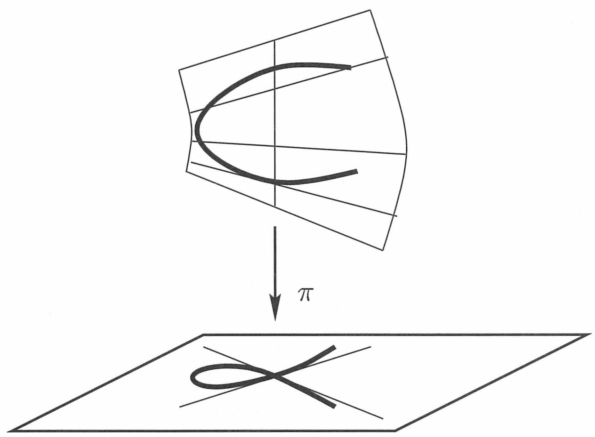


Resolution of a singularity by blowing up – a node

Alterating singularities

by Frans Oort

Recently the problem of resolution of singularities, solved by Hironaka in 1964, was considered from a completely new, fresh point of view. We report on these exciting methods of A. J. de Jong and various results.

1 Singularities of algebraic varieties

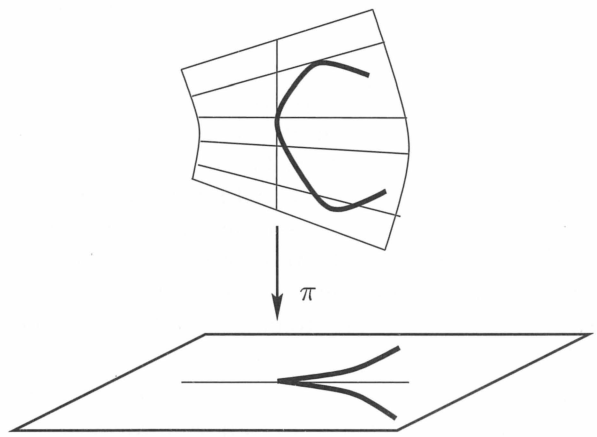
Consider an algebraic variety V of dimension d over some field k. For example this could be an algebraic curve given by an equation like $Y^2 = X^3$. We say that a point $P \in V$ is *non-singular on* V if "V locally around P looks like affine space $\mathbb{A}^d$." This notion can be made more precise, for example, by the Jacobian criterion on the partial derivatives of equations defining the variety, or with the help of the structure of the local ring $\mathcal{O}_{V,P}$ (the ring of germs of functions on V regular at P). Algebraic varieties show up in many disguises (solutions of equations in number theory, spaces where you like to compute integrals, solutions of differential equations, and so on). In practice we see that abstract theory works well on a non-singular variety (e.g., think of Hodge theory), and computations are usually easy to perform when working on regular models. But singularities often cause difficulties. Hence the natural question arises: *given a variety V, can we find a variety V', with a map $\varphi : V' \to V$ such that V' is non-singular, and φ is an isomorphism almost everywhere*? This is the famous problem of ***resolution of singularities***.

2 Algebraic curves

Here is a case where the solution of the problem is not difficult (and classically well known). Let $V = C$ be an *algebraic curve.* The construction of a non-singular model of the function field of C can be produced in several ways.

(a) Here is an algebraic method. If R is is the ring of functions on an algebraic curve, and this ring is integrally closed in its field of fractions $Q(R) = L$, then the curve is nonsingular. Integrally closed ("normal") means that elements in L which are a zero of a monic equation over R are already in R. Taking the integral closure (an algebraic operation) of rings of functions on an algebraic curve produces a new curve which is non-singular. Here is an example: the affine curve $\mathcal{Z}(-Y^2 + X^3) = C \subset \mathbb{A}^2$ is singular in $P = (0,0)$. The integral closure of $R = k[X,Y]/(-Y^2 + X^3) = k[x,y]$ is $R' = k[x,y,y/x] = k[y/x] \cong k[t]$, and the parametrization $\mathbb{A}^1 = C' \to C$ given by $t \mapsto (x = t^2, y = t^3) \in C$ is the desingularization of this curve. The general case of an arbitrary curve is not very much more complicated.

(b) Here is a geometric method. Let $C \subset \mathbb{A}^n$ be an affine curve, singular at P. Perform a "blowing-up" of $\mathbb{A}^n$ at a center containing P. One might hope that taking something like the inverse image of C under a blowing-up (algebraic geometers say: "taking the strict transform") reduces the singular behavior of the curve. One can see that this is indeed the case by a proper choice of the center of blowing up, and we can arrive at a desigularization after a finite number of steps. See [7, page 37, Exercise 5.6] for an explanation in easy cases.



Resolution of a singularity by blowing up – a cusp

3 Resolution of singularities

Soon one found out that the method which works for algebraic curves in general does not work for higher dimensions.
Example: $\mathcal{Z}(UW - V^2) = S \subset \mathbb{A}^3$ is a singular surface, and its coordinate ring is integrally closed. Note that if the characteristic of k is $\neq 2$ this is the quotient singularity of the $\mathbb{Z}/2$-action $X \mapsto -X$, $Y \mapsto -Y$, with $U = X^2$, $V = XY$, $W = Y^2$. For this surface the method of normalization does not produce a non-singular model, but the method of blowing up still works, as is easily seen. However, complications arise in higher dimensions, and a fascinating problem is born, **resolution of singularities:** *for an algebraic variety V, find a modification $V' \to V$ with V' nonsingular.* Here we have:

> **Definition.** *We say that $\varphi : W \to V$ is a* modification *if it isa birational, proper morphism.* (Terms like "blowing up", "dilatation", "quadratic transformation", "Cremona transformation" are used to indicated this or a closely related concept.)

Heisuke Hironaka

In fact one should like to solve a more precise problem: given V, write $V^o := V - \mathrm{Sing}(V)$ for the regular part of V, and try to find a modification $W \to V$, which is an isomorphism on $W \supset (W - \varphi^{-1}(V^o)) \xrightarrow{\sim} V^o \subset V$. Partial results where achieved in early times, e.g., see [15], [16], [17]. The general case *in characteristic zero* was solved: H. Hironaka proved in 1964:

> *In characteristic zero any variety can be modified into a nonsingular variety,*

and in fact, Hironaka proved it in a strong form, namely only blowing up in the singular locus is necessary, see [8]. Since then his method has been analyzed, understood better and made "canonical", see [5], [14]; for more details see various contributions in [12]. It seemed that the topic was getting very complicated. Hironaka's ingenuous proof had many applications. It was not easy to understand the fine details of his proof.

Generalizing that method to varieties in positive characteristic has failed up to now: resolution of singularities *in positive characteristic* has been a topic to which many years of intensive research have been devoted. Abhyankar solved resolution of singularities for low dimensions, see [1], [2]. For the general question of resolution of singularities in positive characteristic we seem to have neither a fully verified theorem nor a counterexample, see [13]. The algorithms involved in Hironaka's theory are difficult to understand in more complicated situations. It seemed we were at the limits of our possibilities ...? Until:

4 Alterations

> **Definition.** *A morphism $W \to V$ is an alteration* (say between varieties, or between integral schemes), *if it is dominant, proper and generically finite* see [pre-9], [9]. – Any modification is an alteration; a finite (ramified) covering is an alteration (and one could "feel" an alteration as a combination of these two concepts).

This new terminology was invented in order to approach the problem of replacing a singular variety by a nonsingular one: In 1995 A. J. de Jong proved:

> *Any variety can be altered into a nonsingular variety.*

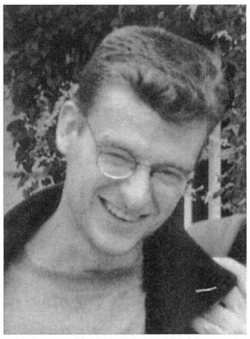
In contrast with the involved proof by Hironaka, the proof by de Jong is clear, geometric, and not very involved, see [9]. For surveys see [4], [11], and the first chapter in the first part of [12]. The proof of this theorem can be refined to a proof of the weak form of resolution of singularities in characteristic zero, see [6], [3].

Over a field of characteristic zero the method by de Jong gives a slightly weaker result (no control over where the blow ups take place). However, the method of alterations thus invented works in geometric situations, also in positive characteristic, or in mixed characteristics. In relative situations it proves much more than what could be achieved by Hironaka's method, and in many cases it is good enough for applications.

We like to mention a clear difference between the two methods. In the approach taken by Hironaka singularities of a variety are studied closely, invariants are defined, and methods are applied (blowing up, algorithms involved) in order to improve the situation, in the sense that the given invariants get "better". The algorithm then should terminate (and in fact, in characteristic zero it does), resulting in the construction of a regular variety. A big advantage of this process developed by Hironaka (and by many others) is the fact that usually it is very explicit, it is canonical in a certain sense, and once it works, the result is in its strongest form.

In the approach by Johan de Jong singularities in the beginning are completely ignored. A variety is fibered by curves, and certain operations are performed, creating possibly many more singularities, until this fibering is in a manageable form. Induction on the dimension allows us to assume the base space is regular, and only then, finally, attention is paid to the singularities. But these are like normal crossings singularities of an algebraic curve, and an easy, explicit blowing up finishes the job.

5 Epilogue



A. Johan de Jong

The original paper [9] by de Jong is written in a clear style, and we advice the reader to consult that; you can find surveys in [4], [12]. Or, if you know some algebraic geometry, start with the gist of the strategy (e.g., as exposed in [11]), and try to finish the whole proof of obtaining a nonsingular variety by an alteration of a singular one; it is a rewarding exercise!

From a technical point of view the new method initiated by de Jong throws a completely new light on various situations, and allows several applications, not possible with earlier results, see, e.g., [10].

From a philosophical point of view this new approach is refreshing and promising. It shows the power of geometric methods, it tells us to postpone difficult algorithms and computation in proofs to a later stage, and it also shows that a combination of modern thinking and well-known, established techniques can solve difficult problems.

References

[1] S. Abhyankar: *Resolution of singularities of arithmetic surfaces,* Arithm. Algebr. Geom. (Ed. O. F. G. Schilling) (Proc. Conf. Arithm. Algebr. Geom. at Purdue, 1963), Harper & Row, 1965, pp. 111–152 .

[2] S. Abhyankar: *Local uniformization of algebraic surfaces over ground fields of characteristic $p \neq 0$,* Ann. Math. **63** (1965), 491–526.

[3] D. Abramovich & A. J. de Jong: *Smoothness, semistability, and toroidal geometry,* Manuscript, April 1996, 12 pp.

[4] P. Berthelot: *Altérations de variétés algébriques [d'après A. J. de Jong],* Sém. Bourbaki 1995–1996, Exp. 815.

[5] E. Bierstone & P. D. Milman: *Canonical desingularization in characteristic zero by blowing up the maximal strata of a local invariant,* Invent. Math. **128** (1997), 207–302.

[6] F. Bogomolov & T. Pantev: *Weak Hironaka theorem,* Math. Res. Lett. **3** (1996), 299–307.

[7] R. Hartshorne: *Algebraic geometry,* Grad. Texts Math. **52**, Springer-Verlag, 1977.

[8] H. Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II,* Ann. of Math. **79** (1964), 109–326.

[pre-9] A. J. de Jong: *Smoothness, semi-stability and alterations,* Preprint 916, Utrecht, June 1995, 32 pp.

[9] A. J. de Jong: *Smoothness, semi-stability and alterations,* Publications Mathématiques I.H.E.S. **83**, 1996, pp. 51–93.

[10] A. J. de Jong: *Families of curves and alterations,* Manuscript, February 1996, 16 pp.

[11] F. Oort: *Alterations can remove singularitites.* Featured review. Bull. AMS (New Series) **35** (1998), 319–331.

[12] *Resolution of Singularities,* a research text book in tribute to Oscar Zariski. (H. Hauser, J. Lipman, F. Oort, A. Quirós, eds.), Birkhäuser, to appear.
In this volume: D. Abramovich & F. Oort: *Alterations and resolution of singularities.*, pp. 37–106.

[13] M. Spivakovsky: *Resolution of singularities I: Local Uniformization,* Manuscript, September 1996, 96 pp.

[14] O. Villamayor: *Cronstructiveness of Hironaka's resolution,* Ann. Scient. Ec. Norm. Sup., 4^e Ser. **22** (1989), 1–32.

[15] O. Zariski: *The reduction of the singularities of an algebraic surface,* Ann. Mat. **40** (1939), 639–689.

[16] O. Zariski: *Local uniformization on algebraic varieties,* Ann. Math. **41** (1940), 852–860.

[17] O. Zariski: *Reduction of the singularities of algebraic three dimensional varieties,* Ann. Math. **45** (1944), 472–542.

Address of author
Dr. Frans Oort
Mathematisch Instituut
Budapestlaan 6
NL – 3508 TA Utrecht
The Netherlands
oort@math.uu.nl