \tilde{i}_4^{\text{neu}}); (T_{\tilde{i}^{\text{neu}}}, I_{\tilde{i}^{\text{neu}}}, F_{\tilde{i}^{\text{neu}}}) \rangle.$$

2.3.3 Proposed Model for Find Total Cost while Considering the Uncertainty

The neutrosophic total cost, optimal order quantity, and optimal shortage quantity are denoted as $\tilde{\tau}^{\text{neu}}$, $\tilde{v}^{\text{neu}}$, and $\tilde{v}_s^{\text{neu}}$ respectively where $\tilde{i}^{\text{neu}}$, $\tilde{k}^{\text{neu}}$ & $\tilde{n}^{\text{neu}}$ are neutrosophic variables.

The optimal order quantity ($\tilde{v}^{\text{neu}}$) is as follows (equation (A5) Appendix B):

$$\tilde{v}^{\text{neu}} = \sqrt{\frac{2\left[\left[\begin{pmatrix}\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} \\ + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}}\end{pmatrix} \oplus \begin{pmatrix}\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} \\ + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}}\end{pmatrix}\right]\right] \cdot \begin{pmatrix}\tilde{n}_1^{\text{neu}} + \tilde{n}_2^{\text{neu}} \\ + \tilde{n}_3^{\text{neu}} + \tilde{n}_4^{\text{neu}}\end{pmatrix} \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ - (I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ - (F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix} \cdot a}{\left[(\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}}) \cdot (\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}}) \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ - (I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ - (F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\right] \cdot \ell}}$$

Table 2: Tabular comparison study with some of the existing methods such as those of Rajput et al. (2019), Saranya and Varadarajan (2018), Sen and Malakar (2015)

Demand	Total cost				
	Classical environment	**Sen and Malakar (2015)**	**Saranya and Varadarajan (2018)**	**Rajput et al. (2019)**	**Proposed method**
1,000	828.078	NA	815.5122568	809.039	809.039
1,025	838.3657572	NA	825.6432323	819.0904	819.0904
1,125	878.3100657	NA	864.9813704	858.1163	858.1163
1,225	916.515139	NA	902.6066669	895.443	895.443
1,325	953.1901324	NA	938.7251031	931.2748	931.2748
1,425	988.5053653	NA	973.5044136	965.7781	965.7781

The optimal shortage quantity ($\tilde{\nu}_s^{\text{neu}}$) is as follows (equation (A6) Appendix B):

$$\tilde{\nu}_s^{\text{neu}} = \sqrt{\frac{2\left[\begin{pmatrix}\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} \\ + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}}\end{pmatrix} \cdot \begin{pmatrix}\tilde{n}_1^{\text{neu}} + \tilde{n}_2^{\text{neu}} \\ + \tilde{n}_3^{\text{neu}} + \tilde{n}_4^{\text{neu}}\end{pmatrix} \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ - (I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ - (F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\right] \cdot a}{\begin{pmatrix}\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} \\ + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}}\end{pmatrix} \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ - (I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ - (F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix} \cdot \left[\begin{pmatrix}\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} \\ + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}}\end{pmatrix} + \begin{pmatrix}\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} \\ + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}}\end{pmatrix}\right] \cdot \ell}}$$

Therefore, optimal (minimum) total cost while considering uncertainty is as follows (Appendix B equation (A7)):

$$\frac{\partial(\tilde{\tau}^{\text{neu}})^R(\nu, \nu_s)}{\partial \nu} = \frac{1}{12}\left[\left\langle\left[\begin{array}{l}(\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}}) \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ -(I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ -(F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix} \\ \otimes \left\{\dfrac{2\nu(\nu - \nu_s) - (\nu - \nu_s)^2}{2\nu^2}\right\}\ell \\ \oplus (\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}}) \cdot \begin{pmatrix}2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ -(I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ -(F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix} \cdot \left\{\dfrac{-(\nu_s)^2}{2\nu^2}\right\}\ell \\ \oplus (\tilde{n}_1^{\text{neu}} + \tilde{n}_2^{\text{neu}} + \tilde{n}_3^{\text{neu}} + \tilde{n}_4^{\text{neu}}) \cdot \left(-\dfrac{a}{\nu^2}\right)\end{array}\right]\right\rangle\right]$$

3 Numerical Analysis and Result Discussion

The purpose of introducing this section is to explain the validation of our proposed model. Additionally, we have attempted to compare our proposed model with some of the existing models. This comparison helps demonstrate the superiority of our proposed model, and we have used two examples to illustrate this. Specifically, we conducted two case studies using existing literature datasets (such as those of Rajput et al. (2019), Saranya and Varadarajan (2018) and Sen and Malakar (2015). In Example 3.1, our goal is to establish the approach of our proposed model by considering the existing dataset and comparing it with the current existing method. Furthermore, it is evident that from Example 3.1, our proposed method not only addresses existing problems but also solves a new type of environment, as discussed below in Example 3.2.

Example 3.1: Comparison with the existing method: As per Rajput et al.'s (2019) consideration, A manufacturing facility must establish an EOQ model to maximize the product's total cost. The cycle length is 6 months, with ordering costs of Rs. 20, holding costs of Rs. 04, and

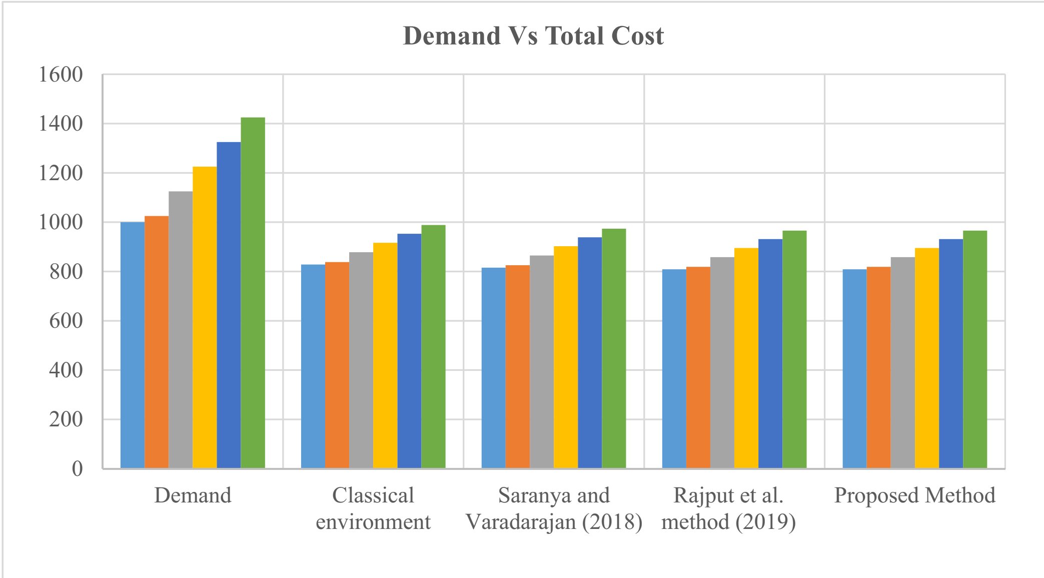

Figure 1: Clustered comparison with existing methods.

shortage costs of Rs. 10 per unit. We use an SVTpN membership function to capture the data's inherent uncertainty when addressing this challenge using Neutrosophic parameters. For every feasible cost, the corresponding membership functions are specified as follows: $\tilde{i}^{neu} = \langle(1, 3, 5, 6);\ (1, 0, 0)\rangle$, $\tilde{k}^{neu} = \langle(8, 9, 11, 12);\ (1, 0, 0)\rangle$, and $\tilde{n}^{neu} = \langle(15, 18, 22, 25);\ (1, 0, 0)\rangle$.

Solution: In Example 3.1, we are demonstrating that our proposed method not only solves the new type of problem but also addresses the problem solved by the existing method, as shown in Table 2. In this table, we can see that the Total Cost (TC) of the existing method is equal to the TC of our proposed method.

In addition to tabular comparison, we also conducted a pictorial comparison study with some of the existing methods such as those of Rajput et al. (2019), Saranya and Varadarajan (2018), and Sen and Malakar (2015).

In Figures 1 and 2, we have attempted to compare two approaches: the first using clusters and the second using a line graph. In Figure 1, we can clearly see that as we changed the demand in increasing order, the same trend was observed in all other clusters. This indicates that demand plays a significant role in terms of total cost, a finding consistent with what other authors have observed in their studies. Additionally, in Figure 2, we aim to demonstrate that our proposed method is either equivalent to or provides a better solution than other methods. The dominance of the yellow-colored graph suggests that our proposed method provides the least value in comparison to the other method. Finally, after doing the logical comparison, i.e. the classical total cost is greater than Rajput et al. (2019) proposed method but it is equal to our proposed method.

In Example 3.1, many authors have proposed different methods to solve Rajput's numerical problem. In the comparison study of tabular, pictorial, and logical methods, it is observed that our proposed method provides an optimal solution similar to that of Rajput et al. (2019). Our proposed method not only solves existing problems but also solves a new type of environment, which is discussed below in Example 3.2.

Example 3.2: A manufacturing facility must create an EOQ model to maximize the product's total cost. The cycle length is 6 months, with ordering costs of Rs. 20, holding costs of Rs. 04, and shortage costs of Rs. 10 per unit. We use an SVTpN membership function to capture the data's inherent uncertainty when addressing this challenge using Neutrosophic parameters by considering the six cases discussed in Table 3.

Solution: After implementing equations (A5), (A6), and (A7) of our proposed method, the final total cost (TC) that we obtained is displayed in Table 4.

We compared our proposed method with some of the existing methods such as those of Rajput et al. (2019), Saranya and Varadarajan (2018), and Sen and Malakar (2015) using both tabular and pictorial comparison studies.

In Figures 3 and 4, we can see that both figures are represented in green colour. This indicates that only our proposed method can handle this type of problem. Additionally, we observe that representations in red, blue, and light black colours do not exist in either figure. This suggests that all other existing methods are unable to handle

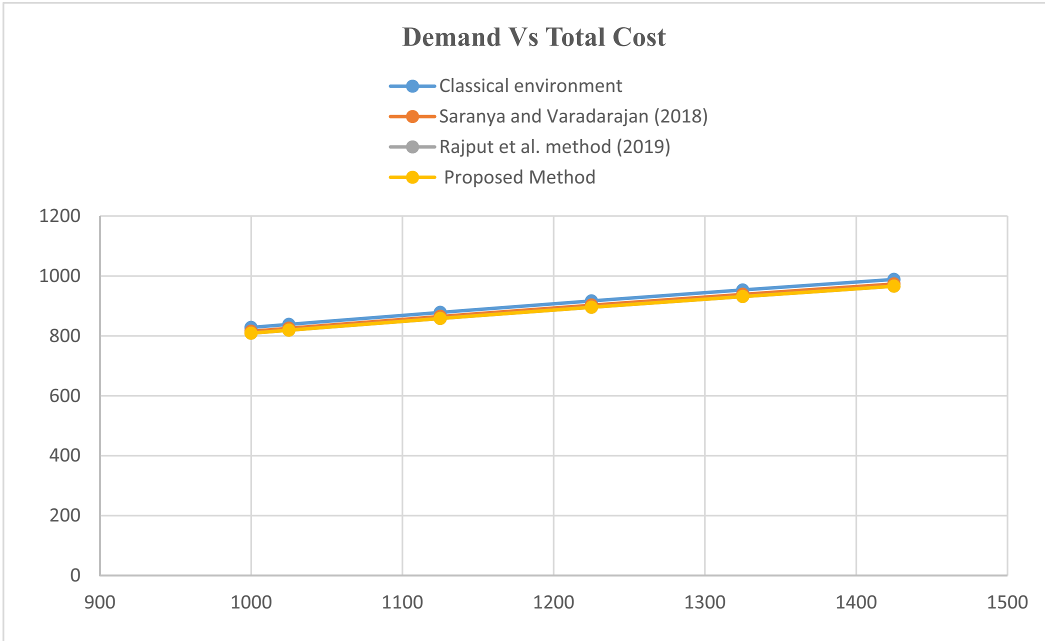

Figure 2: Scatter representation of existing methods.

Table 3: Finding the total optimal cost under different cases

Different cases	$\tilde{i}^{neu} = \langle(\tilde{i}_1^{neu}, \tilde{i}_2^{neu}, \tilde{i}_3^{neu}, \tilde{i}_4^{neu});(T_{\tilde{i}^{neu}}, I_{\tilde{i}^{neu}}, F_{\tilde{i}^{neu}})\rangle$, $\tilde{k}^{neu} = \langle(\tilde{k}_1^{neu}, \tilde{k}_2^{neu}, \tilde{k}_3^{neu}, \tilde{k}_4^{neu});(T_{\tilde{k}^{neu}}, I_{\tilde{k}^{neu}}, F_{\tilde{k}^{neu}})\rangle$, **and** $\tilde{n}^{neu} = \langle(n_1^{neu}, n_2^{neu}, n_3^{neu}, n_4^{neu});(T_{\tilde{n}^{neu}}, I_{\tilde{n}^{neu}}, F_{\tilde{n}^{neu}})\rangle$
Case 1	$\langle(1, 3, 5, 6);\ (0.99, 0.98, 0.73)\rangle$, $\langle(8, 9, 11, 12);\ (0.98, 0.91, 0.71)\rangle$, $\langle(15, 18, 22, 25);\ (0.99, 0.97, 0.7)\rangle$
Case 2	$\langle(7, 9, 11, 12);\ (0.83, 0.82, 0.61)\rangle$, $\langle(14, 15, 17, 18);\ (0.83, 0.85, 0.56)\rangle$, $\langle(21, 24, 28, 3);\ (0.85, 0.82, 0.53)\rangle$
Case 3	$\langle(8, 10, 12, 13);\ (0.79, 0.81, 0.58)\rangle$, $\langle(15, 16, 18, 19);\ (0.81, 0.81, 0.54)\rangle$, $\langle(22, 25, 29, 32);\ (0.82, 0.79, 0.51)\rangle$
Case 4	$\langle(9, 11, 13, 14);\ (0.77, 0.79, 0.57)\rangle$, $\langle(16, 17, 19, 20);\ (0.79, 0.78, 0.53)\rangle$, $\langle(23, 26, 30, 33);\ (0.8, 0.75, 0.49)\rangle$
Case 5	$\langle(10, 12, 14, 15);\ (0.73, 0.77, 0.55)\rangle$, $\langle(17, 18, 20, 21);\ (0.77, 0.75, 0.51)\rangle$, $\langle(24, 27, 31, 34);\ (0.79, 0.73, 0.46)\rangle$
Case 6	$\langle(11, 13, 15, 16);\ (0.71, 0.75, 0.51)\rangle$, $\langle(18, 19, 21, 22);\ (0.74, 0.71, 0.49)\rangle$, $\langle(25, 28, 32, 35);\ (0.7, 0.69, 0.44)\rangle$

Table 4: Tabular comparison study with some of the existing methods such as those of Sen and Malakar (2015), Saranya and Varadarajan (2018), Rajput et al. (2019)

Demand	**Sen and Malakar (2015), Saranya and Varadarajan (2018), Rajput et al. (2019)**	**Our proposed method**
1,000	NA	342.4935301
1,025	NA	632.5781217
1,125	NA	719.6689366
1,225	NA	800.3864355
1,325	NA	881.5672484
1,425	NA	981.3623184

similar uncertain situations. Hence, it is clear that our proposed method not only addresses existing numerical challenges but also tackles new types of uncertain problem types. Moreover, from the above comparison, it is clear that our proposed method is superior to some of the existing methods. Now, we are going to perform a logical comparison of Examples 3.1 and 3.2, as discussed in Table 5.

3.1 Logical Comparison

Table 5 lists a comparative analysis of total cost across various environmental scenarios. In Example 3.1, our observations reveal that in a fuzzy environment, the value of total cost is lower than that in a classical environment. Furthermore, in our proposed model, the total cost is observed to be lower than the classical and equal to the fuzzy environments. Additionally, in Example 3.2 it is clear that the classical and fuzzy approach is unable to provide the total cost

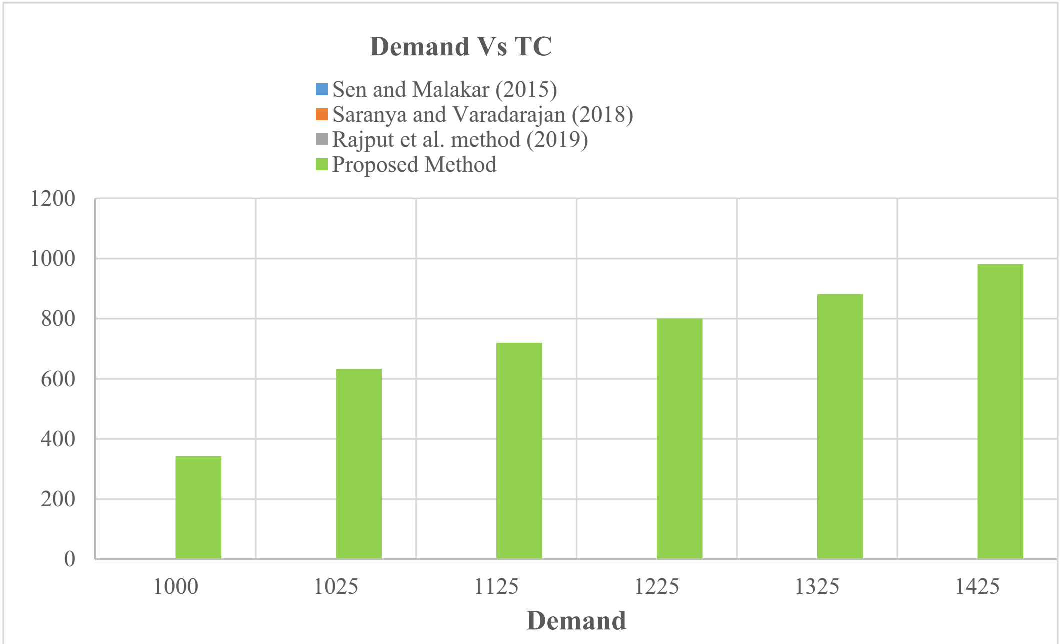

Figure 3: Column representation (ref Table 4).

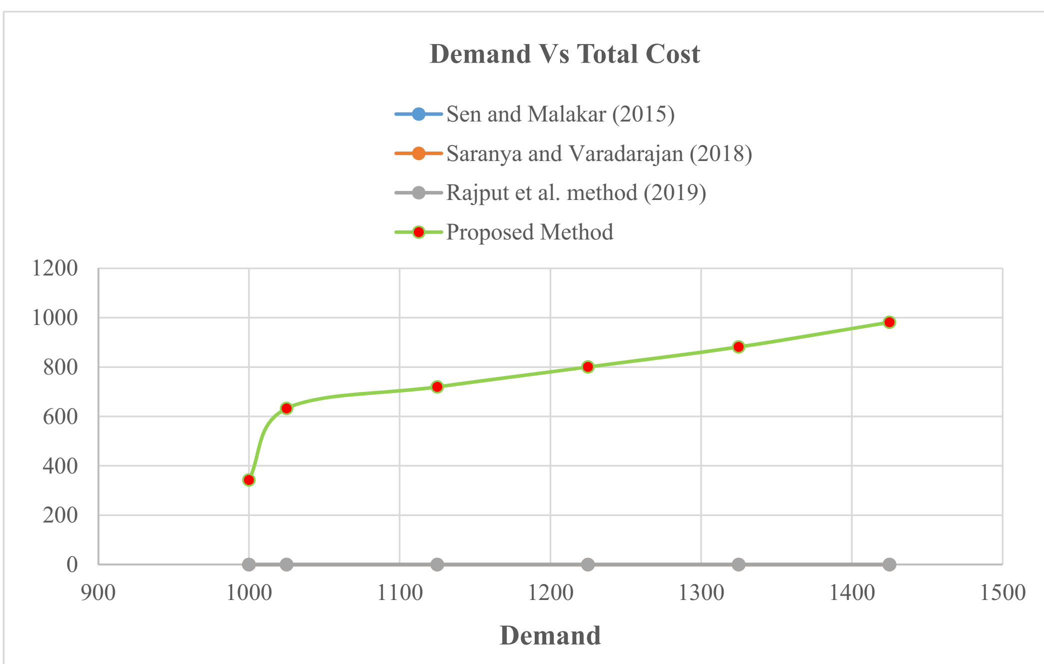

Figure 4: Graphical interpretation of Table 4.

Table 5: Logical comparison with TC

Examples	Comparison
Example 3.1	$\text{Classical TC}(828.078) > \text{Fuzzy TC}(809.039) \approx \text{Ourproposed TC}(809.039)$
Example 3.2	$\text{Classical TC}(\text{NA}) - \text{Fuzzy TC}(\text{NA}) - \text{Ourproposed TC}(342.4935)$

as compared with our proposed model. That is why we said that our proposed approach not only handled the existing problem but also solved the new type of problem. To explain more about our method, we have also conducted a sensitive analysis in below Section 4.

4 Sensitive Analysis

A manufacturing facility develops an EOQ model to maximize the product's total cost. The cycle length is 6 months, and the ordering price is Rs. $\langle(15, 18, 22, 25);\ (0.99, 0.97, 0.70)\rangle$ per unit, the holding cost is Rs. $\langle(1, 3, 5, 6);\ (0.99, 0.98, 0.73)\rangle$ per unit, and the shortage cost is Rs. $\langle(8, 9, 11, 12);\ (0.98, 0.91, 0.71)\rangle$ per unit.

Table 6 provides insights into optimizing total costs in a manufacturing setting, specifically focusing on cost-effective IM using neutrosophic theory. This sensitive analysis is crucial for decision-making in IM. Therefore, optimizing these parameters ($\tilde{i}^{neu}$, $\tilde{k}^{neu}$, and $\tilde{n}^{neu}$) can lead to substantial savings or expenditures, depending on their values.

Based on the information from Table 6, we can conclude that the model is extremely sensitive to the holding cost, shortage cost, and total cost parameters.

- The model is highly sensitive to holding cost $\tilde{i}^{neu} = \langle(1, 3, 5, 6);\ (0.99, 0.98, 0.73)\rangle$; i.e. If we increase the percentage in the holding cost parameter (i.e. 13, 29, and 37%), we observe that the optimal order quantity decreases, while the optimal shortage quantity increases and the total cost also increases. Conversely, if we decrease the percentage in the holding cost (i.e. −13, −29, and −37%), we find that the optimal order quantity increases, the optimal shortage quantity decreases and the total cost decreases as well.
- Similarly, if we increase the percentage in the shortage cost $\tilde{k}^{neu}$ parameter (i.e. 13, 29, and 37%), we observe that the optimal order quantity decreases, and the optimal shortage quantity also decreases, while the total cost increases. Conversely, if we decrease the percentage (i.e. −13, −29, and −37%) in the shortage cost, we find that the optimal order quantity increases, the optimal shortage quantity increases, and the total cost decreases as well.
- Changing the $\tilde{n}^{neu}$ ordering cost parameter will have an effect, with an increase of 13, 29, and 37% the observations raise in overall cost, optimal shortage quantity, and ideal order quantity, respectively. On the other hand, decreasing the proportion of the ordering cost by −13, −29, or −37% results in a reduction of the overall cost, optimal shortage quantity, and optimal order quantity.

Table 6: Sensitive analysis on proposed method

Parameter	Change in parameters (%)	v Proposed method	v_s Proposed method	Proposed TC Proposed method
$\tilde{i}^{neu}$ $\langle(1, 3, 5, 6);\ (0.99, 0.98, 0.73)\rangle$	13	47.32780723	14.08615158	357.7882502
	29	44.83152683	14.87048766	377.7103867
	37	43.0753968	15.47673884	393.1091665
	−13	48.68631097	13.69310292	347.804814
	−29	53.37369196	12.4905481	317.2599217
	−37	57.98788415	11.49665446	292.0150232
$\tilde{k}^{neu}$ $\langle(8, 9, 11, 12);\ (0.98, 0.91, 0.71)\rangle$	13	48.65951376	12.12446356	347.9963531
	29	47.90170815	10.78867301	353.5016597
	37	47.58566783	10.22614638	355.8494418
	−13	50.43868855	15.19237607	335.7211264
	−29	52.12240319	18.01465548	324.8762969
	−37	53.25390911	19.87086161	317.9735275
$\tilde{n}^{neu}$ $\langle(15, 18, 22, 25);\ (0.99, 0.97, 0.70)\rangle$	13	52.55684753	14.33368569	364.0756165
	29	56.15454864	15.3148769	388.9978733
	37	57.86958518	15.78261414	400.8783992
	−13	46.11579628	12.57703535	319.4566979
	−29	41.65999947	11.36181804	288.5901781
	−37	39.24283374	10.70259102	271.8458119

5 Conclusion

The investigation of neutrosophic set theories defines a potential use in handling inventories in our research study. Current FIMs have problems with handling uncertainty, inaccurate data, and imprecise timing, in contrast to classical models that have optimized order amounts, timing, and safety stock levels. To address these challenges, our novel approach focuses on the uncertainty associated with holding costs, shortage costs, and ordering costs, utilizing trapezoidal neutrosophic numbers. We improve decision-making procedures and offer more reliable inventory optimization solutions by utilizing neutrosophic reasoning. Notably, our method demonstrates promising results in managing inventory under uncertainty a milestone in the literature of neutrosophic sets. Additionally, Our research contributes valuable insights to enhance inventory management strategies and moderate uncertainties in EPQ operations through graphical, logical, and tabular comparisons. Despite our significant progress, several avenues for future exploration and improvement remain. These include investigating Dynamic Demand and Supply, Multi-Objective Optimization, and conducting Empirical Validation. However, it's crucial to acknowledge the limitations of our model. Its success hinges on accurate data related to costs, demand, and lead times. Obtaining precise data, especially under conditions of uncertainty, remains a challenge. Like any model, our approach is based on certain assumptions, and ongoing validation against various scenarios is essential.

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Data availability statement: All data supporting the reported findings in this research article are provided within the manuscript.

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Appendix

A Neutrosophic number and its Arithmetic and logical operators

Definition A.1. Normalized Fuzzy set (Zadeh, 1965): A fuzzy set $\tilde{\xi} = \{(\omega, \sigma_{\tilde{\xi}}(\omega)) : \omega \in W, \sigma_{\tilde{\xi}}(\omega) \in [0,1]\}$ is called a normalized fuzzy set if and only if $\text{Sup}_{\omega \in W}\{\sigma_{\tilde{\xi}}(\omega)\} = 1$.

Definition A.2. Neutrosophic Set (Wang et al., 2010, 2011): From the universal discourse W, a set $\widehat{\text{neu}S}$ symbolically represented as ω known to be Neutrosophic set. The condition $\widehat{\text{neu}S} = \{(\omega, [t_{\widehat{\text{neu}S}}(\omega), i_{\widehat{\text{neu}S}}(\omega), f_{\widehat{\text{neu}S}}(\omega)]) : \omega \in W\}$ defines the discussion of truth, indeterminacy, and falsity degrees as $t_{\widehat{\text{neu}S}}(\omega), f_{\widehat{\text{neu}S}}(\omega), i_{\widehat{\text{neu}S}}(\omega) : W \to [0,1]$ by exhibiting the following relation:

$$0 \le \text{Sup}\{T_{\widehat{\text{neu}S}}(\omega)\} + \text{Sup}\{F_{\widehat{\text{neu}S}}(\omega)\} + \text{Sup}\{I_{\widehat{\text{neu}S}}(\omega)\} \le 3$$

Definition A.3. (Liang et al., 2018a,b): Let $T_{\hat{d}}, I_{\hat{d}}, F_{\hat{d}} \in [0,1]$, then a Single-valued trapezoidal neutrosophic (SVTpN) number $\hat{d} = \langle(\hat{d}^a, \hat{d}^s, \hat{d}^h, \hat{d}^o);\ (T_{\hat{d}}, I_{\hat{d}}, F_{\hat{d}})\rangle$ is a special Ns on the real number set R, whose indeterminacy-MF $\xi_{\hat{d}}(x)$, truth-MF $\psi_{\hat{d}}(x)$, and falsity-MF $\zeta_{\hat{d}}(x)$ are given as follows:

$$\xi_{\hat{d}}(x) = \begin{cases} \dfrac{(\hat{d}^s - x + I_{\hat{d}}(x - \hat{d}^a))}{(\hat{d}^s - \hat{d}^a)}, \hat{d}^a \le x \le \hat{d}^s \\ I_{\hat{d}}, \hat{d}^s \le x \le \hat{d}^h \\ \dfrac{(x - \hat{d}^h + I_{\hat{d}}(\hat{d}^o - x))}{(\hat{d}^o - \hat{d}^h)}, \hat{d}^h \le x \le \hat{d}^o \\ 1, \text{otherwise} \end{cases},$$

$$\psi_{\hat{d}}(x) = \begin{cases} \dfrac{T_{\hat{d}}(x - \hat{d}^a)}{(\hat{d}^s - \hat{d}^a)}, \hat{d}^a \le x \le \hat{d}^s \\ T_{\hat{d}}, \hat{d}^s \le x \le \hat{d}^h \\ \dfrac{T_{\hat{d}}(\hat{d}^o - x)}{(\hat{d}^o - \hat{d}^h)}, \hat{d}^h \le x \le \hat{d}^o \\ 0, \text{otherwise} \end{cases},$$

$$\zeta_{\hat{d}}(x) = \begin{cases} \dfrac{(\hat{d}^s - x + F_{\hat{d}}(x - \hat{d}^a))}{(\tilde{d}^s - \tilde{d}^a)}, \hat{d}^a \le x \le \hat{d}^s \\ F_{\hat{d}}, \hat{d}^s \le x \le \hat{d}^h \\ \dfrac{(x - \hat{d}^h + F_{\hat{d}}(\hat{d}^o - x))}{(\hat{d}^o - \hat{d}^h)}, \hat{d}^h \le x \le \hat{d}^o \\ 1, \text{otherwise} \end{cases}$$

Note: Special Case

Case 1. When $\hat{d}^a > 0$, then $\hat{d} = \langle(\hat{d}^a, \hat{d}^s, \hat{d}^h, \hat{d}^o);\ (T_{\hat{d}}, I_{\hat{d}}, F_{\hat{d}})\rangle$ is called a positive TpNN.

Case 2. If $I_{\hat{d}} = 0, F_{\hat{d}} = 0\ \&\ T_{\hat{d}} = 1$, a TpNN is reduced to General Trapezoidal Fuzzy Number (GTpFN), $\hat{d} = \langle(\hat{d}^a, \hat{d}^s, \hat{d}^h, \hat{d}^o)\rangle$.

Definition A.4. (Ye, 2017): Let $\underset{\sim}{\tilde{A}} = \langle[\tilde{a}_1^a, \tilde{a}_2^n, \tilde{a}_3^k, \tilde{a}_4^i];(T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}})\rangle$ and $\underset{\sim}{\tilde{B}} = \langle[\tilde{b}_1^a, \tilde{b}_2^n, \tilde{b}_3^k, \tilde{b}_4^i];(T_{\tilde{B}}, I_{\tilde{B}}, F_{\tilde{B}})\rangle$ are two trapezoidal neutrosophic numbers, then

$$\begin{aligned} &\tilde{A} \otimes \tilde{B} = \langle[\tilde{a}_1^a\tilde{b}_1^a, \tilde{a}_2^n\tilde{b}_2^n, \tilde{a}_3^k\tilde{b}_3^k, \tilde{a}_4^i\tilde{b}_4^i]; \\ &(T_{\tilde{A}} \wedge T_{\tilde{B}}, I_{\tilde{A}} \vee I_{\tilde{B}}, F_{\tilde{A}} \vee F_{\tilde{B}})\rangle \\ &\alpha \otimes \tilde{A} = \langle[\alpha\tilde{a}_1^a, \alpha\tilde{a}_2^n, \alpha\tilde{a}_3^k, \alpha\tilde{a}_4^i];\ (T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}})\rangle, \quad \alpha \ge 0 \\ &\alpha \otimes \tilde{A} = \langle[\alpha\tilde{a}_4^i, \alpha\tilde{a}_3^k, \alpha\tilde{a}_2^n, \alpha\tilde{a}_1^a];\ (T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}})\rangle, \quad \alpha < 0, \end{aligned}$$

$$\begin{aligned} &\tilde{A} \oplus \tilde{B} = \langle[\tilde{a}_1^a + \tilde{b}_1^a, \tilde{a}_2^n + \tilde{b}_2^n, \tilde{a}_3^k + \tilde{b}_3^k, \tilde{a}_4^i + \tilde{b}_4^i]; \\ &(T_{\tilde{A}} \wedge T_{\tilde{B}}, I_{\tilde{A}} \vee I_{\tilde{B}}, F_{\tilde{A}} \vee F_{\tilde{B}})\rangle, \end{aligned}$$

$$\frac{\tilde{A}}{\tilde{B}} = \begin{cases} \left\langle \left(\dfrac{a_1^a}{b_4^i}, \dfrac{a_2^n}{b_3^k}, \dfrac{a_3^k}{b_2^n}, \dfrac{a_4^i}{b_1^a}\right);\ T_{\tilde{A}} \wedge T_{\tilde{B}}, I_{\tilde{A}} \vee I_{\tilde{B}}, F_{\tilde{A}} \vee F_{\tilde{B}} \right\rangle, \\ \text{if } a_4^i > 0, b_4^i > 0 \\ \left\langle \left(\dfrac{a_4^i}{b_4^i}, \dfrac{a_3^k}{b_3^k}, \dfrac{a_2^n}{b_2^n}, \dfrac{a_1^a}{b_1^a}\right);\ T_{\tilde{A}} \wedge T_{\tilde{B}}, I_{\tilde{A}} \vee I_{\tilde{B}}, F_{\tilde{A}} \vee F_{\tilde{B}} \right\rangle, \\ \text{if } a_4^i < 0, b_4^i > 0 \\ \left\langle \left(\dfrac{a_4^i}{b_1^a}, \dfrac{a_3^k}{b_2^n}, \dfrac{a_2^n}{b_3^k}, \dfrac{a_1^a}{b_4^i}\right);\ T_{\tilde{A}} \wedge T_{\tilde{B}}, I_{\tilde{A}} \vee I_{\tilde{B}}, F_{\tilde{A}} \vee F_{\tilde{B}} \right\rangle, \\ \text{if } a_4^i < 0, b_4^i < 0. \end{cases}$$

Definition A.5. (Ye, 2017): Let $\tilde{A}_d = \langle(a^a, b^n, c^k, d^i);(T_{\tilde{A}_d}, I_{\tilde{A}_d}, F_{\tilde{A}_d})\rangle$ be TpFN then the score function of $\tilde{A}_d$ is defined as $\text{Sc}(\tilde{A}_d) = \frac{1}{12}(a^a + b^n + c^k + d^i)(2 + T_{\tilde{A}_d} - I_{\tilde{A}_d} - F_{\tilde{A}_d})$.

Let $\tilde{A}_d = \langle(a_1^a, b_1^n, c_1^k, d_1^i);(T_{\tilde{A}_d}, I_{\tilde{A}_d}, F_{\tilde{A}_d})\rangle$ and $\tilde{B}_d = \langle(a_2^a, b_2^n, c_2^k, d_2^i);(T_{\tilde{B}_d}, I_{\tilde{B}_d}, F_{\tilde{B}_d})\rangle$ be two SVTN-number. Then,

1. If $\text{Sc}(\tilde{A}_d) < \text{Sc}(\tilde{B}_d)$, then $\tilde{A}_d$ is smaller than $\tilde{B}_d$, denoted by $\tilde{A}_d < \tilde{B}_d$.
2. If $\text{Sc}(\tilde{A}_d) = \text{Sc}(\tilde{B}_d)$;

   (a) If $Ac(\tilde{A}_d) < Ac(\tilde{B}_d)$, then $\tilde{A}_d$ is smaller than $\tilde{B}_d$, denoted by $\tilde{A}_d < \tilde{B}_d$.

   (b) If $Ac(\tilde{A}_d) = Ac(\tilde{B}_d)$, then $\tilde{A}_d$ and $\tilde{B}_d$ are the same, denoted by $\tilde{A}_d = \tilde{B}_d$.

B EOQ Model in Neutrosophic Environment

In an environment characterized by its clarity and precision, we can determine the total cost using equation (1). However, in real-world scenarios, this cost may exhibit slight fluctuations, thereby impacting the quantity ordering (n), shortage (v_s), and holding (i). To address this variability, we adopt a model that accounts for permissible shortages within a Neutrosophic framework. The neutrosophic methodology necessitates carrying cost (known as holding cost) per unit, ordering cost per order, and shortage cost per unit quantity. Consequently, we transform the values of n, v_s, and i into Neutrosophic numbers using a trapezoidal Neutrosophic representation as follows:

$$\tilde{n}^{\text{neu}} = \langle (n_1^{\text{neu}}, n_2^{\text{neu}}, n_3^{\text{neu}}, n_4^{\text{neu}}); (T_{\tilde{n}^{\text{neu}}}, I_{\tilde{n}^{\text{neu}}}, F_{\tilde{n}^{\text{neu}}}) \rangle,$$
$$\tilde{k}^{\text{neu}} = \langle (\tilde{k}_1^{\text{neu}}, \tilde{k}_2^{\text{neu}}, \tilde{k}_3^{\text{neu}}, \tilde{k}_4^{\text{neu}}); (T_{\tilde{k}^{\text{neu}}}, I_{\tilde{k}^{\text{neu}}}, F_{\tilde{k}^{\text{neu}}}) \rangle, \text{ and}$$
$$\tilde{i}^{\text{neu}} = \langle (\tilde{i}_1^{\text{neu}}, \tilde{i}_2^{\text{neu}}, \tilde{i}_3^{\text{neu}}, \tilde{i}_4^{\text{neu}}); (T_{\tilde{i}^{\text{neu}}}, I_{\tilde{i}^{\text{neu}}}, F_{\tilde{i}^{\text{neu}}}) \rangle.$$

Proposed model for finding total cost while considering the uncertainty

The neutrosophic total cost is denoted as $\tilde{\tau}^{\text{neu}}$**, where** $\tilde{i}^{\text{neu}}$, $\tilde{k}^{\text{neu}}$ & $\tilde{n}^{\text{neu}}$ **are neutrosophic variables**

$$\tilde{\tau}^{\text{neu}} = \tilde{i}^{\text{neu}} \frac{(v - v_s)^2}{2v} \ell \oplus \tilde{k}^{\text{neu}} \frac{(v_s)^2}{2v} \ell \oplus \tilde{n}^{\text{neu}} \frac{a}{v}, \tag{A1}$$

$$\tilde{\tau}^{\text{neu}} = \begin{bmatrix} \langle (\tilde{i}_1^{\text{neu}}, \tilde{i}_2^{\text{neu}}, \tilde{i}_3^{\text{neu}}, \tilde{i}_4^{\text{neu}}); (T_{\tilde{i}^{\text{neu}}}, I_{\tilde{i}^{\text{neu}}}, F_{\tilde{i}^{\text{neu}}}) \rangle \frac{(v - v_s)^2}{2v} \ell \oplus \left\langle \begin{matrix} (\tilde{k}_1^{\text{neu}}, \tilde{k}_2^{\text{neu}}, \tilde{k}_3^{\text{neu}}, \tilde{k}_4^{\text{neu}}); \\ (T_{\tilde{k}^{\text{neu}}}, I_{\tilde{k}^{\text{neu}}}, F_{\tilde{k}^{\text{neu}}}) \end{matrix} \right\rangle \frac{(v_s)^2}{2v} \ell \\ \oplus \langle (n_1^{\text{neu}}, n_2^{\text{neu}}, n_3^{\text{neu}}, n_4^{\text{neu}}); (T_{\tilde{n}^{\text{neu}}}, I_{\tilde{n}^{\text{neu}}}, F_{\tilde{n}^{\text{neu}}}) \rangle \frac{a}{v} \end{bmatrix}. \tag{A2}$$

Now applying the def. A.4 and def. A.5, respectively, on equation (A2), we get

$$\tilde{\tau}^{\text{neu}} = \left\langle \begin{matrix} \left[\tilde{i}_1^{\text{neu}} \cdot \frac{(v - v_s)^2}{2v} \ell, \tilde{i}_2^{\text{neu}} \cdot \frac{(v - v_s)^2}{2v} \ell, \tilde{i}_3^{\text{neu}} \cdot \frac{(v - v_s)^2}{2v} \ell, \tilde{i}_4^{\text{neu}} \cdot \frac{(v - v_s)^2}{2v} \ell \right]; (T_{\tilde{i}^{neu}}, I_{\tilde{i}^{neu}}, F_{\tilde{i}^{neu}}) \\ \oplus \left[\tilde{k}_1^{\text{neu}} \cdot \frac{(v_s)^2}{2v} \ell, \tilde{k}_2^{\text{neu}} \cdot \frac{(v_s)^2}{2v} \ell, \tilde{k}_3^{\text{neu}} \cdot \frac{(v_s)^2}{2v} \ell, \tilde{k}_4^{\text{neu}} \cdot \frac{(v_s)^2}{2v} \ell \right]; (T_{\tilde{k}^{neu}}, I_{\tilde{k}^{neu}}, F_{\tilde{k}^{neu}}) \\ \oplus \left[n_1^{\text{neu}} \cdot \frac{a}{v}, n_2^{\text{neu}} \frac{a}{v}, n_3^{\text{neu}} \frac{a}{v}, n_4^{\text{neu}} \frac{a}{v} \right]; (T_{\tilde{n}^{\text{neu}}}, I_{\tilde{n}^{\text{neu}}}, F_{\tilde{n}^{\text{neu}}}) \end{matrix} \right\rangle,$$

$$(\tilde{\tau}^{\text{neu}})^R = \frac{1}{12} \left[\left\langle \begin{bmatrix} \left(\tilde{i}_1^{\text{neu}} + \tilde{i}_2^{\text{neu}} + \tilde{i}_3^{\text{neu}} + \tilde{i}_4^{\text{neu}} \right) \cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ -(I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ -(F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}}) \end{pmatrix} \frac{(v - v_s)^2}{2v} \ell \\ \oplus \left(\tilde{k}_1^{\text{neu}} + \tilde{k}_2^{\text{neu}} + \tilde{k}_3^{\text{neu}} + \tilde{k}_4^{\text{neu}} \right) \cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ -(I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ -(F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}}) \end{pmatrix} \frac{(v_s)^2}{2v} \ell \\ \oplus \left(\tilde{n}_1^{\text{neu}} + \tilde{n}_2^{\text{neu}} + \tilde{n}_3^{\text{neu}} + \tilde{n}_4^{\text{neu}} \right) \cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{\text{neu}}} \wedge T_{\tilde{k}^{\text{neu}}} \wedge T_{\tilde{n}^{\text{neu}}}) \\ -(I_{\tilde{i}^{\text{neu}}} \vee I_{\tilde{k}^{\text{neu}}} \vee I_{\tilde{n}^{\text{neu}}}) \\ -(F_{\tilde{i}^{\text{neu}}} \vee F_{\tilde{k}^{\text{neu}}} \vee F_{\tilde{n}^{\text{neu}}}) \end{pmatrix} \frac{a}{v} \end{bmatrix} \right\rangle \right] \tag{A3}$$

$$= (\tilde{\tau}^{\text{neu}})^R (v, v_s) \text{ (Say).}$$

$(\tilde{\tau}^{\text{neu}})^R(v, v_s)$ is minimum when $\frac{\mathrm{d}(\tilde{\tau}^{\text{neu}})^R(v, v_s)}{\mathrm{d}v} = 0$, $\frac{\mathrm{d}^2(\tilde{\tau}^{\text{neu}})^R(v, v_s)}{\mathrm{d}v^2} > 0$, to optimize the total cost.

Initially, we have consider $\frac{\partial(\tilde{\tau}^{\text{neu}})^R}{\partial v} = 0$

$$\frac{\partial(\tilde{\tau}^{\text{neu}})^R(\nu,\nu_s)}{\partial\nu}\approx\frac{1}{12}\left[\left\langle\begin{bmatrix}\left(\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}}\right)\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot\left\{\frac{2\nu(\nu-\nu_s)-(\nu-\nu_s)^2}{2\nu^2}\right\}\ell\\ \oplus\left(\tilde{k}_1^{\text{neu}}+\tilde{k}_2^{\text{neu}}+\tilde{k}_3^{\text{neu}}+\tilde{k}_4^{\text{neu}}\right)\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot\left\{\frac{-(\nu_s)^2}{2\nu^2}\right\}\ell\\ \oplus\left(\tilde{n}_1^{\text{neu}}+\tilde{n}_2^{\text{neu}}+\tilde{n}_3^{\text{neu}}+\tilde{n}_4^{\text{neu}}\right)\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot\left(-\frac{a}{\nu^2}\right)\end{bmatrix}\right\rangle\right].$$

Now differentiating equation (A3) with respect to ν to get equation (A4).

$$\begin{bmatrix}\begin{pmatrix}\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}\\+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot\nu^2\\ -\begin{bmatrix}\begin{pmatrix}\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}\\+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\\ \oplus\begin{pmatrix}\tilde{k}_1^{\text{neu}}+\tilde{k}_2^{\text{neu}}\\+\tilde{k}_3^{\text{neu}}+\tilde{k}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\end{bmatrix}\cdot\nu_s^2\end{bmatrix}=\frac{2\begin{pmatrix}\tilde{n}_1^{\text{neu}}+\tilde{n}_2^{\text{neu}}\\+\tilde{n}_3^{\text{neu}}+\tilde{n}_3^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot a}{\ell}$$

$$\nu^2=\frac{1}{\begin{pmatrix}\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}\\+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}}\otimes\begin{bmatrix}\dfrac{2\begin{pmatrix}\tilde{n}_1^{\text{neu}}+\tilde{n}_2^{\text{neu}}\\+\tilde{n}_3^{\text{neu}}+\tilde{n}_3^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\cdot a}{\ell}\\ \otimes\begin{bmatrix}\begin{pmatrix}\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}\\+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\\ \oplus\begin{pmatrix}\tilde{k}_1^{\text{neu}}+\tilde{k}_2^{\text{neu}}\\+\tilde{k}_3^{\text{neu}}+\tilde{k}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\end{bmatrix}\cdot\nu_s^2\end{bmatrix}. \tag{A4}$$

Now, similarly differentiating equation (A3) with respect to ν_s, i.e. $\frac{\partial(\tilde{\tau}^{\text{neu}})^R}{\partial\nu_s}=0$

$$\frac{\partial(\tilde{\tau}^{\text{neu}})^R(\nu,\nu_s)}{\partial\nu_s}\approx-\frac{\begin{pmatrix}\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}\\+\tilde{i}_3^{\text{neu}}\\+\tilde{i}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}}{\nu}\cdot(\nu-\nu_s)\ell+\frac{\begin{pmatrix}\tilde{k}_1^{\text{neu}}+\tilde{k}_2^{\text{neu}}\\+\tilde{k}_3^{\text{neu}}\\+\tilde{k}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}}{\nu}\cdot\nu_s\ell=0$$

$$\nu_s=\frac{(\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}})\cdot(2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}}))\cdot\nu}{\left[(\tilde{i}_1^{\text{neu}}+\tilde{i}_2^{\text{neu}}+\tilde{i}_3^{\text{neu}}+\tilde{i}_4^{\text{neu}})\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\oplus\begin{pmatrix}\tilde{k}_1^{\text{neu}}+\tilde{k}_2^{\text{neu}}\\+\tilde{k}_3^{\text{neu}}+\tilde{k}_4^{\text{neu}}\end{pmatrix}\cdot\begin{pmatrix}2+(T_{\tilde{i}^{\text{neu}}}\wedge T_{\tilde{k}^{\text{neu}}}\wedge T_{\tilde{n}^{\text{neu}}})\\-(I_{\tilde{i}^{\text{neu}}}\vee I_{\tilde{k}^{\text{neu}}}\vee I_{\tilde{n}^{\text{neu}}})\\-(F_{\tilde{i}^{\text{neu}}}\vee F_{\tilde{k}^{\text{neu}}}\vee F_{\tilde{n}^{\text{neu}}})\end{pmatrix}\right]}$$

Putting the value of ν_s in equation (A4), we have

$$v = \sqrt{\frac{2[\{(\tilde{i}_1^{neu} + \tilde{i}_2^{neu} + \tilde{i}_3^{neu} + \tilde{i}_4^{neu}) \oplus (\tilde{k}_1^{neu} + \tilde{k}_2^{neu} + \tilde{k}_3^{neu} + \tilde{k}_4^{neu})\}]\cdot(\tilde{n}_1^{neu} + \tilde{n}_2^{neu} + \tilde{n}_3^{neu} + \tilde{n}_4^{neu}) \cdot(2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}))\cdot a}{\left[(\tilde{i}_1^{neu} + \tilde{i}_2^{neu} + \tilde{i}_3^{neu} + \tilde{i}_4^{neu})\cdot(\tilde{k}_1^{neu} + \tilde{k}_2^{neu} + \tilde{k}_3^{neu} + \tilde{k}_4^{neu})\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\right]\cdot \ell}}$$

Similarly,

$$v_s = \sqrt{\frac{2\left[(\tilde{i}_1^{neu} + \tilde{i}_2^{neu} + \tilde{i}_3^{neu} + \tilde{i}_4^{neu})\cdot(\tilde{n}_1^{neu} + \tilde{n}_2^{neu} + \tilde{n}_3^{neu} + \tilde{n}_4^{neu})\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\right]\cdot a}{(\tilde{k}_1^{neu} + \tilde{k}_2^{neu} + \tilde{k}_3^{neu} + \tilde{k}_4^{neu})\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix} \cdot \left[\begin{pmatrix} \tilde{i}_1^{neu} + \tilde{i}_2^{neu} \\ + \tilde{i}_3^{neu} + \tilde{i}_4^{neu} \end{pmatrix} + \begin{pmatrix} \tilde{k}_1^{neu} + \tilde{k}_2^{neu} \\ + \tilde{k}_3^{neu} + \tilde{k}_4^{neu} \end{pmatrix}\right]\cdot \ell}}$$

Therefore, optimal quantity while considering uncertainty

$$\tilde{v}^{neu} = \sqrt{\frac{2\left[\left\{\begin{pmatrix} \tilde{i}_1^{neu} + \tilde{i}_2^{neu} \\ + \tilde{i}_3^{neu} + \tilde{i}_4^{neu} \end{pmatrix} \oplus \begin{pmatrix} \tilde{k}_1^{neu} + \tilde{k}_2^{neu} \\ + \tilde{k}_3^{neu} + \tilde{k}_4^{neu} \end{pmatrix}\right\}\right]\cdot \begin{pmatrix} \tilde{n}_1^{neu} + \tilde{n}_2^{neu} \\ + \tilde{n}_3^{neu} + \tilde{n}_4^{neu} \end{pmatrix}\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\cdot a}{\left[(\tilde{i}_1^{neu} + \tilde{i}_2^{neu} + \tilde{i}_3^{neu} + \tilde{i}_4^{neu})\cdot(\tilde{k}_1^{neu} + \tilde{k}_2^{neu} + \tilde{k}_3^{neu} + \tilde{k}_4^{neu})\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\right]\cdot \ell}}. \tag{A5}$$

And, similarly, optimal shortage quantity while considering uncertainty:

$$\tilde{v}_s^{neu} = \sqrt{\frac{2\left[\begin{pmatrix} \tilde{i}_1^{neu} + \tilde{i}_2^{neu} \\ + \tilde{i}_3^{neu} + \tilde{i}_4^{neu} \end{pmatrix}\cdot \begin{pmatrix} \tilde{n}_1^{neu} + \tilde{n}_2^{neu} \\ + \tilde{n}_3^{neu} + \tilde{n}_4^{neu} \end{pmatrix}\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\right]\cdot a}{\begin{pmatrix} \tilde{k}_1^{neu} + \tilde{k}_2^{neu} \\ + \tilde{k}_3^{neu} + \tilde{k}_4^{neu} \end{pmatrix}\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ - (I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ - (F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\cdot \left[\begin{pmatrix} \tilde{i}_1^{neu} + \tilde{i}_2^{neu} \\ + \tilde{i}_3^{neu} + \tilde{i}_4^{neu} \end{pmatrix} + \begin{pmatrix} \tilde{k}_1^{neu} + \tilde{k}_2^{neu} \\ + \tilde{k}_3^{neu} + \tilde{k}_4^{neu} \end{pmatrix}\right]\cdot \ell}}. \tag{A6}$$

This shows that $\psi(v, v_s)$ is minimum at $\tilde{v}^{neu}\&\tilde{v}_s^{neu}$.
Optimal (minimum) total cost while considering uncertainty,

$$\frac{\partial(\tilde{\tau}^{neu})^R(v, v_s)}{\partial v} = \frac{1}{12}\left[\left\langle \begin{bmatrix} \left(\tilde{i}_1^{neu} + \tilde{i}_2^{neu} + \tilde{i}_3^{neu} + \tilde{i}_4^{neu}\right)\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ -(I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ -(F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix} \otimes \left\{\frac{2v(v - v_s) - (v - v_s)^2}{2v^2}\right\}\ell \\ \oplus\left(\tilde{k}_1^{neu} + \tilde{k}_2^{neu} + \tilde{k}_3^{neu} + \tilde{k}_4^{neu}\right)\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ -(I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ -(F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\cdot \left\{\frac{-(v_s)^2}{2v^2}\right\}\ell \\ \oplus\left(\tilde{n}_1^{neu} + \tilde{n}_2^{neu} + \tilde{n}_3^{neu} + \tilde{n}_4^{neu}\right)\cdot \begin{pmatrix} 2 + (T_{\tilde{i}^{neu}} \wedge T_{\tilde{k}^{neu}} \wedge T_{\tilde{n}^{neu}}) \\ -(I_{\tilde{i}^{neu}} \vee I_{\tilde{k}^{neu}} \vee I_{\tilde{n}^{neu}}) \\ -(F_{\tilde{i}^{neu}} \vee F_{\tilde{k}^{neu}} \vee F_{\tilde{n}^{neu}}) \end{pmatrix}\cdot \left(-\frac{a}{v^2}\right) \end{bmatrix} \right\rangle\right]. \tag{A7}$$