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Constructing models for spherical and elliptical densities

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Abstract: The article provides construction algorithms for consistent model classes of continuous spherical and elliptical distributions. The algorithms are based on characterization theorems for consistent families of generator functions. This characterization uses the term of complete monotonicity. The algorithms are applied to several examples of generators leading to consistent families of generators with explicit formulas for marginal densities of arbitrary dimension.

Keywords: elliptical distributions, spherical distributions, generator functions, complete monotonicity

MSC 2020: 60E99, 62H99

1 Introduction

Apart from copulas, elliptically contoured distributions (abbreviated: elliptical distributions) play an important role in modelling multivariate distributions. Models for elliptical distributions are applied in various fields, such as finance (see McNeil et al. [13], for example), ecological science, econometry, engineering, and other disciplines. The classical theory is presented in the popular textbooks by Anderson [2], Fang and Zhang [6], and Fang et al. [5]. Fundamental properties of elliptical distributions were published in the papers by Kelker [9], Cambanis et al. [3], and Kano [8], among others. Kano [8] introduced the term of consistent spherical distributions. Since the publication of the Kano article, it is an open problem whether there are principles for constructing consistent families of spherical distributions. Based on such a family, a corresponding family of elliptical distributions can be established. In this article, we show that the consistency of families of spherical distributions is closely related with complete monotonicity of generator functions. Contrary to most of the other papers, we define the continuous elliptical distribution by use of the density instead of characteristic functions.

Parametric estimation problems in the context of elliptical distributions were examined by a series of authors, for example, see the studies by Maruyama and Seo [12] and Srivastava and Bilodeau [17]. Semiparametric estimators for elliptical distributions were considered in the paper by Liebscher [10], where the generator function is estimated by a kernel estimator.

The classes of distributions that we focus on in this article are introduced next.

DEFINITIONS: A p-dimensional random vector X follows a *continuous spherical distribution* $\mathrm{SC}_p(g_p)$ if its density function exists and is given by $g_p(t^T t)$ for $t \in \mathbb{R}^p$ with a function $g_p : [0, \infty) \to [0, \infty)$. We write $X \sim \mathrm{SC}_p(g_p)$.

Function g_p is referred to as a *density generator* for dimension p if g_p is non-negative, and fulfills

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$$\int_0^\infty r^{p/2-1} g_p(r) \mathrm{d}r = \frac{\Gamma(p/2)}{\pi^{p/2}} = s_p^{-1} \quad \text{for } p \geq 1. \tag{1.1}$$

A random vector $Y \in \mathbb{R}^p$ has a *continuous elliptical distribution* (of full rank) with parameters $\mu \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p,p}$, $\operatorname{rank}(\Sigma) = p$, if Y has the same distribution as $\mu + A^T Z$, where $Z \sim \mathrm{SC}_p(g_p)$ and $A \in \mathbb{R}^{p,p}$ is a matrix such that $A^T A = \Sigma$. In symbols $Y \sim \mathrm{EC}_p(\mu;\, \Sigma;\, g_p)$.

For the density f_Y of Y, we have

$$f_Y(y) = |\Sigma|^{-1/2} g_p((y-\mu)^T \Sigma^{-1} (y-\mu)) \tag{1.2}$$

for $y \in \mathbb{R}^p$. Let $s_p = \frac{\pi^{p/2}}{\Gamma(p/2)}$. Condition (1.1) ensures that $t \mapsto g_p(t^T t)$ ($t \in \mathbb{R}^p$) and f_Y according to (1.2) represent a density. Next, we provide the theorems about the representation of spherical and elliptical random vectors.

Theorem 1.1. *Let $X \sim \mathrm{SC}_p(g_p)$ and U be a random vector having uniform distribution on the (p-1)-dimensional unit sphere. Then we have*

$$X \stackrel{d}{=} RU, \tag{1.3}$$

where $R = \|X\|$ is a real non-negative random variable independent of U. The symbol $\stackrel{d}{=}$ means equality of the distributions. Moreover, R has the density

$$h_p(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} g_p(r^2) \quad (r \geq 0). \tag{1.4}$$

From (1.4), it follows that

$$f_{R^2}(r) = \frac{\pi^{p/2}}{\Gamma(p/2)} r^{p/2-1} g_p(r) \tag{1.5}$$

is the density of R^2, where R is the random radius in (1.3). Considering specific families of generator functions, this formula will be used in the generation of spherical random variates according to Equation (1.3). Theorem 1.2 is a consequence of Theorem 1.1 (cf. [6], Corollary 1 to Theorem 2.6.1):

Theorem 1.2. *Let $Y \sim \mathrm{EC}_p(\mu;\, \Sigma;\, g_p)$ with $\operatorname{rank}(\Sigma) = p$. Assume that U is a random vector having uniform distribution on the (p-1)-dimensional unit sphere. Then*

$$Y \stackrel{d}{=} \mu + RAU$$

holds, where R is a real non-negative random variable independent of U and having density (1.4), $\Sigma = A^T A$ *holds for a matrix $A \in \mathbb{R}^{p,p}$.*

Unfortunately, the parametrization $(\mu;\, \Sigma;\, g_p)$ of EC_p densities is not unique. This is stated in Theorem 1.3 (cf. [6], Theorem 2.6.2):

Theorem 1.3. *Suppose that $Y \sim \mathrm{EC}_p(\mu;\, \Sigma;\, g_p)$ and $Y \sim \mathrm{EC}_p(\mu^*;\, \Sigma^*;\, g_p^*)$. Then there is a constant $c > 0$, such that*

$$\mu^* = \mu, \quad \Sigma^* = c\Sigma,$$
$$g_p^*(z) = c^{p/2} g_p(cz) \quad \textit{for almost every } z \in \mathbb{R}.$$

Theorem 1.3 is the reason that we do not consider the scale parametrization of g_p.

In this article, the aim is to provide classes of continuous spherical and elliptical distributions with explicit formulas for marginal densities of arbitrary dimension. We are interested in consistent families of generators for random vectors $X \sim \mathrm{SC}_p(g_p)$, where every marginal distribution of X of dimension m is distributed as

$\mathrm{SC}_m(h)$ and h also belongs to the consistent family. We search for an algorithm by which, starting from g_1, a family $\{g_p\}_{p\geq 1}$ of density generators can be derived. Here, ideas of Kano's [8] paper are utilized. In Section 2, we study the properties of consistent families of generator functions. Section 3 is devoted to the construction of families. The random number generation of spherically distributed vectors is briefly discussed in Section 4. A survey of consistent families of elliptical distributions with explicit formulas can be found in Sections 5 and 6. Section 7 presents the proofs of the presented theorems.

2 Characterization of consistent generator families

In this section, we consider a family of non-negative functions $\{g_p\}_{p\in I}$, where $I = \{1, 2, \ldots, p_0\}$, $p_0 \geq 3$, or $I = \{1, 2, \ldots\} = \mathbb{N}$, i.e., $p_0 = \infty$. We call a family $\{g_p\}_{p\in I}$ of density generators *consistent* if

$$g_p(z_1^2 + \ldots + z_p^2) = \int_{-\infty}^{\infty} g_{p+1}(z_1^2 + \ldots + z_{p+1}^2)\mathrm{d}z_{p+1} = 2\int_0^{\infty} g_{p+1}(z_1^2 + \ldots + z_p^2 + t^2)\mathrm{d}t \tag{2.1}$$

for all $z_1, \ldots, z_p \in \mathbb{R}$, $1 \leq p < p_0$. The notion of consistency goes back to Kano [8]. Now we give the most important statements of Kano's paper.

Theorem 2.1. [8], *Theorem* 1: (*a*) *The family* $\{g_p\}_{p\in I}$ *of density generators is consistent if and only if*

$$g_p(r) = 2\int_0^{\infty} g_{p+1}(r + t^2)\mathrm{d}t = \int_r^{\infty} \frac{1}{\sqrt{y - r}} g_{p+1}(y)\mathrm{d}y \tag{2.2}$$

for $r \geq 0, 1 \leq p < p_0$.

(*b*) *Let* $\{g_p\}_{p\geq 1}$ *be an infinite family of density generators. This family is consistent if and only if there exists a random variable* $\xi > 0$ *with distribution function* F_0, *unrelated to p, such that for* $p \geq 1$,

$$X_p \overset{d}{=} \frac{1}{\sqrt{\xi}} Z_p,$$

where X_p *has density* $t \mapsto g_p(t^T t)$, Z_p *is a p-dimensional standard normal random vector,* Z_p *and* ξ *are independent, and*

$$g_p(r) = \int_0^{\infty} \left(\frac{t}{2\pi}\right)^{p/2} e^{-rt/2}\mathrm{d}F_0(t) \tag{2.3}$$

for almost all $r \geq 0$.

Equation (2.3) can be interpreted in a manner that allows g_p to be represented as a Laplace-Stieltjes transform:

$$g_p(r) = \int_0^{\infty} \left(\frac{t}{\pi}\right)^{p/2} e^{-rt}\mathrm{d}F_0(2t) = \int_0^{\infty} e^{-rt}\mathrm{d}\tilde{F}_p(t),$$

where $\tilde{F}_p(t) = \int_0^t (s/\pi)^{p/2}\mathrm{d}F_0(2s)$. Unfortunately, in many cases, F_0 and $\tilde{F}_p$ are not of explicit form. If F_0 has a density, then it is denoted by f_0. In this case, the function $t \mapsto \mathcal{L}^{-1}(g_p)(t) = 2\left(\frac{t}{\pi}\right)^{p/2} f_0(2t)$ is the inverse Laplace transform of g_p, which implies

$$f_0(t) = \mathcal{L}^{-1}(g_p)(t/2)\frac{2^{p/2-1}\pi^{p/2}}{t^{p/2}} \quad (t \geq 0). \tag{2.4}$$

The following Proposition 2.2 provides a formula for the marginal densities of spherical distributions. This statement follows directly from (2.1).

Proposition 2.2. *Let $\{g_p\}_{p\in I}$ be a consistent family of density generators. We consider a p-dimensional random vector X with continuous spherical distribution* $\mathrm{SC}_p(g_p)$.

Then $z \longmapsto g_q(z_1^2 + \ldots + z_q^2)$ $(z = (z_1, \ldots, z_q)^T \in \mathbb{R}^q)$ *is the q-dimensional marginal density of X for* $q < p$.

3 Construction of consistent density generators

Consistent families are connected with completely monotonic functions on $[0, \infty)$ (cf. [11]):

DEFINITION: A function $f : [0, \infty) \to [0, \infty)$ is *completely monotonic* if all derivatives exists on $[0, \infty)$ and

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for } x \geq 0, n = 1, 2, \ldots$$

The Bernstein-Widder theorem (cf. [11], Theorem A) gives an important assertion about the characterization of completely monotonic functions:

Theorem 3.1. *The function f is completely monotonic on* $[0, \infty)$ *if and only if*

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}F(t) \quad (x \geq 0)$$

with a suitable function $F : [0, \infty) \to \mathbb{R}$, *which is bounded and non-decreasing.*

The central idea in this section is to determine the function g_1 and to provide evaluation formulas for the other generators $\{g_p\}_{p\geq 2}$ to obtain a consistent family. The following Theorem 3.2 provides an equivalent characterization of families of functions satisfying the consistency condition (2.2) in the case of an infinite family $\{g_p\}$ with $p_0 = \infty$.

Theorem 3.2. *Suppose that* $\lim_{r\to\infty} g_1(r) = 0$. *Then the family* $\{g_p\}_{p\geq 1}$ *of functions* $g_p : [0, \infty) \to [0, \infty)$ *fulfills* (2.2) *for* $p \geq 1$, *and* $g_p(0) = \lim_{u\to 0+} g_p(u) < +\infty$ *for* $p \geq 1$ *if and only if*

$g_1 : [0, \infty) \to [0, \infty)$ *is completely monotonic on* $[0, \infty)$, *the functions* g_p $(p \geq 1)$ *are differentiable on* $[0, \infty)$ *and the following two conditions are satisfied*:

$$g_{p+2}(r) = -\frac{1}{\pi} g_p'(r), \tag{3.1}$$

for $p \geq 1, r \geq 0$ *and*

$$g_2(r) = 2\int_0^\infty g_3(r + y^2)\mathrm{d}y \quad (r \geq 0). \tag{3.2}$$

Furthermore, an equation describing the relation between g_p and g_{p+1} can be established using fractional derivatives:

Proposition 3.3. *Assume that* $g_1 : [0, \infty) \to [0, \infty)$ *is completely monotonic on* $[0, \infty)$ *with* $\lim_{r\to\infty} g_1(r) = 0$, *and* $\{g_p\}_{p\geq 1}$ *is a family of functions* $g_p : [0, \infty) \to [0, \infty)$ *satisfying* (2.2) *and* $g_p(0+) < +\infty$ *for* $p \geq 1$. *Then*

$$(g_p(-u))^{(1/2)} = \sqrt{\pi}\, g_{p+1}(-u) \quad (u \leq 0, p \geq 1),$$

where the derivative of half order is meant with respect to u, and it is a fractional derivative in the sense of Caputo, see the definition in the Appendix.

Considering the construction of elliptical densities, functions g_p have to satisfy the additional condition (1.1). Theorem 3.4 contains the corresponding statement.

Theorem 3.4. *Let $\{g_p\}_{p\in I}$ be a family of functions.*

In the case $p_0 = \infty$, suppose that $g_1 : [0,\infty) \to [0,\infty)$ is completely monotonic on $[0,\infty)$.

In the case $p_0 < \infty$, assume that $\{g_p\}_{p\in I}$ is a finite family of non-negative functions on $[0,\infty)$ with $g_p(0) = g_p(0+) < +\infty$ for $p \in I$, and $g_1, \dots, g_{p_0-2}$ are differentiable.

Further assume that identities $\lim_{r\to\infty} g_1(r) = 0$, (3.1) for $p + 2 \le p_0$, and (3.2) are fulfilled. Let

$$\int_0^\infty r^{-1/2} g_1(r)\mathrm{d}r = 1 \tag{3.3}$$

be satisfied. Then $\{g_p\}_{p\ge 1}$ is a consistent family of density generators.

On the basis of Theorem 3.4, we can provide algorithms for the construction of functions g_p.

Algorithm for infinite p_0

(1) Choose a completely monotonic function g_1 with $g_1(0+) < +\infty$, (3.3) and $\lim_{r\to\infty} g_1(r) = 0$.

(2) Evaluate $g_3(r) = -\frac{1}{\pi} g_1'(r)$.

(3) Evaluate $g_2(r) = 2\int_0^\infty g_3(r + y^2)\mathrm{d}y$.

(4) For $k = 1, 2, \dots,$ do.

$$g_{2k+2}(r) = -\frac{1}{\pi} g_{2k}'(r), \quad g_{2k+3}(r) = -\frac{1}{\pi} g_{2k+1}'(r).$$

Algorithm for finite p_0

(1) Choose a non-negative differentiable function g_1 with $g_1(0+) < +\infty$, (3.3) and $\lim_{r\to\infty} g_1(r) = 0$.

(2) Evaluate $g_3(r) = -\frac{1}{\pi} g_1'(r)$ and check $g_3(r) \ge 0$ for $r \ge 0$.

(3) Evaluate $g_2(r) = 2\int_0^\infty g_3(r + y^2)\mathrm{d}y$ and check $g_2(r) \ge 0$ for $r \ge 0$.

(4) For $k = 1, 2, \dots, \lfloor (p_0 - 3)/2 \rfloor$, do

$$g_{2k+2}(r) = -\frac{1}{\pi} g_{2k}'(r), \quad g_{2k+3}(r) = -\frac{1}{\pi} g_{2k+1}'(r),$$

and check $g_{2k+2}(r), g_{2k+3}(r) \ge 0$ for $r \ge 0$.

At first glance, it seems that for finite p_0, more checks are required in the algorithms than in the infinite case. Here, it should be noticed that completely monotonicity of g_1 has a lot of consequences under (3.1) and (3.2): g_p is completely monotonic and $g_p(r) \ge 0$ in view of Lemma 7.2, $g_p(0) = g_p(0+) < +\infty$, and $g_p(r) \ge 0$ for $r \ge 0$. The remarks on page 2 of the study by Miller and Samko [14] show that completely monotonic functions on $[0,\infty)$ cannot vanish on an interval and are strictly positive on $[0,\infty)$. Therefore, in the case $p_0 = \infty$, the density generators from the aforementioned algorithm are strictly positive on $[0,\infty)$.

Now we consider the problem of how to obtain other consistent families from two given families. One approach is to deal with mixtures of scale transformed generators.

Proposition 3.5. *Let $\{g_p\}_{p\in I}$ and $\{h_p\}_{p\in I}$ be consistent families of density generators. Then the family $\{\bar{g}_p\}_{p\ge 1}$ defined by*

$$\bar{g}_p(r) = q g_p(r) + (1-q) h_p(r) \quad (r \ge 0)$$

for $q \in [0,1]$ is consistent, too.

In this proposition, a special case is given by $h_p(r) = a^{p/2} g_p(ar)$, where $a > 0$ is a parameter. The proof can be done in a straightforward way.

4 Random number generation

Theorems 1.1 and 1.2 can be used for the generation of spherical or elliptical distributions. Now we show how to generate $X \sim \mathrm{SC}_d(g_d)$.

Algorithm

(1) Generate $V \sim \mathcal{N}(0, I)$ deploying standard algorithms, e.g., the Box-Muller method

(2) $U = \frac{1}{\| V \|} V$

(3) Generate R having density according to (1.4)

(4) $X = RU$

Variable V contains independently standard normally distributed components. The formula in step 2 can be used since U has the uniform distribution on the p-dimensional unit sphere and U is independent of g_p and parameters.

5 Models for consistent generator functions on the infinite interval $[0, \infty)$

In the following, we do not consider scale parameters for the generator families because they are taken into account in the definition of elliptical densities. Moreover, the normalizing constants are chosen such that (3.3) is satisfied. All computations were done by the computer algebra system (CAS) Mathematica. Let R_p denote the random radius of a p -dimensional spherical random vector. The Appendix contains figures of a couple of the generator functions of this section.

5.1 Normal distribution generator

The density generator is given by

$$g_p(r) = \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{r}{2}\right) \quad (r \geq 0).$$

For the density f_{R^2} of R_p^2, we obtain

$$f_{R^2}(r) = \frac{1}{2^{p/2}\Gamma(p/2)} r^{p/2-1} \exp\left(-\frac{r}{2}\right) \quad (r \geq 0)$$

in view of (1.5). Here, f_{R^2} is the χ_p^2 density. The density of the one-dimensional marginal distribution is the standard normal one.

5.2 Normal mixture generator

We introduce the normal mixture model with parameters $q \in [0, 1]$ and $a > 0$, $a \neq 1$ as follows:

$$g_p(r) = \frac{1}{(2\pi)^{p/2}} \left(q a^{p/2} \exp\left(-\frac{ar}{2}\right) + (1-q) \exp\left(-\frac{r}{2}\right) \right) \quad (r \geq 0),$$

where q is the mixture fraction. Here, R_p^2 is a mixture of $W_1 \sim \Gamma(\frac{p}{2}, \frac{2}{a})$ and $W_2 \sim \Gamma(\frac{p}{2}, 2)$ with mixture fraction q.

5.3 *t*-distribution (Pearson VII) generator

The generator function for the multivariate t_{2m}-distribution with parameter $m > 0$ is given by (cf. [6], p.71, Example 2.6.2)

$$g_p(r) = \frac{\Gamma(p/2+m)}{(2m)^{p/2}\pi^{p/2}\Gamma(m)}\left(1+\frac{r}{2m}\right)^{-p/2-m} \quad (r \geq 0).$$

For the density of $W = \frac{1}{2m}R_p^2$, we obtain

$$f_W(r) = \frac{\Gamma(p/2+m)}{\Gamma(p/2)\Gamma(m)} r^{p/2-1}(1+r)^{-p/2-m} \quad (r \geq 0),$$

which is the density of a beta-prime distribution with parameters $p/2$ and m (also called compound beta distribution, cf. [4]). The one-dimensional marginal density is just the t_{2m} density.

5.4 Difference of powers

Here, we consider the difference of Pearson VII generators defined by ($m > 0$, $a > 1$, $0 < c \leq 1$)

$$g_p(r) = ((r+1)^{-m-p/2} - c(r+a)^{-m-p/2})\gamma_{a,m,c,p} \tag{5.1}$$

for $r \geq 0$, $p \geq 1$, where

$$\gamma_{a,m,c,p} = \frac{\Gamma(m+p/2)}{\pi^{-p/2}\Gamma(m)(1-ca^{-m})}.$$

The consistency of this generator is proved in Lemma 7.4. We evaluate the inverse Laplace transform of g_p:

$$t \mapsto \frac{e^{-t} - ce^{-at}}{\Gamma(m)(1-ca^{-m})\pi^{p/2}} t^{-1+m+p/2},$$

which implies a formula for the density f_0 of F_0 by (2.4):

$$f_0(t) = \frac{e^{-t/2} - ce^{-at/2}}{2^m\Gamma(m)(1-ca^{-m})} t^{m-1}$$

for $t \geq 0$. This formula could be the basis for the generation of spherical random variables $X \sim \mathrm{SC}_p(g_p)$, g_p as in (5.1) using Theorem 2.1b.

5.5 Logarithmic generator I

We introduce the generator function g_1 with parameter $a > 1$, $b > 0$, $b \leq a - 1$:

$$g_1(r) = \frac{\sqrt{b}\ln(r+a)}{2\pi\ln(\sqrt{a}+\sqrt{b})(r+b)} \tag{5.2}$$

for $r \geq 0$. In the cases $p = 2, 3$, we obtain

$$g_2(r) = \frac{\sqrt{b}}{2\pi\ln(\sqrt{a}+\sqrt{b})} \times \left(-\frac{1}{\sqrt{r+a}(r+b)} + \frac{\ln(\sqrt{a+r}+\sqrt{r+b})}{(r+b)^{3/2}}\right), \text{ and}$$

$$g_3(r) = \frac{\sqrt{b}}{2\pi^2\ln(\sqrt{a}+\sqrt{b})}\left(-\frac{1}{(r+a)(r+b)} + \frac{\ln(r+a)}{(r+b)^2}\right)$$

for $r \geq 0$. The generator functions g_p, $p > 3$ can be derived from these formulas. In Lemma 7.5, it is proven that the resulting family $\{g_p\}_{p\geq 1}$ of generators is consistent.

5.6 Logarithmic generator II

Here, we consider a modification of the logarithmic generator I:

$$g_1(r) = \frac{1}{2\pi r}\ln(1+r),$$
$$g_2(r) = \frac{1}{2\pi}\left(-\sqrt{\frac{1}{r^2(r+1)}} + 2r^{-3/2}\ln(\sqrt{r}+\sqrt{r+1})\right),$$
$$g_3(r) = \frac{-r+(r+1)\ln(r+1)}{2\pi^2 r^2(r+1)}$$

for $r \geq 0$. Analogously to Lemma 7.5, one can show that the resulting family $\{g_p\}_{p\geq 1}$ of generators is consistent.

5.7 Logarithmic generator III

We introduce the generator function g_1 with parameters $a, c > 0$:

$$g_1(r) = \frac{\ln(1+c(r+a)^{-1})}{2\pi(\sqrt{a+c}-\sqrt{a})}$$

for $r \geq 0$. Function g_1 is completely monotonic according to formula (1.2) in [14]. Condition (3.3) is fulfilled. In the cases $p = 2, 3$, we obtain

$$g_2(r) = \frac{\sqrt{a+c}+\sqrt{a}}{2\pi\sqrt{(r+a)(r+a+c)}(\sqrt{r+a}+\sqrt{r+a+c})}, \text{ and}$$
$$g_3(r) = \frac{\sqrt{a+c}+\sqrt{a}}{2\pi^2(r+a)(r+a+c)}$$

for $r \geq 0$. We obtain the other generator functions by using (3.1). The inverse Laplace transform of g_1 can be evaluated and gives

$$t \mapsto \frac{e^{-at} - e^{-(a+c)t}}{2\pi(\sqrt{a+c}-\sqrt{a})t}.$$

Hence,

$$f_0(t) = \frac{e^{-at/2} - e^{-(a+c)t/2}}{\sqrt{2\pi}(\sqrt{a+c}-\sqrt{a})t^{3/2}}$$

holds for $s \geq 0$.

5.8 Fractional-exponential generator

The generator function g_1 with parameter $a > 0$ is introduced as follows:

$$g_1(r) = \frac{e^{-a-r}\sqrt{a}}{(r+a)\operatorname{erfc}(\sqrt{a})\pi}, \tag{5.3}$$

where $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}\mathrm{d}x$ is the complementary error function. Further, we have

$$g_2(r) = \frac{\sqrt{a}}{\pi^{3/2}\operatorname{erfc}(\sqrt{a})}\left(\frac{e^{-a-r}}{r+a} + \frac{\sqrt{\pi}\operatorname{erfc}(\sqrt{r+a})}{2(r+a)^{3/2}}\right),$$

$$g_3(r) = \frac{(r+1+a)e^{-a-r}\sqrt{a}}{(r+a)^2\operatorname{erfc}(\sqrt{a})\pi^2}.$$

We show in Lemma 7.8 that the resulting family $\{g_p\}_{p\geq 1}$ is consistent. By using identity (2.4) for $p = 1$, we obtain the density f_0 as follows:

$$f_0(t) = \frac{e^{-at/2}\sqrt{a}H\left(\frac{t}{2}-1\right)}{\sqrt{2\pi t}\operatorname{erfc}(\sqrt{a})},$$

where $H(t) = 1$ for $t \geq 0$, and $H(t) = 0$ for $t < 0$. This implies

$$g_p(r) = \frac{\sqrt{a}\,\bar{\Gamma}\left(\frac{p+1}{2}, a+r\right)}{\operatorname{erfc}(\sqrt{a})\pi^{(p+1)/2}(a+r)^{(p+1)/2}},$$

where $\bar{\Gamma}(b, z) = \int_z^\infty x^{b-1}e^{-x}\mathrm{d}x$ is the incomplete gamma function.

5.9 Difference-exponential distribution

We introduce the generator function g_1 with parameter $a > 1$:

$$g_1(r) = \frac{e^{-r} - e^{-ar}}{r \cdot 2(\sqrt{a}-1)\sqrt{\pi}} \tag{5.4}$$

for $r \geq 0$. Obviously, the equation

$$\int_1^a e^{-rt}\mathrm{d}t = \frac{e^{-r} - e^{-ar}}{r}$$

holds true, and therefore, by the Bernstein-Widder Theorem 3.1, g_1 has an inverse Laplace transform and is completely monotonic. Further, we obtain

$$g_3(r) = \frac{e^{-r}(1+r) - e^{-ar}(1+ar)}{2r^2(\sqrt{a}-1)\sqrt{\pi^3}},$$

$$g_2(r) = \frac{2e^{-r}\sqrt{r} - 2e^{-ar}\sqrt{ar} + \sqrt{\pi}(-2\Phi(\sqrt{2r}) + 2\Phi(\sqrt{2ar}))}{4r^{3/2}(\sqrt{a}-1)\pi},$$

where Φ is the distribution function of the standard normal distribution. By identity (2.4) for $p = 1$, we can derive a formula for the density f_0 (function H as mentioned earlier):

$$f_0(t) = \frac{H\left(\frac{t}{2}-1\right) - H\left(\frac{t}{2}-a\right)}{\sqrt{8t}(\sqrt{a}-1)}.$$

For g_p, we obtain

$$g_p(r) = \frac{\bar{\Gamma}\left(\frac{p+1}{2}, r\right) - \bar{\Gamma}\left(\frac{p+1}{2}, ar\right)}{r^{(p+1)/2} \cdot 2(\sqrt{a}-1)\pi^{p/2}}.$$

By means of a CAS, it is shown that identities (3.1)–(3.3) hold true. Then $\{g_p\}_{p\geq 1}$ is a consistent family of density generators in view of Theorem 3.4.

5.10 Polylogarithmic generator

The family of polylogarithmic generator functions g_p with parameters $a \in (0,1)$, $m \in \mathbb{R}$ is defined by

$$g_p(r) = \frac{\mathrm{Li}_{m-(p-1)/2}(ae^{-r})}{\pi^{p/2}\mathrm{Li}_{m+1/2}(a)} \tag{5.5}$$

for $r \geq 0$, where $z \mapsto \mathrm{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m}$ is the polylogarithmic function. Some interesting special values are

$$\mathrm{Li}_1(x) = -\ln(1-x), \quad \mathrm{Li}_0(x) = \frac{x}{1-x}, \quad \mathrm{Li}_{-1}(x) = \frac{x}{(1-x)^2}.$$

In the case of an integer parameter $m \leq 1$, the generators g_p with odd indices can be represented by a rationale function of x or by a logarithmic one. The proof that this family is a consistent family of density generators can be found in Section 7.2, Lemma 7.6.

5.11 Bessel I generator

By using modified Bessel functions K_m of the second kind (cf. [1], pp. 374), we define generator functions for dimension p with parameter $m > -1/2$ ([6], p. 72, Example 2.6.4)

$$g_p(r) = \frac{r^{m/2-(p-1)/4}}{2^{m+(p-1)/2}\pi^{p/2}\Gamma(m+1/2)} K_{m-(p-1)/2}(\sqrt{r}) \tag{5.6}$$

for $r \geq 0$. Notice that $K_{1/2}(z) = \sqrt{\frac{\pi}{2}}\, e^{-z} z^{-1/2}$ and $K_{3/2}(z) = \sqrt{\frac{\pi}{2}}\, e^{-z}(1+z)z^{-3/2}$ such that explicit formulas for $K_{\nu+1/2}$ ($\nu \in \mathbb{N}$) are available. The consistency proof for $\{g_p\}_{p\geq 1}$ is provided in Lemma 7.7.

5.12 Bessel II generator

We introduce generator functions for dimensions 1–3:

$$\begin{aligned} g_1(r) &= \frac{e^{-r-a/2}}{(r+a)^{1/2}K_0(a/2)}, \\ g_2(r) &= \frac{e^{-r/2}}{2\pi K_0(a/2)}\left(K_0\left(\frac{a+r}{2}\right) + K_1\left(\frac{a+r}{2}\right)\right), \\ g_3(r) &= \frac{e^{-r-a/2}(1+2a+2r)}{2\pi(r+a)^{3/2}K_0(a/2)} \quad (r \geq 0). \end{aligned} \tag{5.7}$$

The generator functions for higher dimensions p can be evaluated straightforwardly. The consistency proof for $\{g_p\}_{p\geq 1}$ is located in Lemma 7.8. For the Bessel II generator, we are able to provide function f_0:

$$f_0(t) = \frac{e^{-a(t-1)/2} H\left(\frac{t}{2} - 1\right)}{(t(t-2))^{1/2} K_0(a/2)}.$$

5.13 Linear-exponential generator

Define

$$g_p(r) = \frac{(r + a - p + 1)e^{-r/2}}{(2\pi)^{p/2}(a + 1)}$$

for $p \geq 1$, $r \geq 0$, $a > 2$. Utilizing a CAS, it is shown that (3.1)–(3.3) hold true. Then $\{g_p\}_{p=1\dots p_0}$ is a finite consistent family of density generators if $a + 1 > p_0$. In principle, it is possible to generalize the linear factor to a polynomial factor but then the choice of the parameters has to be considered carefully to obtain a family of density generators.

5.14 Kotz-type generator

The article by Nadarajah [15] gives a good review of properties of Kotz-type elliptical distributions. The general formula for the generator function of dimension p with parameters $N > (2 - p)/2$, $a, s > 0$ is given by

$$g_p(r) = r^{N-1} \exp(-ar^s). \tag{5.8}$$

Unfortunately, starting from a Kotz-type g_1 according to (5.8), only generators of odd index can be represented by an explicit formula. Furthermore, starting from a Kotz-type g_2 according to (5.8), there are no explicit formulas for the marginal densities of dimension 1 and odd dimensions. Exceptions exist only in some very special cases, e.g., $N = 1$, $s = 1$ (normal distribution). A series-integral representation of the one-dimensional density is provided in the study by Nadarajah [15]. Therefore, the formula for Kotz generator g_p cannot be used as a basis for a spherical family represented by explicit formulas.

Logistic and exponential power generator functions g_1 lead to very sophisticated formulas for g_p or the corresponding g_p cannot represented by an explicit formula. Elliptical distributions with logistic generators are examined in the study by Wang and Yin [18]. We will not go into further detail here.

6 Models for generator functions on the finite interval [0, 1]

In a remark below the algorithms in Section 3, it was stated that the algorithm for the case $p_0 = \infty$ gives only strictly positive generator functions on $[0, +\infty)$. Therefore, only finite families $\{g_p\}_{p:1\leq p\leq p_0}$ are suitable for modelling spherical and elliptical densities concentrated on finite intervals. We consider only two of them.

6.1 Power distribution on [0, 1]

The density generator is given by

$$g_p(r) = (1 - r)^{b-(p-1)/2} \cdot \frac{\Gamma(b + \frac{3}{2})}{\pi^{p/2}\Gamma\left(b + \frac{3-p}{2}\right)} \quad \text{for} \quad r \in [0, 1], p : 1 \leq p \leq 2b + 1, \tag{6.1}$$

where $b > 0$ is the parameter. The proof that this family is consistent for $p_0 = \lceil 2b + 3 \rceil - 1$ can be found in Section 7, Lemma 7.9. The corresponding one-dimensional density is given by

$$g_1(z^2) = (1 - z^2)^b \frac{\Gamma(b + \frac{3}{2})}{\sqrt{\pi}\,\Gamma(b + 1)} \quad \text{for } z \in [-1, 1].$$

The density of R_p can immediately be derived:

$$h_p(r) = 2r^{p-1}(1-r^2)^{b-(p-1)/2} \cdot \frac{\Gamma(b+\frac{3}{2})}{\Gamma(\frac{p}{2})\Gamma\left(b+\frac{3-p}{2}\right)} \quad \text{for } r \in [0,1].$$

6.2 Uniform distribution

The uniform distribution ([6], p. 71, Example 2.6.3) is a special case of the power distribution in the $b = (p-1)/2$. We have

$$g_p(r) = \pi^{p/2}\Gamma\left(1+\frac{p}{2}\right), \quad h_p(r) = pr^{p-1} \quad \text{for } 0 \le r \le 1.$$

7 Proofs

7.1 Proofs of the results in Sections 1–3

Proof of Theorem 1.1. Let $X \sim SC_p(g_p)$ and $Q \in \mathbb{R}^{p,p}$ be any orthogonal matrix. Then $Y = QX$ has the density

$$f_Y(y) = g_p(y^T QQ^T y) = g_p(y^T y) \quad (y \in \mathbb{R}^p).$$

Since this density does not depend on Q, Theorem 1.1 follows by applying Theorems 2.5.3 and 2.5.5 in the paper by Fang and Zhang [6]. □

Proof of Theorem 1.3. By Theorem 2.6.2 of [6], $\mu^* = \mu$ and $\Sigma^* = c\Sigma$ with a constant c. Then

$$\begin{aligned} f_Y(y) &= c^{-p/2}\,|\Sigma|^{-1/2} g_p^*(c^{-1}(y-\mu)^T\Sigma^{-1}(y-\mu)) \\ &= |\Sigma|^{-1/2} g_p((y-\mu)^T\Sigma^{-1}(y-\mu)) \end{aligned}$$

for almost every $y \in \mathbb{R}^p$, which implies $g_p^*(z) = c^{p/2} g_p(cz)$ for almost every $z \in \mathbb{R}^p$. □

Next, we prove the complete monotonicity of functions g_p and a further auxiliary statement.

Lemma 7.1. *Let $g_1, g_2, g_3 : [0,\infty) \to [0,\infty)$ be functions fulfilling* (3.1) *for $p = 1$,* (3.2) *and* $\lim_{r\to\infty} g_1(r) = 0$. *Then*

$$g_1(r) = 2\int_0^\infty g_2(r+y^2)\,\mathrm{d}y \quad (r \ge 0). \tag{7.1}$$

Proof. By (3.1) for $p = 1$ and (3.2), we obtain

$$\begin{aligned} 2\int_0^\infty g_2(u+y^2)\,\mathrm{d}y &= 4\int_0^\infty\int_0^\infty g_3(u+y^2+t^2)\,\mathrm{d}t\,\mathrm{d}y \\ &= 4\int_0^{\pi/2}\int_0^\infty r g_3(u+r^2)\,\mathrm{d}r\,\mathrm{d}\alpha \text{ (polar coordinates)} \\ &= \pi\int_0^\infty g_3(u+r)\,\mathrm{d}r \\ &= \pi\left[-\frac{1}{\pi} g_1(u+r)\right]_{r=0}^\infty \\ &= g_1(u) - g_1(\infty) = g_1(u). \end{aligned}$$

□

Lemma 7.2. *Suppose that function $g_1 : [0, \infty) \to [0, \infty)$ is completely monotonic with $g_1(\infty) = 0$. Let* (3.1) *and* (3.2) *be fulfilled. Then all functions g_p are non-negative and completely monotonic and*

$$g_{2k+p}(r) = g_p^{(k)}(r)(-\pi)^{-k} \tag{7.2}$$

for $r, k \geq 0, p \geq 1$.

Proof. Assuming (3.1), one can easily show (7.2) by induction. Observe that

$$g_3(r) = -\frac{1}{\pi} g_1'(r) \geq 0$$

for $r \geq 0$. By (7.2), g_{2k+1} is non-negative for $k \geq 0$, and we obtain

$$g_{2k+1}^{(m)}(r)(-1)^m = g_1^{(k+m)}(r)\pi^{-k}(-1)^{k+m} \geq 0$$

for $r, k, m \geq 0$, which is the assertion of the lemma for odd p. Equation (7.2) for $p = 2$ shows that g_2 is infinitely differentiable. Condition (3.2) implies that

$$g_2(r) = 2\int_0^\infty g_3(r + y^2)\mathrm{d}y = -\frac{2}{\pi}\int_0^\infty g_1'(r + y^2)\mathrm{d}y \geq 0.$$

Hence, by taking derivatives and applying the Leibniz rule, we derive

$$g_2^{(m)}(r)(-1)^m = \frac{2}{\pi}(-1)^{m+1}\int_0^\infty g_1^{(m+1)}(r + y^2)\mathrm{d}y \geq 0$$

for $r, m \geq 0$, since $\tilde{g}(t) := (-1)^{m+1} g_1^{(m+1)}(t)$ $(t, m \geq 0)$ is non-negative and monotonously non-increasing $(\tilde{g}'(t) \leq 0)$, $(-1)^{m+1} g_1^{(m+1)}(r + t^2) \leq \tilde{g}(t)$ for $r \geq 0, t \geq 1$ and $\int_1^\infty \tilde{g}(t)\mathrm{d}t = (-1)^{m+1}(g_1^{(m)}(\infty) - g_1^{(m)}(1)) < \infty$. Therefore, g_2 is completely monotonic. Further from (7.2), we obtain

$$g_{2k+2}^{(m)}(r)(-1)^m = g_2^{(k+m)}(r)\pi^{-k}(-1)^{k+m} \geq 0$$

for $r, k, m \geq 0$. This proves the lemma. □

Proof of Theorem 3.2. (a) part ⇒: Assume that the family $\{g_p\}_{p\geq 1}$ of non-negative functions fulfills (2.2), and $g_p(0) = g_p(0+) < +\infty$ for $p \geq 1$. Hence, (3.2) holds true. Changing the coordinate system to the polar one, we obtain

$$\begin{aligned} g_p(u) &= 4\int_0^\infty\int_0^\infty g_{p+2}(u + y^2 + t^2)\mathrm{d}t\mathrm{d}y \\ &= 4\int_0^{\pi/2}\int_0^\infty g_{p+2}(u + r^2) r\mathrm{d}r\mathrm{d}\alpha \\ &= \pi\int_0^\infty g_{p+2}(u + r)\mathrm{d}r \end{aligned} \tag{7.3}$$

for $p \geq 1, u \geq 0$. We denote the antiderivative of g_p by G_p where $G_p(0) = 0$. Note that by (7.3),

$$\lim_{u\to\infty} G_{p+2}(u) = \int_0^\infty g_{p+2}(r)\mathrm{d}r = g_p(0)\pi^{-1} < +\infty$$

for $p \geq 1$. Further by (7.3), we derive

$$\begin{aligned} \frac{g_p(u + h) - g_p(u)}{h} &= \frac{\pi}{h}\int_0^\infty (g_{p+2}(u + h + r) - g_{p+2}(u + r))\mathrm{d}r \\ &= \frac{\pi}{h}(G_{p+2}(\infty) - G_{p+2}(u + h) - (G_{p+2}(\infty) - G_{p+2}(u))) \\ &= \frac{\pi}{h}(-G_{p+2}(u + h) + G_{p+2}(u)) \end{aligned}$$

for $p \geq 1$, $u \geq 0$. Hence, taking the limit $h \to 0$ leads to identity (3.1). By induction, we obtain

$$g_1^{(k)}(r)(-\pi)^{-k} = g_{2k+1}(r) \geq 0 \quad (k \geq 0)$$

from (3.1) and the proof of part a) is complete.

(b) part $\Leftarrow$: Assume that identities (3.1) and (3.2) are fulfilled and g_1 is completely monotonic. Claim: (2.2) holds true.

In view of Lemma 7.1, Equation (7.1) holds true. Hence,

$$g_j(r) = 2\int_0^\infty g_{j+1}(r + y^2)\mathrm{d}y$$

holds for $j = 1, 2$. Observe that g_2 and g_3 are completely monotonic in view of Lemma 7.2. Taking derivatives in (7.1) and applying the Leibniz rule and (7.2) (see Lemma 7.2), we obtain

$$g_{2k+j}(r) = g_j^{(k)}(r)(-\pi)^{-k} = 2\int_0^\infty g_{j+1}^{(k)}(r + t^2)\mathrm{d}t(-\pi)^{-k} = 2\int_0^\infty g_{2k+j+1}(r + t^2)\mathrm{d}t$$

for since $\tilde{g}(t) := (-1)^k g_{j+1}^{(k)}(t)$ $(t \geq 0, k \geq 1, j = 1, 2)$ is non-negative and monotonously non-increasing ($\tilde{g}'(t) \leq 0$), $(-1)^k g_{j+1}^{(k)}(r + t^2) \leq \tilde{g}(t)$ for $r \geq 0, t \geq 1$ and $\int_1^\infty \tilde{g}(t)\mathrm{d}t = (-1)^k(g_{j+1}^{(k-1)}(\infty) - g_{j+1}^{(k-1)}(1)) < \infty$. Hence, the claim is proved. Since all functions g_p are completely monotonic on $[0, \infty)$ by Lemma 7.2, and hence continuous at 0, condition $g_p(0+) < +\infty$ is fulfilled for $p \geq 1$, and functions g_p are non-negative. The proof of the theorem is now complete. □

Proof of Proposition 3.3. Observe that for the Caputo-derivative with respect to u,

$$(g_p(-u))^{(1/2)} = -\frac{1}{\Gamma(1/2)}\int_{-\infty}^{u}\frac{1}{\sqrt{u - y}}g_p'(-y)\mathrm{d}y \quad (u \leq 0).$$

Equations (3.1) and (2.2) imply

$$(g_p(-u))^{(1/2)} = -\frac{1}{\sqrt{\pi}}\int_{-u}^{\infty}\frac{1}{\sqrt{y + u}}g_p'(y)\mathrm{d}y = \sqrt{\pi}\int_{-u}^{\infty}\frac{1}{\sqrt{y + u}}g_{p+2}(y)\mathrm{d}y = \sqrt{\pi}\,g_{p+1}(-u)$$

for $u \leq 0$. □

The following Lemma 7.3 provides sufficient conditions for (1.1).

Lemma 7.3. *Let $\{g_p\}_{p \in I}$ be a family of functions $g_p : [0, \infty) \to [0, \infty)$. Assume that* (2.2) *and* (3.3) *are satisfied. Then* (1.1) *is satisfied for $p \in I$.*

Proof. In view of equation (2.2), we obtain

$$\begin{aligned}\int_0^\infty r^{p/2-1}g_p(r)\mathrm{d}r &= \int_0^\infty r^{p/2-1}\int_r^\infty (t - r)^{-1/2}g_{p+1}(t)\mathrm{d}t\mathrm{d}r \\ &= \int_0^\infty\int_0^t r^{p/2-1}(t - r)^{-1/2}\mathrm{d}r g_{p+1}(t)\mathrm{d}t \\ &= \frac{\sqrt{\pi}\Gamma(p/2)}{\Gamma((p + 1)/2)}\int_0^\infty r^{(p-1)/2}g_{p+1}(r)\mathrm{d}r\end{aligned}$$

for $p + 1 \in I$. By induction and by (3.3), we obtain

$$\int_0^\infty r^{p/2-1} g_p(r)\mathrm{d}r = \pi^{-p/2}\Gamma\left(\frac{p}{2}\right) = s_p^{-1}.$$

The proof of the lemma is now complete. □

Now we are in a position to prove Theorem 3.4.

Proof of Theorem 3.4. (i) Case $p_0 = \infty$: Theorem 3.4 is a consequence of Theorem 3.2 and Lemma 7.3.

(ii) Case $p_0 < \infty$:

Claim: Equation (2.2) holds true for $p + 1 \in I$.

Similarly to the proof of Theorem 3.2, part ⇐, we obtain

$$g_{2k+j}(r) = g_j^{(k)}(r)(-\pi)^{-k} \geq 0 \quad \text{for } r \geq 0, j = 1, 2, 3, k \geq 1, 2k + j \in I\text{, and}$$

$$g_{2k+j}(r) = 2\int_0^\infty g_{j+1}^{(k)}(r + t^2)\mathrm{d}t(-\pi)^{-k} = 2\int_0^\infty g_{2k+j+1}(r + t^2)\mathrm{d}t$$

for $r \geq 0, j = 1, 2, k \geq 1, 2k + j + 1 \in I$. Hence, the claim is proved.

An application of Lemma 7.3 yields the validity of (1.1) for $p \in I$. Therefore, the proof is complete. □

7.2 Proofs of the results in Section 5

In this section, we show consistency of several families of generators:

Lemma 7.4. *The family* $\{g_p\}_{p\geq 1}$ *defined in* (5.1) *is a consistent family of non-negative density generators.*

Proof. Obviously, $g_p(r) \geq 0$ and g_1 is completely monotonic. It can be checked by using a CAS that conditions (3.1)–(3.3) hold. Consistency of $\{g_p\}$ follows in view of Theorem 3.4. □

Lemma 7.5. *The family* $\{g_p\}_{p\geq 1}$ *resulting from* (5.2) *by the algorithm is a consistent family of non-negative density generators.*

Proof. By [14], p. 390, the function $u \mapsto \ln(a + u)/(u + a - 1)$ is completely monotonic for $a \geq 1, u \geq 0$. Further $u \mapsto (u + a - 1)/(u + b)$ is completely monotonic for $b \leq a - 1, u \geq 0$. The product of these two functions is again completely monotonic (cf. Theorem 1 of [14]). By using a CAS, one can show that (3.3) is satisfied. Apply Theorem 3.4 to obtain the lemma. □

Lemma 7.6. *The family* $\{g_p\}_{p\geq 1}$ *defined in* (5.5) *is consistent.*

Proof. By using the definition of the Li function, we obtain

$$g_p(r) = \frac{\mathrm{Li}_{m-(p-1)/2}(ae^{-r})}{\pi^{p/2}\mathrm{Li}_{m+1/2}(a)} = \frac{1}{\pi^{p/2}\mathrm{Li}_{m+1/2}(a)}\sum_{k=1}^\infty k^{-m+p/2-1/2}a^k e^{-kr} \tag{7.4}$$

for $r \geq 0$. Note that $\mathrm{Li}_{m-(p-1)/2}(0) = 0$ such that $\lim_{r\to\infty} g_1(r) = 0$. Obviously, $r \mapsto e^{-kr}$ is completely monotonic. Since the coefficients in the series in (7.4) are positive, g_p is a completely monotonic function in view of Theorem 3 of [14]. Notice that $\int_{-\infty}^\infty e^{-kr^2}\mathrm{d}r = \sqrt{\pi/k}$ holds true. We have

$$\int_{-\infty}^\infty g_1(r^2)\mathrm{d}r = \frac{1}{\sqrt{\pi}\,\mathrm{Li}_{m+1/2}(a)}\sum_{k=1}^\infty k^{-m}a^k\int_{-\infty}^\infty e^{-kr^2}\mathrm{d}r = \frac{1}{\mathrm{Li}_{m+1/2}(a)}\sum_{k=1}^\infty k^{-m-1/2}a^k = 1.$$

Further

$$g_{p+2}(r) = -\frac{1}{\pi} g_p'(r) = \frac{1}{\pi^{(p+2)/2}\mathrm{Li}_{m+1/2}(a)} \mathrm{Li}_{m-(p+1)/2}(ae^{-r})$$

holds for $r \geq 0$. It remains to show that (3.2) is valid. By (7.4),

$$\begin{aligned} 2\int_0^\infty g_3(r + y^2)\mathrm{d}y &= \frac{2}{\pi^{3/2}\mathrm{Li}_{m+1/2}(a)} \sum_{k=1}^\infty k^{-m+1} a^k \int_0^\infty e^{-kr-ky^2}\mathrm{d}y \\ &= \frac{1}{\pi \mathrm{Li}_{m+1/2}(a)} \sum_{k=1}^\infty k^{-m+1/2} a^k e^{-kr} = g_2(r). \end{aligned}$$

An application of Theorem 3.4 leads to the lemma. □

Lemma 7.7. *The family* $\{g_p\}_{p\geq 1}$ *defined by* (5.6) *is consistent and the functions* g_p *are density generators.*

Proof. Ismail showed in [7] on p. 354 that the function $r \mapsto r^{\nu/2}K_\nu(\sqrt{r})$ is completely monotonic for $\nu > -1/2$. Hence, g_1 is completely monotonic. Utilizing a CAS, it is shown that (3.1)–(3.3) hold true. Therefore, the lemma follows by Theorem 3.4. □

Lemma 7.8. *The two families* $\{g_p\}_{p\geq 1}$ *resulting from* (5.3) *and* (5.7) *by the algorithm are consistent and the functions* g_p *are density generators.*

Proof. In view of the study by Miller and Samko [14], the functions $r \mapsto (a + r)^{-1}$ and $r \mapsto (a + r)^{-1/2}$ are completely monotonic. Since $r \mapsto e^{-r}$ is a completely monotonic function and products of completely monotonic functions have this property (cf. Theorem 1 of [14]), g_1 is completely monotonic. The validity of (3.1)–(3.3) is shown by using CAS. Theorem 3.4 is obtained to obtain the lemma. □

Lemma 7.9. *The family* $\{g_p\}_{1\leq p\leq p_0}$ *defined in* (6.1) *is consistent for* $p_0 = \lceil 2b + 3\rceil - 1$.

Proof. Note that

$$-\frac{1}{\pi} g_p'(r) = \left(b - \frac{p-1}{2}\right)(1 - r)^{b-(p+1)/2} \cdot \frac{\Gamma\left(b + \frac{3}{2}\right)}{\pi^{1+p/2}\left(b - \frac{p-1}{2}\right)\Gamma\left(b + \frac{1-p}{2}\right)} = g_{p+2}(r)$$

for $r \geq 0$. Further, we derive

$$\begin{aligned} 2\int_0^{\sqrt{1-u}} g_{p+1}(u + t^2)\mathrm{d}t &= 2\int_0^{\sqrt{1-u}} (1 - u - t^2)^{b-p/2}\mathrm{d}t \cdot \frac{\Gamma\left(b + \frac{3}{2}\right)}{\pi^{(p+1)/2}\Gamma\left(b + \frac{2-p}{2}\right)} \\ &= (1 - u)^{b-(p-1)/2} \frac{\Gamma\left(b + \frac{2-p}{2}\right) \cdot \Gamma\left(b + \frac{3}{2}\right)}{\Gamma\left(b + \frac{3-p}{2}\right) \cdot \pi^{p/2}\Gamma\left(b + \frac{2-p}{2}\right)} \\ &= g_p(u). \end{aligned}$$

Moreover, g_p satisfies identity (1.1). This completes the proof in view of Theorem 3.4. □

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Appendix

A Definition of the fractional derivative

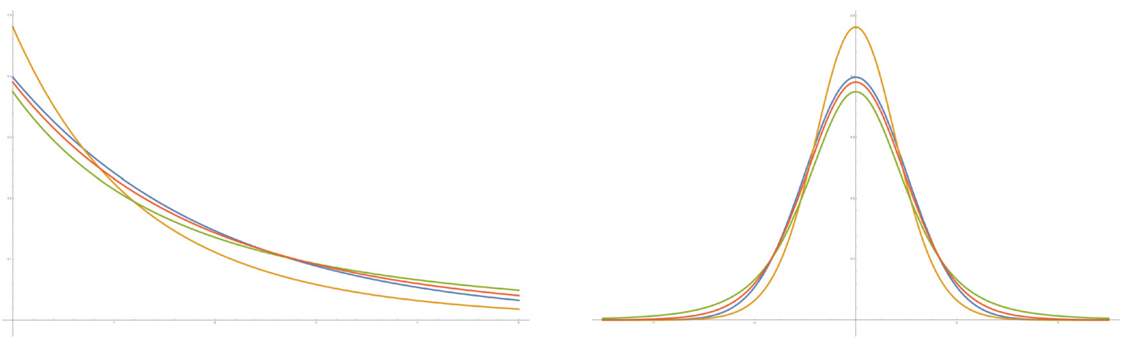
DEFINITION: (According to Oliveira and de Oliveira [16]) Let $a \in [-\infty, \infty)$, $\nu > 0$, $\nu \notin \mathbb{N}$, and $n = [\nu] + 1$, where $[\alpha]$ is the integer part of α. The fractional derivative of order ν in the sense of Caputo of function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by

$$ {}_C D_a^{\nu} \varphi(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x \frac{\varphi^{(n)}(t)}{(x - t)^{\nu - n + 1}} \mathrm{d}t \quad \text{for } x > a. $$

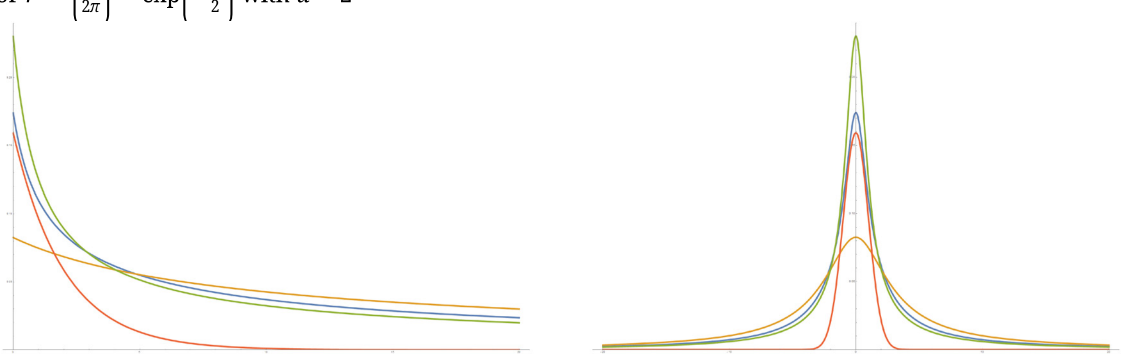
B Graphics of generator functions

This section provides figures of the generator functions on the left and the one-dimensional marginal density on the right.

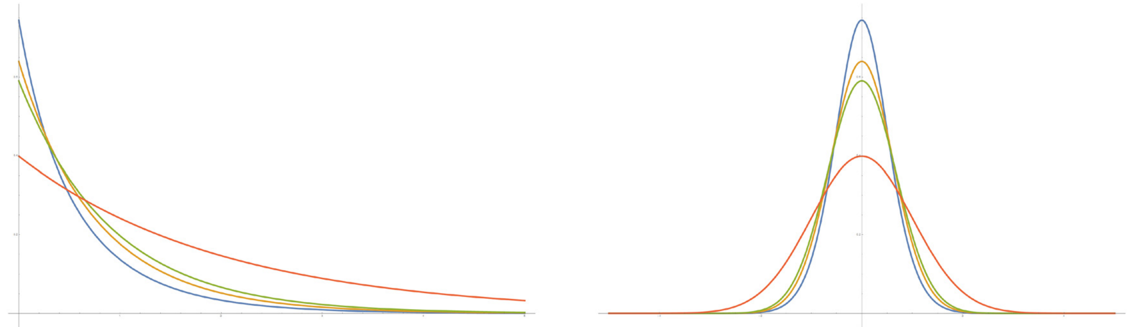
(1) Normal generator (blue), mixed normal generator $q = \frac{1}{2}$, $a = 2$ (orange), power generator (green $m = 2$, red $m = 6$)



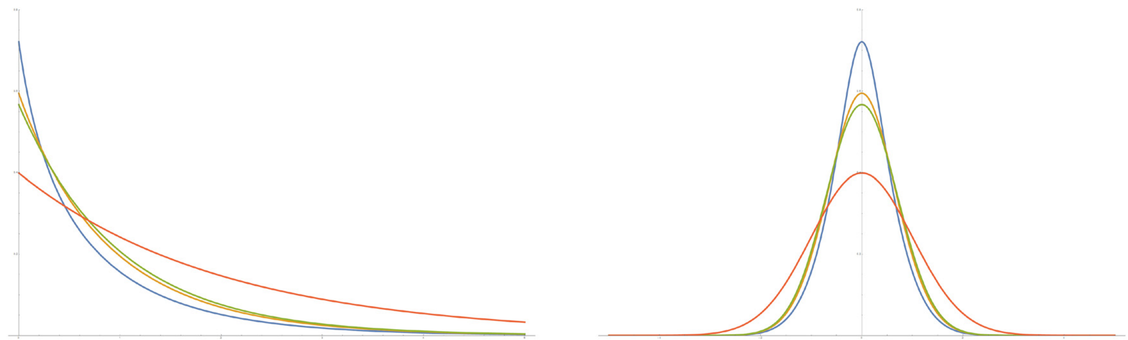
(2) Logarithmic generator I, blue: $a = 3$, $b = 1$, orange: $a = 6$, $b = 5$, green: $a = 6$, $b = 1$, red: normal generator $r \mapsto \left(\frac{a}{2\pi}\right)^{p/2} \exp\left(-\frac{ar}{2}\right)$ with $a = 2$



(3) Fractional-exponential generator blue: $a = 1$, orange: $a = 3$, green: $a = 10$, red: normal generator



(4) polylogarithmic generator blue: $a = 1$, orange: $a = 3$, green: $a = 10$, red: normal generator



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