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Abel-Gontcharoff polynomials, parking trajectories and ruin probabilities

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Abstract: The central mathematical tool discussed is a non-standard family of polynomials, univariate and bivariate, called Abel-Goncharoff polynomials. First, we briefly summarize the main properties of this family of polynomials obtained in the previous work. Then, we extend the remarkable links existing between these polynomials and the parking functions which are a classic object in combinatorics and computer science. Finally, we use the polynomials to determine the non-ruin probabilities over a finite horizon for a bivariate risk process, in discrete and continuous time, assuming that claim amounts are dependent via a partial Schur-constancy property.

Keywords: remarkable polynomials, parking functions, ruin probability, partial Schur-constancy

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1 Introduction

Goncharoff [19] introduced a new family of univariate polynomials in order to solve some interpolation problems in numerical analysis. Their interest in probability was first shown by Daniels [12] and Picard [32] and then developed by Lefèvre and Picard [25] and Picard and Lefèvre [33] for the study of epidemic processes. In these two articles, a multivariate version of the polynomials was also constructed to be able to deal with heterogeneous populations. Since then, these polynomials have been widely used, in particular to study first crossing problems, epidemic models and insurance risk processes. Among numerous articles, we refer to the studies by Picard and Lefèvre [36], Lefèvre and Picard [27], Ball [4] and Britton and Pardoux (editors) [6].

In the following, we give these polynomials the name Abel-Gontcharoff (A-G) polynomials because of the key role played by Abel-type expansions with respect to the A-G polynomials. The basic elements of the theory have already been presented in previous works (see, e.g. Lefèvre and Picard [27,28]). A brief updated summary is provided in Section 2.

Quite unexpectedly, Kung and Yan [23] and Khare et al. [21] independently encountered and studied this family of A-G polynomials, univariate and multivariate, in a purely mathematical framework. They notably proved that the polynomials make it possible to count so-called parking functions, an object with multiple applications in combinatorics and computer science. In parallel, we will show in Section 3 how the A-G polynomials lead to naturally defining a new concept of stochastic parking trajectories that takes into account ordered sequences of independent uniform random variables on $(0, 1)$.

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In the initial theory of insurance risk, the amounts of claims are assumed to be independent and identically distributed, which may prove restrictive in reality. Recently, dependent claims modelling has been integrated in various ways for univariate risk processes (e.g. Albrecher et al. [2], Constantinescu et al. [10], and Lefèvre and Simon [29]). For the multivariate case, the literature on the quantities linked to ruin remains limited (see the study by Albrecher et al. [3] and the references therein). In particular, results over a finite horizon are quite few (e.g. Picard et al. [37], and Dimitrova and Kaishev [16], and Castañer et al. [7]). In Section 4, we will consider a bivariate risk process, in discrete and continuous time, where claim amounts for the two risks are dependent on each other by satisfying a partial Schur-constancy property (see the studies by Castañer et al. [8] and Lefèvre [24]). A short presentation of this form of dependency is recalled in the Appendix. By using the A-G polynomials, we can then derive compact formulas for the non-ruin probabilities over a finite horizon that exhibit the hidden underlying algebraic structure.

It is worth mentioning that the family of A-G polynomials can be generalized to a family of functions that we have called A-G pseudopolynomials. For the theory with applications in applied probability, we refer to Picard and Lefèvre [34] and Lefèvre and Picard [26].

2 Abel-Gontcharoff polynomials

The concepts and results that we briefly present below essentially come from the study by Lefèvre and Picard [25,27,28]. Further details, including proofs, can be found in these articles.

2.1 Univariate A-G polynomials

We first deal with the construction of univariate A-G polynomials. To do this, we begin with a family $\mathcal{U}$ of reals, which will play the role of parameters:

$$\mathcal{U} = \{u_i,\ i \in \mathbb{N}_0\}, \tag{1}$$

where $\mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}$. We also introduce the shift operators $\mathcal{E}^k$, $k \in \mathbb{N}_0$, which shift the parameters of $\mathcal{U}$ to start with the kth, i.e.

$$\mathcal{E}^k\mathcal{U} = \{u_{k+i},\ i \in \mathbb{N}_0\}.$$

The associated family of A-G polynomials of degrees n in the argument x, denoted $\{G_n(x|\mathcal{U}), n \in \mathbb{N}_0\}$, is then defined as follows.

Definition 2.1. Starting from $G_0(x|\mathcal{U}) = 1$, the polynomials $G_n(x|\mathcal{U})$ are constructed recursively by using the differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}G_n(x|\mathcal{U}) = G_{n-1}(x|\mathcal{E}\mathcal{U}), \quad n \geq 1, \tag{2}$$

with the boundary conditions

$$G_n(u_0|\mathcal{U}) = 0, \quad n \geq 1. \tag{3}$$

So, for $n = 1$, as $G_0 = 1$, (2) gives $G_1(x|\mathcal{U}) = g_1 + x$, where g_1 is a constant such that by (3), $g_1 + u_0 = 0$, and hence, $g_1 = -u_0$. Applying the recursion with $n = 2$ and 3 then yields the following formulas for illustration.

Examples 2.2.

$$\begin{aligned}
G_1(x|\mathcal{U}) &= -u_0 + x,\\
G_2(x|\mathcal{U}) &= (-u_0^2/2 + u_0u_1) - u_1x + x^2/2,\\
G_3(x|\mathcal{U}) &= (-u_0^3/6 + u_0^2u_2/2 + u_0u_1^2/2 - u_0u_1u_2) + (-u_1^2/2 + u_1u_2)x - u_2x^2/2 + x^3/6.
\end{aligned}$$

Therefore, G_n, $n \geq 1$, depends only on the first n parameters $u_0, \dots, u_{n-1}$ of the family $\mathcal{U}$. The insertion of $\mathcal{U}$ in the notation is justified to take into account the whole family of polynomials. Note that in Definition 2.1, a factor n sometimes multiplies the right-hand sie of (2).

We easily see that an affine transformation of $\mathcal{U}$ has a simple effect on G_n:

$$G_n(x|a + b\mathcal{U}) = b^n G_n\left(\frac{x-a}{b} \,\middle|\, \mathcal{U}\right), \quad n \in \mathbb{N}_0. \tag{4}$$

Below, we state two alternative definitions, (5) and (7), of the A-G polynomials. As usual, $f^{(k)}(a)$ denotes the k-th derivative of a function $f(x)$ evaluated at the point $x = a$. The first is directly related to the original interpolation problem.

Proposition 2.3. *Each G_n is the unique polynomial of degree n satisfying the biorthogonality condition*

$$G_n^{(k)}(u_k|\mathcal{U}) = \delta_{n,k}, \quad k \in \mathbb{N}_0. \tag{5}$$

The second is a consequence of a key property of the A-G polynomials, namely, that they allow one to write an Abel-type expansion, rather than a Taylor expansion, for any polynomial in x (among others).

Proposition 2.4. *A polynomial $R_n(x)$ of degree n in x can be expanded as follows*:

$$R_n(x) = \sum_{k=0}^{n} R_n^{(k)}(u_k)\; G_k(x|\mathcal{U}). \tag{6}$$

So, taking $R_n(x) = x^n/n!$ in (6) gives

$$G_n(x|\mathcal{U}) = \frac{x^n}{n!} - \sum_{k=0}^{n-1} \frac{(u_k)^{n-k}}{(n-k)!}\, G_k(x|\mathcal{U}), \quad n \geq 1, \tag{7}$$

hence a simple recursion for the polynomials G_n starting from $G_0 = 1$.

A determinantal formula exists for G_n but is cumbersome to apply. In general, a recursive method is more efficient. An explicit expression is available when the u_i are of affine form.

Special case 2.5. If $u_i \equiv u$, $i \in \mathbb{N}_0$, are constant parameters,

$$G_n(x|\mathcal{U}) = \frac{(x-u)^n}{n!}, \quad n \in \mathbb{N}_0,$$

and (6) reduces to the Taylor expansion of $R_n(x)$.

More generally, if $u_i \equiv ui$ is a linear homogeneous function of $i \in \mathbb{N}_0$,

$$G_n(x|\mathcal{U}) = x\, \frac{(x-un)^{n-1}}{n!}, \quad n \in \mathbb{N}_0, \tag{8}$$

and (6) becomes the Abel expansion of $R_n(x)$.

In some cases, it may be useful to consider parameters which are no longer real numbers but random variables. The elegant identity below has various applications and will be used here in Section 4.2.

Proposition 2.6. *Let $\{Y_i, i \geq 1\}$ be a sequence of exchangeable random variables, of partial sums $W_i = Y_1 + \ldots + Y_i$, $i \geq 1$. Then, for any integer $l \geq n$,*

$$E[G_n(x|W_n, \dots, W_1)|W_l] = \frac{x^{n-1}}{(n-1)!}\left(\frac{x}{n} - \frac{W_l}{l}\right), \quad a.s. \tag{9}$$

2.2 Bivariate A-G polynomials

The generalization to polynomials with several arguments is fairly easy to perform. We limit ourselves to the bivariate case for simplicity. This time, we consider two families $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ of real parameters with double indices:

$$\mathcal{U}^{(1)} = \{u^{(1)}_{i_1,i_2},\ i_1, i_2 \in \mathbb{N}_0\}, \quad \text{and} \quad \mathcal{U}^{(2)} = \{u^{(2)}_{i_1,i_2},\ i_1, i_2 \in \mathbb{N}_0\}. \tag{10}$$

Here too, we introduce shift operators $\mathcal{E}^{k_1,k_2}$, for $k_1, k_2 \in \mathbb{N}_0$, such that

$$\mathcal{E}^{k_1,k_2}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = \{(u^{(1)}_{k_1+i_1,k_2+i_2}, u^{(2)}_{k_1+i_1,k_2+i_2}),\ i_1, i_2 \in \mathbb{N}_0\}.$$

The associated bivariate A-G polynomials of degrees n_1 in x_1 and n_2 in x_2 form a family denoted $\{G_{n_1,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}), n_1, n_2 \in \mathbb{N}_0\}$.

Definition 2.7. Starting from $G_{0,0}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = 1$, the bivariate A-G polynomials are constructed recursively by using the partial differential equations

$$\begin{aligned}
&\frac{\partial}{\partial x_1} G_{n_1,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = G_{n_1-1,n_2}(x_1, x_2|\mathcal{E}^{1,0}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})), \quad n_1 \geq 1, n_2 \geq 0,\\
&\frac{\partial}{\partial x_2} G_{n_1,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = G_{n_1,n_2-1}(x_1, x_2|\mathcal{E}^{0,1}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})), \quad n_1 \geq 0, n_2 \geq 1,
\end{aligned} \tag{11}$$

with the boundary conditions

$$G_{n_1,n_2}(u^{(1)}_{0,0}, u^{(2)}_{0,0}|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = 0, \quad n_1 + n_2 \geq 1. \tag{12}$$

Obviously, when $n_1 = 0$, $G_{0,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$ becomes the univariate polynomial $G_{n_2}(x_2|\mathcal{U}^{(2)})$; similarly when $n_2 = 0$. Considering $n_1 = n_2 = 1$, we obtain from (11)

$$\begin{aligned}
(\partial/\partial x_1)\ G_{1,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = G_{0,1}(x_1, x_2|\mathcal{E}^{1,0}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})) = -u^{(2)}_{1,0} + x_2,\\
(\partial/\partial x_2)\ G_{1,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = G_{1,0}(x_1, x_2|\mathcal{E}^{0,1}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})) = -u^{(1)}_{0,1} + x_1,
\end{aligned}$$

which implies that

$$G_{1,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = g_{1,1} - u^{(2)}_{1,0}x_1 - u^{(1)}_{0,1}x_2 + x_1x_2,$$
$$\text{where } g_{1,1} \text{ is a constant such that } G_{1,1}(u^{(1)}_{0,0}, u^{(2)}_{0,0}|U^{(1)}, U^{(2)}) = 0.$$

By way of illustration, here are the first bivariate A-G polynomials.

Example 2.8.

$$\begin{aligned}
&G_{1,0}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = -u^{(1)}_{0,0} + x_1,\\
&G_{0,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = -u^{(2)}_{0,0} + x_2,\\
&G_{2,0}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = -(u^{(1)}_{0,0})^2/2 + u^{(1)}_{1,0}u^{(1)}_{0,0} - u^{(1)}_{1,0}x_1 + x_1^2/2,\\
&G_{0,2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = -(u^{(2)}_{0,0})^2/2 + u^{(2)}_{0,1}u^{(2)}_{0,0} - u^{(2)}_{0,1}x_2 + x_2^2/2,\\
&G_{1,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = u^{(1)}_{0,1}u^{(2)}_{0,0} + u^{(1)}_{0,0}u^{(2)}_{1,0} - u^{(1)}_{0,0}u^{(2)}_{0,0} - u^{(2)}_{1,0}x_1 - u^{(1)}_{0,1}x_2 + x_1x_2,\\
&G_{2,1}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = g_{2,1} + (u^{(1)}_{1,1}u^{(2)}_{1,0} + u^{(1)}_{1,0}u^{(2)}_{2,0} - u^{(1)}_{1,0}u^{(2)}_{1,0})x_1 - u^{(2)}_{2,0}x_1^2/2 + [-(u^{(1)}_{0,1})^2/2 + u^{(1)}_{0,1}u^{(1)}_{1,1}]x_2 - u^{(1)}_{1,1}x_1x_2\\
&\qquad + x_1^2x_2/2,
\end{aligned}$$

where $g_{2,1}$ is a constant such that $G_{2,1}(u^{(1)}_{0,0}, u^{(2)}_{0,0}|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = 0$.

Note that for $n_1 + n_2 \geq 1$, each G_{n_1,n_2} depends on the family $\mathcal{U}^{(1)}$ only through the terms $u^{(1)}_{i_1,i_2}$ for $0 \leq i_1 \leq n_1 - 1$, $0 \leq i_2 \leq n_2$, and on the family $\mathcal{U}^{(2)}$ only through the terms $u^{(2)}_{i_1,i_2}$ for $0 \leq i_1 \leq n_1$, $0 \leq i_2 \leq n_2 - 1$.

As in (4), an affine transformation of $(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$ has a simple effect on G_{n_1,n_2}:

$$G_{n_1,n_2}(x_1, x_2|a_1 + b_1\,\mathcal{U}^{(1)}, a_2 + b_2\,\mathcal{U}^{(2)}) = b_1^{n_1} b_2^{n_2}\;\; G_{n_1,n_2}\left(\frac{x_1 - a_1}{b_1}, \frac{x_2 - a_2}{b_2} | \mathcal{U}^{(1)}, \mathcal{U}^{(2)}\right), \quad n_1, n_2 \in \mathbb{N}_0. \tag{13}$$

The two equivalent definitions (5) and (7) generalize to (14) and (16). For a function $f(x_1, x_2)$, let $f^{(k_1,k_2)}(a_1, a_2)$ be the partial derivative order k_1 in x_1 and k_2 in x_2 calculated at point (a_1, a_2).

Proposition 2.9. *For $k_1, k_2 \in \mathbb{N}_0$,*

$$G^{(k_1,k_2)}_{n_1,n_2}(u^{(1)}_{k_1,k_2}, u^{(2)}_{k_1,k_2}|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = \delta_{n_1,k_1}\delta_{n_2,k_2}, \quad n_1, n_2 \in \mathbb{N}_0, \tag{14}$$

and this property characterizes the bivariate A-G polynomials.

Proposition 2.10. *A polynomial $R_{n_1,n_2}(x_1, x_2)$ of degree n_1 in x_1 and n_2 in x_2 admits a bivariate Abel-type expansion*

$$R_{n_1,n_2}(x_1, x_2) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} R^{(k_1,k_2)}_{n_1,n_2}(u^{(1)}_{k_1,k_2}, u^{(2)}_{k_1,k_2})\; G_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}). \tag{15}$$

So, taking $R_{n_1,n_2}(x_1, x_2) = x_1^{n_1} x_2^{n_2}/n_1!n_2!$ in (15) *yields*

$$\frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!} = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{(u^{(1)}_{k_1,k_2})^{n_1-k_1}}{(n_1 - k_1)!}\frac{(u^{(2)}_{k_1,k_2})^{n_2-k_2}}{(n_2 - k_2)!}\; G_{k_1,k_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}), \quad n_1 + n_2 \geq 1, \tag{16}$$

hence a simple recursion for the polynomials G_{n_1,n_2} starting from $G_{0,0} = 1$.

An explicit expression is available when $u^{(1)}_{i_1,i_2}$ and $u^{(2)}_{i_1,i_2}$ are again of affine form.

Special case 2.11. If for $j = 1, 2$, $u^{(j)}_{i_1,i_2} \equiv u^{(j)}$, $i_1, i_2 \in \mathbb{N}_0$, are constant parameters,

$$G_{n_1,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = \frac{(x_1 - u^{(1)})^{n_1}}{n_1!}\frac{(x_2 - u^{(2)})^{n_2}}{n_2!}, \quad n_1, n_2 \in \mathbb{N}_0.$$

If for $j = 1, 2$, $u^{(j)}_{i_1,i_2} = u^{(j)}_{i_j}$, $i_j \in \mathbb{N}_0$, are one-dimensional parameters,

$$G_{n_1,n_2}(x_1, x_2, |\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = G_{n_1}(x_1|\mathcal{U}^{(1)})\; G_{n_2}(x_2|\mathcal{U}^{(2)}), \quad n_1, n_2 \in \mathbb{N}_0,$$

where G_{n_1} and G_{n_2} are univariate A-G polynomials.

If for $j = 1, 2$, $u^{(j)}_{i_1,i_2} = u_1^{(j)} i_1 + u_2^{(j)} i_2$ are linear homogeneous functions of $i_1, i_2 \in \mathbb{N}_0$,

$$G_{n_1,n_2}(x_1, x_2|\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = \frac{(x_1 - u^{(1)}_{n_1,n_2})^{n_1-1}}{n_1!}\frac{(x_2 - u^{(2)}_{n_1,n_2})^{n_2-1}}{n_2!}(x_1x_2 - x_1u^{(2)}_{n_1,0} - x_2u^{(1)}_{0,n_2}), \quad n_1, n_2 \in \mathbb{N}_0. \tag{17}$$

3 Ordered uniforms and parking trajectories

Parking functions are a classical object in mathematics with applications in combinatorics, group theory and computer science. They are traditionally presented in the following way.

Suppose that there are n cars and n parking spots on a one way street. Each car $i = 1, \dots, n$ has a preferred parking spot $a_i \in \{1, \dots, n\}$. Car 1 first enters the street and goes to its preferred spot a_1. Car 2 then goes to its preferred spot a_2: it parks there if this is empty, otherwise it takes the first available spot to the right of a_2. The next cars try to park following the same rule. If for any car i, there is no available spot at a_i or further right, the parking process stops.

A sequence $(a_1, \dots, a_n)$ in $\{1, \dots, n\}$ is said to be a parking function if every car succeeds in parking. It is easily seen that $(a_1, \dots, a_n)$ is a parking function iff its rearrangement in the ascending order, denoted $(a_{1:n}, \dots, a_{n:n})$, satisfies the constraints $a_{i:n} \le i$, $i = 1, \dots, n$.

Parking functions were originally introduced by Konheim and Weiss [22] for investigating the storage device of hashing. They proved, inter alia, that there are $(n+1)^{n-1}$ parking functions. Since then, parking functions have been encountered in many mathematical uses. A comprehensive survey was done by Yan [41]; see, e.g. Stanley [40] for supplements. Moreover, Diaconis and Hicks [14] studied links with probability, which led them to find new combinatorics involved.

3.1 Unidimensional trajectories

Various extensions of parking functions have been proposed and discussed in the literature. In particular, Kung and Yan [23] have shown that the univariate A-G polynomials make it possible to count the parking functions, which are upper bounded by any non-decreasing integer sequence u_{i-1} (instead of just i), $i = 1, \dots, n$. Specifically, they proved that the number of parking functions, denoted $PK_n(u_1, \dots, u_n)$, is given by

$$PK_n(u_1, \dots, u_n) = (-1)^n n!\ G_n(0|\{u_0, \dots, u_{n-1}\}), \tag{18}$$

where G_n is a univariate A-G polynomial as in Section 2.1, the factor $n!$ on the right-hand side being due to our definition (2).

Of course, the combinatorial result (18) requires that the upper bounds u_i are all (non-decreasing) positive integers. This assumption may seem somewhat surprising because the polynomials G_n are defined from arbitrary real parameters.

We are going to derive a different interpretation of the right-hand side of (18) when the non-decreasing upper bounds u_i are this time reals in $(0, 1)$. So, let $\mathcal{U}$ be the family (1) of these u_i, and consider the associated A-G polynomials $G_n(x|\mathcal{U})$, where $x \ge u_0$. From (2) and (3), we directly see that G_n admits the following integral representation:

$$G_n(x|\mathcal{U}) = \int\limits_{y_0=u_0}^{x} dy_0 \int\limits_{y_1=u_1}^{y_0} dy_1 \dots \int\limits_{y_{n-1}=u_{n-1}}^{y_{n-2}} dy_{n-1}, \quad n \ge 1. \tag{19}$$

Now, we introduce a sample $(U_1, \dots, U_n)$ of n independent $(0, 1)$-uniform random variables. Denote by $(U_{1:n}, \dots, U_{n:n})$ the corresponding order statistics. By using (19), we directly obtain the following representation of G_n.

Proposition 3.1. *For* $0 \le x \le u_0$,

$$P(U_{1:n} \le u_0, \dots, U_{n:n} \le u_{n-1}, \quad \textit{and} \quad U_{1:n} \ge x) = (-1)^n n!\ G_n(x|\{u_0, \dots, u_{n-1}\}). \tag{20}$$

We observe that the right-hand side of (18) and that of (20) when $x = 0$ are identical, but u_i are no longer necessarily integers here. By mimicry with the parking functions, we will say that a sample $(U_1, \dots, U_n)$ of independent $(0, 1)$-uniforms is a *parking trajectory* if its order statistics $(U_{1:n}, \dots, U_{n:n})$ satisfy the constraints $U_{i:n} \le u_{i-1}$, $i = 1, \dots, n$. Therefore, the formula (20) gives, for $x = 0$, the probability that such a parking trajectory does exist.

Note that (20) can presumably be derived from formula (18) of PK_n by using an appropriate limit argument. However, such a method would be unnecessarily complicated.

3.2 Bidimensional trajectories

As noted in Section 1, Khare et al. [21] have recently introduced, independently, the family of bivariate A-G polynomials (see also the study by Adeniran et al. [1]). Their motivation was mainly combinatorial and a key result concerns bivariate parking functions, which are defined, as in the univariate case, from sequences of non-decreasing positive integer parameters.

In fact, they generalized the result (18) by proving that the number of bivariate parking functions is given by an appropriate bivariate A-G polynomial. Their starting point is a two-dimensional lattice $\{(i_1, i_2), i_1 = 0, \ldots, n_1, i_2 = 0, \ldots, n_2\}$. Consider all possible paths going from $(0, 0)$ to (n_1, n_2) by successive steps which are either horizontal (i.e. of the type $(i_1, i_2) \to (i_1 + 1, i_2)$) or vertical (i.e. of the type $(i_1, i_2) \to (i_1, i_2 + 1)$). To this two-dimensional lattice are attached two sets of upper bounds, $\{u^{(1)}_{i_1,i_2}\}$ and $\{u^{(2)}_{i_1,i_2}\}$, which must be satisfied at each horizontal and vertical step, respectively. These bounds are positive integers, which are non-decreasing in i_1 and i_2 (i.e. $u^{(j)}_{i_1,i_2} \le u^{(j)}_{i_1',i_2'}$ when $i_1 \le i_1'$ and $i_2 \le i_2'$, for $j = 1, 2$).

A pair of integer sequences $[(a^{(1)}_1, \ldots, a^{(1)}_{n_1}), (a^{(2)}_1, \ldots, a^{(2)}_{n_2})]$ is a bivariate parking function iff its rearrangement in ascending order, denoted $[(a^{(1)}_{1:n_1}, \ldots, a^{(1)}_{n_1:n_1}), (a^{(2)}_{1:n_2}, \ldots, a^{(2)}_{n_2:n_2})]$, satisfies the corresponding upper bounds constraints. Let us denote

$$\mathcal{U}^{(1)} = \{u^{(1)}_{i_1,i_2}, 0 \le i_1 \le n_1 - 1, 0 \le i_2 \le n_2\}, \quad \mathcal{U}^{(2)} = \{u^{(2)}_{i_1,i_2}, 0 \le i_1 \le n_1, 0 \le i_2 \le n_2 - 1\}.$$

Khare et al. [21] proved that the number of bivariate parking functions, $PK_{n_1,n_2}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$, is given by

$$PK_{n_1,n_2}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = (-1)^{n_1+n_2} n_1! n_2! \; G_{n_1,n_2}(0, 0 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}), \tag{21}$$

where G_{n_1,n_2} is a bivariate A-G polynomial as in Section 2.2, the factor $n_1!n_2!$ on the right-hand side being due to our definition (11).

We are going to show that here too, a formula like (21) is true in a probabilistic context, for sequences of parameters, which are (non-decreasing) reals in $(0, 1)$. This result was proved in the study by Lefevre and Picard [28] in a sometimes imprecise way. It is argued in detail below.

Let us look at the same two-dimensional lattice as mentioned earlier, and consider all the paths going from $(0, 0)$ to (n_1, n_2) by successive steps, either vertical or horizontal. The lattice is again completed by upper bounds to be satisfied at each step: $u^{(1)}_{i_1,i_2}$ for a horizontal step $(i_1, i_2) \to (i_1 + 1, i_2)$ and $u^{(2)}_{i_1,i_2}$ for a vertical step $(i_1, i_2) \to (i_1, i_2 + 1)$. These bounds are always non-decreasing in i_1 and i_2 but are now reals in $(0, 1)$.

We now introduce a pair of independent samples of (n_1, n_2) independent $(0, 1)$-uniform random variables $[(U^{(1)}_1, \ldots, U^{(1)}_{n_1}), (U^{(2)}_1, \ldots, U^{(2)}_{n_2})]$. The question we ask is whether the order statistics of these two samples, denoted $[(U^{(1)}_{1:n_1}, \ldots, U^{(1)}_{n_1:n_1}), (U^{(2)}_{1:n_2}, \ldots, U^{(2)}_{n_2:n_2})]$, make it possible to generate a path to (n_1, n_2), which satisfies the upper bound constraints. A trajectory which does so is called a *bivariate parking trajectory*.

As an illustration, suppose $(n_1, n_2) = (3, 2)$, and consider the associated lattice for which the fixed upper bounds are given in the left part of Figure 1. Assume the two ordered uniform samples, $(U^{(1)}_{1:3}, U^{(1)}_{2:3}, U^{(1)}_{3:3})$ and $(U^{(2)}_{1:2}, U^{(2)}_{2:2})$, generate the path which is represented in the right part of Figure 1. Then this path will be a bivariate parking trajectory iff

$$(U^{(1)}_{1:3} \le u^{(1)}_{0,0}, \quad U^{(1)}_{2:3} \le u^{(1)}_{1,1}, \quad U^{(1)}_{3:3} \le u^{(1)}_{2,1}), \quad \text{and} \quad (U^{(2)}_{1:2} \le u^{(2)}_{1,0}, \quad U^{(2)}_{2:2} \le u^{(2)}_{3,1}).$$

We will prove that the probability that a bivariate parking trajectory does exist is given by formula (22), where $(x_1, x_2) = (0, 0)$. Note that the right-hand side has exactly the same expression as (21).

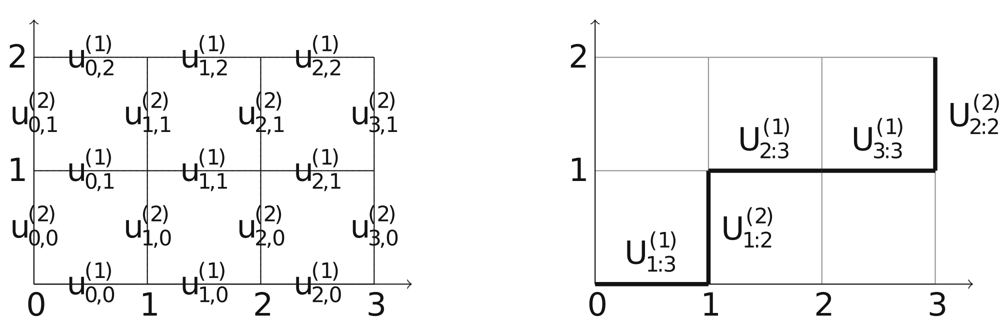

Figure 1: Lattice from $(0, 0)$ to $(n_1, n_2) = (3, 2)$: the set of upper bounds to satisfy (on the left side), and a path of two ordered uniform samples (on the right side).

Proposition 3.2. *For* $0 \le x_1 \le u_{0,0}^{(1)}$ *and* $0 \le x_2 \le u_{0,0}^{(2)}$,

$$\begin{aligned} &P(\text{a bivariate parking trajectory exists to } (n_1, n_2), \quad \text{and} \quad U_{1:n_1}^{(1)} \ge x_1, U_{1:n_2}^{(2)} \ge x_2) \\ &= (-1)^{n_1+n_2} n_1! n_2!\ G_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}). \end{aligned} \tag{22}$$

Proof. Denote by $B_{n_1,n_2}(x_1, x_2)$ the probability $P(\dots)$ defined in the light-hand side of (22). Its increment in x_1 is given by

$$B_{n_1,n_2}(x_1 + dx_1, x_2) - B_{n_1,n_2}(x_1, x_2) = -P(\text{a parking trajectory}, \text{ and } x_1 \le U_{1:n_1}^{(1)} \le x_1 + dx_1,\ U_{1:n_2}^{(2)} \ge x_2). \tag{23}$$

As $U_{1:n_1}^{(1)} \le U_{2:n_1}^{(1)}$, the constraint $U_{2:n_1}^{(1)} \ge x_1$ may be added in the right-hand side of (23), which so becomes

$$-P(\text{a parking trajectory}, \text{ and } x_1 \le U_{1:n_1}^{(1)} \le x_1 + dx_1, U_{2:n_1}^{(1)} \ge x_1,\ U_{1:n_2}^{(2)} \ge x_2). \tag{24}$$

Moreover, the event $(x_1 \le U_{1:n_1}^{(1)} \le U_{2:n_1}^{(1)} \le x_1 + dx_1)$ is of order $o(dx_1)$, implying that $U_{2:n_1}^{(1)} \ge x_1$ can be reinforced as $U_{2:n_1}^{(1)} \ge x_1 + dx_1$. Thus, at first order, (24) is equivalent to

$$-P(\text{a parking trajectory}, \text{ and } x_1 \le U_{1:n_1}^{(1)} \le x_1 + dx_1, U_{2:n_1}^{(1)} \ge x_1 + dx_1,\ U_{1:n_2}^{(2)} \ge x_2). \tag{25}$$

Among the uniforms $U_1^{(1)}, \dots, U_{n_1}^{(1)}$, only one can take a value in $(x_1, x_1 + dx_1)$, each with the probability dx_1. Suppose, for example, it is $U_1^{(1)}$. The remaining variables then form a set of $n_1 - 1$ independent uniforms independent of $U_1^{(1)}$, denoted $(\tilde{U}_1^{(1)}, \dots, \tilde{U}_{n_1-1}^{(1)})$. Substituting $\tilde{U}_{1:n_1-1}^{(1)}$ for $U_{2:n_1}^{(1)}$ in (25) then yields

$$-n_1 dx_1\ P(\text{a parking trajectory}, \text{ and } \tilde{U}_{1:n_1-1}^{(1)} \ge x_1 + dx_1,\ U_{1:n_2}^{(2)} \ge x_2). \tag{26}$$

After division by $dx_1 \to 0$, we obtain from (23) and (26)

$$\frac{\partial}{\partial x_1} B_{n_1,n_2}(x_1, x_2) = -n_1\ P(\text{a parking trajectory}, \text{ and } \tilde{U}_{1:n_1-1}^{(1)} \ge x_1,\ U_{1:n_2}^{(2)} \ge x_2). \tag{27}$$

Now, let us look at the probability term $P(\dots)$ on the right-hand side of (27). This time, the existence of a parking trajectory concerns the two independent ordered sequences of uniforms, $[(\tilde{U}_{1:n_1-1}^{(1)}, \dots, \tilde{U}_{n_1-1:n_1-1}^{(1)}), (U_{1:n_2}^{(2)}, \dots, U_{n_2:n_2}^{(2)})]$. By definition, they are required to reach the point (n_1, n_2) while satisfying the upper bound constraints given by the lattice. We note, however, that these constraints no longer apply from the origin $(0, 0)$ but from the point $(1, 0)$ because $U_1^{(1)} = x_1 \le u_{0,0}^{(1)}$ and the upper bounds are non-decreasing. Therefore, it suffices to consider the paths from $(1, 0)$ to (n_1, n_2) or, equivalently, from $(0, 0)$ to $(n_1 - 1, n_2)$ after applying the shift operator $\mathcal{E}^{1,0}$ to $(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$.

With the notation of B_{n_1,n_2} completed accordingly, we obtain from (27) that

$$\frac{\partial}{\partial x_1} B_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = -n_1\ B_{n_1-1,n_2}(x_1, x_2 \mid \mathcal{E}^{1,0}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})), \quad n_1 \ge 1. \tag{28}$$

Writing

$$B_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = (-1)^{n_1+n_2} n_1! n_2!\ H_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}), \tag{29}$$

(28) leads to

$$\frac{\partial}{\partial x_1} H_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = H_{n_1-1,n_2}(x_1, x_2 \mid \mathcal{E}^{1,0}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})), \quad n_1 \ge 1. \tag{30}$$

Similarly, we find that

$$\frac{\partial}{\partial x_2} H_{n_1,n_2}(x_1, x_2 \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = H_{n_1,n_2-1}(x_1, x_2 \mid \mathcal{E}^{0,1}(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})), \quad n_2 \ge 1. \tag{31}$$

Moreover, from (29) and the definition of B_{n_1,n_2}, we have

$$H_{0,0} = 1, \quad \text{and} \quad H_{n_1,n_2}(u_{0,0}^{(1)}, u_{0,0}^{(2)} \mid \mathcal{U}^{(1)}, \mathcal{U}^{(2)}) = 0, \quad n_1 + n_2 \ge 1. \tag{32}$$

We therefore deduce from (30)–(32) that each H_{n_1,n_2} corresponds precisely to the A-G polynomial G_{n_1,n_2}. □

Here too, (22) could be obtained from the formula (21) for PK_{n_1,n_2}, but this is superfluous.

4 Ruin in a bidimensional risk model

The A-G polynomials are well known to highlight the hidden algebraic structure in first-crossing problems and epidemic processes. Our objective in this section is to show that they are also a valuable tool to determine non-ruin probabilities over a finite horizon in bidimensional risk processes when claim amounts exhibit a particular form of dependence. For greater clarity, we will only discuss here the case of *univariate polynomials A-G* because they arise in a very simple and natural way.

The notion of dependence used is that, called *partial Schur-constancy*, which was studied by Lefèvre [24] in the continuous case (see the study by Castañer et al. [8] for the discrete case). A short presentation is given in the Appendix. This form of dependence has the advantage of incorporating a large class of partially exchangeable dependencies while leading to compact expressions for the non-ruin probabilities in finite time.

Let us mention that other particular families of polynomials have proven useful in the mathematical theory of insurance. The reader is referred to the studies by Picard and Lefèvre [35], Goffard et al. [17], Dimitrova et al. [15], and Albrecher et al. [3].

4.1 A discrete-time process

To begin, we examine a bidimensional risk process that is formulated on a discrete-time scale $\mathbb{N}_0$. For a single risk, the study of a discrete-time model is fairly standard and well documented (see e.g. the review by Li et al. [30]). For two risks, the discrete-time model below has a structure similar to the one considered in the study by Castañer et al. [7].

Risk model. Specifically, the insurance concerns two risk processes in parallel which are observed on a horizon of length n, say, at equidistant instants $i = 0, 1, \dots, n$. Let $c_0^{(1)}$ (resp. $c_0^{(2)}$) be the initial reserves of the company for the risk labelled 1 (resp. 2). The premium received during period $(i-1, i)$, $1 \le i \le n$, is a constant $c_i^{(1)}$ (resp. $c_i^{(2)}$). Note that these premiums are allowed to vary deterministically over time.

The successive claim amounts for period $(i-1, i)$, $1 \le i \le n$, are random variables $X_i^{(1)}$ (resp. $X_i^{(2)}$). In general, these amounts of claim depend on each other, in a more or less complex way. For the sequel, we will assume that the n claim amounts per unit of time for the risk (1) and those for the risk (2) form two subvectors as defined in (A2) of the Appendix, i.e.

$$[(X_1^{(1)}, \dots, X_n^{(1)}), (X_1^{(2)}, \dots, X_n^{(2)})] \tag{33}$$

constitutes an absolutely continuous vector which is partially Schur-constant. Note that here, $n_1 = n_2 = n$.

So, for each risk $j = 1, 2$, the surplus process at time i is given by

$$R_i^{(j)} = h_i^{(j)} - S_i^{(j)}, \quad 0 \le i \le n, \tag{34}$$

where $h_i^{(j)}$ is the total premiums received including the initial reserve, and $S_i^{(j)}$ is the total claim occurred until time i, i.e.

$$h_i^{(j)} = c_0^{(j)} + c_1^{(j)} + \dots + c_i^{(j)}, \quad \text{and} \quad S_i^{(j)} = X_1^{(j)} + \dots + X_i^{(j)}.$$

Non-ruin probabilities. There are several possible definitions of ruin in multirisk management. Here are the two most standard definitions.

(i) Ruin occurs at time τ_{or} as soon as one of the two surpluses becomes negative. In other words, $\tau_{\text{or}} = \min(\tau_1, \tau_2)$, where τ_j the ruin time for risk j. Thus, the event $(\tau_{\text{or}} > n)$ means that both surpluses remain always non-negative until time n.

(ii) Ruin occurs at time τ_{and} as soon as the two surpluses become negative (not necessarily at the same time). Thus, $\tau_{\text{and}} = \max(\tau_1, \tau_2)$, and the event $(\tau_{\text{and}} > n)$ means that at least one of the two surpluses remains always non-negative until n.

The corresponding probabilities of non-ruin until time n are

$$\phi_{or}(n) = P(\tau_{or} > n), \quad \text{and} \quad \phi_{and}(n) = P(\tau_{and} > n).$$

Of course, they are linked by the identity

$$\phi_{or}(n) = \phi_1(n) + \phi_2(n) - \phi_{and}(n), \tag{35}$$

where $\phi_j(n)$ is the probability of non-ruin until time n for the single risk j.

In practice, it is convenient to deal with the probability of non-ruin $\phi_{or}(n)$, which will also give $\phi_j(n)$, $j = 1, 2$, and thus $\phi_{and}(n)$ using (35). Let $\mathbf{1}(A)$ be the indicator of any event A. We will show that $\phi_{or}(n)$ can be expressed in terms of two A-G polynomials, which are univariate (see Section 2.1).

Proposition 4.1. *If the vector* $[(X_1^{(j)}, \dots, X_n^{(j)}), j = 1, 2]$ *is partially Schur-constant,*

$$\phi_{or}(n) = [(n-1)!]^2\ G_{n-1}(0|\mathcal{U}^{(1)})G_{n-1}(0|\mathcal{U}^{(2)})\ E\{[1/(S_n^{(1)}S_n^{(2)})^{n-1}]\ \mathbf{1}(S_n^{(1)} \le h_n^{(1)},\ S_n^{(2)} \le h_n^{(2)})\}, \tag{36}$$

where $\mathcal{U}^{(j)}$, $j = 1, 2$, *is the family of parameters*

$$u_i^{(j)} = h_{i+1}^{(j)}, \quad 0 \le i \le n-2, \tag{37}$$

and $G_{n-1}(0|\mathcal{U}^{(j)})$ *is the associated univariate A-G polynomial of degree* $n-1$ *at* $x = 0$.

Proof. There is no ruin for either risk up to time n if the two surpluses remain non-negative over the entire horizon. Thus,

$$\phi_{or}(n) = P[(S_1^{(1)} \le h_1^{(1)}, \dots, S_n^{(1)} \le h_n^{(1)}), (S_1^{(2)} \le h_1^{(2)}, \dots, S_n^{(2)} \le h_n^{(2)})].$$

Conditioning on the vector $(S_n^{(1)}, S_n^{(2)})$, of joint density $f_{n,n}$ (see (A8)), we obtain

$$\phi_{or}(n) = \int_{s_1=0}^{h_n^{(1)}}\int_{s_2=0}^{h_n^{(2)}} P[\cap_{i=1}^{n-1}(S_i^{(1)} \le h_i^{(1)}, S_i^{(2)} \le h_i^{(2)})|(S_n^{(1)}, S_n^{(2)}) = (s_1, s_2)]f_{n,n}(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{38}$$

For each risk $j = 1, 2$, let us divide inside (38) all the inequalities $S_i^{(j)} \le h_i^{(j)}$ by $S_n^{(j)}$, $0 \le i \le n$. The claim amount vector (41) being partially Schur-constant, Proposition A.3 with (A6), (A7) applies, so that (38) can be rewritten as follows:

$$\phi_{or}(n) = \int_{s_1=0}^{h_n^{(1)}}\int_{s_2=0}^{h_n^{(2)}} P[\cap_{i=1}^{n-1}(U_{i:n-1}^{(1)} \le h_i^{(1)}/s_1)]\ P[\cap_{i=1}^{n-1}(U_{i:n-1}^{(2)} \le h_i^{(2)}/s_2)]f_{n,n}(s_1, s_2)\ \mathrm{d}s_1\mathrm{d}s_2. \tag{39}$$

By virtue of (20), the probabilities in the integral of (39) are given by univariate A-G polynomials, namely, for $j = 1, 2$,

$$P[\cap_{i=1}^{n-1}(U_{i:n-1}^{(j)} \le h_i^{(j)}/s_j)] = (n-1)!(-1)^{n-1}\ G_{n-1}(0|\mathcal{U}^{(j)}/s_j), \tag{40}$$

where $\mathcal{U}^{(j)}$ is the family of reals $\{u_i^{(j)}\}$ defined in (37). From the relation (4), we have

$$G_{n-1}(0|\mathcal{U}^{(j)}/s_j) = (1/s_j)^{n-1}\ G_{n-1}(0|\mathcal{U}^{(j)}).$$

Thus, the announced formula (36) follows from (39) and (40). □

Remember that the density $f_{n,n}(s_1, s_2)$ of $(S_n^{(1)}, S_n^{(2)})$ is given by (A8), so the expectation in (36) is indeed computable as follows:

$$E\{[1/(S_n^{(1)}S_n^{(2)})^{n-1}]\mathbf{1}(S_n^{(1)} \le h_n^{(1)}, S_n^{(2)} \le h_n^{(2)})\} = \frac{1}{[(n-1)!]^2}\int_{s_1=0}^{h_n^{(1)}}\int_{s_2=0}^{h_n^{(2)}} g^{(n,n)}(s_1, s_2)\ \mathrm{d}s_1\mathrm{d}s_2.$$

Moreover, for a single risk j, $(X_1^{(j)}, \dots, X_n^{(j)})$ is Schur-constant and (36) reduces to

$$\phi_j(n) = (n-1)!\ G_{n-1}(0|\mathcal{U}^{(j)})E\{[1/(S_n^{(j)})^{n-1}]\ \mathbf{1}(S_n^{(j)} \le h_n^{(j)})\},$$

which can also be evaluated, hence $\phi_{\text{and}}(n)$ using (35).

As a special case, suppose that for each risk $j = 1, 2$, the premium per unit of time is constant over time and equal to $c^{(j)}$, starting from an initial capital $v^{(j)}$. This simplifying assumption is found in most of the literature.

Corollary 4.2. *If, in addition,* $h_{i+1}^{(j)} = v^{(j)} + c^{(j)}(i+1)$, $i \in \mathbb{N}_0$,

$$\begin{aligned}\phi_{\text{or}}(n) = (v^{(1)} + c^{(1)})(v^{(2)} + c^{(2)})[(v^{(1)} + c^{(1)}n)(v^{(2)} + c^{(2)}n)]^{n-2}\ E\{[1/(S_n^{(1)}S_n^{(2)})^{n-1}] \\ \mathbf{1}(S_n^{(1)} \le v^{(1)} + c^{(1)}n,\ \ S_n^{(2)} \le v^{(2)} + c^{(2)}n)\}.\end{aligned} \tag{41}$$

Proof. As the function $h_{i+1}^{(j)}$ is of affine form for $i \in \mathbb{N}_0$, the two A-G polynomials $G_{n-1}(0|\mathcal{U}^{(j)})$ in (36) reduce to Abel polynomials. Specifically, by using (4) and (8), we obtain

$$\begin{aligned}G_{n-1}(0|\{v^{(j)} + c^{(j)}(i+1),\ \ i \in \mathbb{N}_0\}) &= G_{n-1}(-v^{(j)} - c^{(j)}|\{c^{(j)}i,\ \ i \in \mathbb{N}_0\}) \\ &= (-v^{(j)} - c^{(j)})(-v^{(j)} - c^{(j)} - c^{(j)}(n-1))^{n-2}/(n-1)! \\ &= (-1)^{n-1}(v^{(j)} + c^{(j)})(v^{(j)} + c^{(j)}n)^{n-2}/(n-1)!.\end{aligned}$$

After substitution in (36), we obtain formula (41). □

4.2 A continuous-time process

We continue by examining a continuous-time version of this risk model. A process of this type has been proposed previously by a number of researchers, e.g. Dang et al. [11] and Gong and Badescu [18]. We thus now reason on a continuous-time scale $t \in \mathbb{R}_+$, and we want to determine the probabilities of non-ruin up to any real time t.

Claim arrivals. A major change is that we have to explicitly introduce the counting processes $\{N_t^{(j)}, t > 0\}$, $j = 1, 2$, which generate the claim arrivals for each risk. To this end, we will consider two processes which are marginally Poisson but can be correlated. More precisely, we assume that if at time t, $(N_t^{(1)}, N_t^{(2)}) = (n_1, n_2)$, then the vectors of claim arrival times for each risk j, denoted by $(T_1^{(j)}, \ldots, T_{n_j}^{(j)})$, are independent of each other and distributed as the order statistics of a sample of n_j independent $(0, t)$-uniform random variables.

For example, we could figure out that each risk j can occur or not during any small interval time $(t, t + \mathrm{d}t)$ with the probabilities

$$\begin{aligned}&P(N_{t+\mathrm{d}t}^{(1)} = k_1 + 1, \quad N_{t+\mathrm{d}t}^{(2)} = k_2|N_t^{(1)} = k_1, \quad N_t^{(2)} = k_2) = (\lambda_1 - \lambda_{1,2})\mathrm{d}t + o(\mathrm{d}t), \\ &P(N_{t+\mathrm{d}t}^{(1)} = k_1, \quad N_{t+\mathrm{d}t}^{(2)} = k_2 + 1|N_t^{(1)} = k_1, \quad N_t^{(2)} = k_2) = (\lambda_2 - \lambda_{1,2})\mathrm{d}t + o(\mathrm{d}t), \\ &P(N_{t+\mathrm{d}t}^{(1)} = k_1 + 1, \quad N_{t+\mathrm{d}t}^{(2)} = k_2 + 1|N_t^{(1)} = k_1, \quad N_t^{(2)} = k_2) = \lambda_{1,2}\mathrm{d}t + o(\mathrm{d}t), \\ &P(N_{t+\mathrm{d}t}^{(1)} = k_1, \quad N_{t+\mathrm{d}t}^{(2)} = k_2|N_t^{(1)} = k_1, \quad N_t^{(2)} = k_2) = 1 - (\lambda_1 + \lambda_2 - \lambda_{1,2})\mathrm{d}t + o(\mathrm{d}t),\end{aligned}$$

with $\lambda_1, \lambda_2 > \lambda_{1,2}$. In this case, each process $\{N_t^{(j)}, t \ge 0\}$ is marginally Poisson of parameter λ_j, and at any time t, the vector $(N_t^{(1)}, N_t^{(2)})$ has a standard bivariate Poisson distribution (see the study by Johnson et al. [20]). In other words, this vector has the representation

$$N_t^{(1)} = M_t^{(1)} + M_t^{(1,2)}, \quad N_t^{(2)} = M_t^{(2)} + M_t^{(1,2)},$$

where $M_t^{(1)}, M_t^{(2)}, M_t^{(1,2)}$ are independent Poisson variables of parameters $(\lambda_1 - \lambda_{1,2})t, (\lambda_2 - \lambda_{1,2})t, \lambda_{1,2}t$, respectively. Note that the correlation coefficient between $N_t^{(1)}$ and $N_t^{(2)}$ is equal to $\lambda_{1,2}/(\lambda_1\lambda_2)^{1/2}$ and is thus always positive, which can be restrictive in certain situations.

Different methods have been developed to overcome this difficulty (see the study by Pfeifer and Neslehová [38] and the references therein). In particular, these authors propose to use copulas while keeping univariate

Poisson margins, which is achievable thanks to Sklar's key theorem. However, a stochastic interpretation as given earlier may no longer exist. We refer the reader to Bäuerle and Grübel [5] for further discussion of this issue.

Risk model. As is often done, we consider here too that for each risk $j = 1, 2$, the successive claim amounts $X_k^{(j)}$, $k \geq 1$, are independent of the claim arrival processes. In addition, as the numbers of claims on $(0, t)$ are integer-valued random variables, it is now the sequence

$$[(X_{k_1}^{(1)}, k_1 \geq 1), (X_{k_2}^{(2)}, k_2 \geq 1)], \tag{42}$$

which is assumed to be absolutely continuous partially Schur-constant. So, the generator $g(x_1, x_2)$ defined through (A2) must be here infinitely monotone function in (x_1, x_2) and thus corresponds to a Laplace transform like (A5). Of course, any pair of subvectors of lengths say (n_1, n_2),

$$[(X_{k_1}^{(1)}, 1 \leq k_1 \leq n_1), (X_{k_2}^{(2)}, 1 \leq k_2 \leq n_2)],$$

then forms a partially Schur-constant vector.

For each risk j, the premiums received up to time t are given by a function $h_t^{(j)}$ including an initial capital. Therefore, the surplus process at time t for risk j is given by

$$R_t^{(j)} = h_t^{(j)} - S_t^{(j)}, \quad t \geq 0, \tag{43}$$

where $S_t^{(j)}$ is the total claim amounts up to time t, i.e.

$$S_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} X_k^{(j)}.$$

Non-ruin probabilities. Ruin is defined as in Section 4.1, and our objective is to determine the non-ruin probabilities $\phi_{\text{or}}(t)$ until any time t. We will see that, as expected, $\phi_{\text{or}}(t)$ can be again expressed in terms of two univariate A-G polynomials, which are defined this time from randomized parameters.

Proposition 4.3. *If the sequence* $[(X_{k_j}^{(j)}, k_j \geq 1), j = 1, 2]$ *is partially Schur-constant,*

$$\phi_{\text{or}}(t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P(N_t^{(1)} = n_1, N_t^{(2)} = n_2)\, E_{(n_1,n_2)}(t), \tag{44}$$

where $E_{(n_1,n_2)}(t)$ *denotes the expectation, taken with respect to* $(S_{n_1}^{(1)}, S_{n_2}^{(2)})$ *and* $(T_1^{(j)}, \dots, T_{n_j}^{(j)})$ for $j = 1, 2$, *defined by*

$$\begin{aligned} E_{(n_1,n_2)}(t) = (-1)^{n_1+n_2}\, (n_1 - 1)!(n_2 - 1)!\; E\{G_{n_1-1}(0|U^{(1)})\, G_{n_2-1}(0|U^{(2)})[1/(S_{n_1}^{(1)})^{n_1-1}(S_{n_2}^{(2)})^{n_2-1}] \\ \mathbf{1}(T_{n_1}^{(1)}, T_{n_2}^{(2)} < t,\ S_{n_1}^{(1)} \leq h_{T_{n_1}^{(1)}}^{(1)},\ S_{n_2}^{(2)} \leq h_{T_{n_2}^{(2)}}^{(2)})\}. \end{aligned} \tag{45}$$

and $U^{(j)}$, $j = 1, 2$, *is the family of randomized parameters*

$$U_i^{(j)} = h_{T_{i+1}^{(j)}}^{(j)}, \quad 0 \leq i \leq n_j - 2. \tag{46}$$

Proof. For each risk j, fix the total number of claims that have occurred up to time t and their successive times of arrival. So, we set $N_t^{(j)} = n_j$ and $(T_1^{(j)}, \dots, T_{n_j}^{(j)}) = (t_1^{(j)}, \dots, t_{n_j}^{(j)})$ with $t_{n_j}^{(j)} \leq t$, and we collect all these values under the label A_t. Denote by $\phi_{\text{or}}(t|A_t)$ the probability of non-ruin until time t conditioned on A_t. Clearly, we have

$$\phi_{\text{or}}(t|A_t) = P[(S_1^{(1)} \leq h_{t_1^{(1)}}^{(1)}, \dots, S_{n_1}^{(1)} \leq h_{t_{n_1}^{(1)}}^{(1)}), (S_1^{(2)} \leq h_{t_1^{(2)}}^{(2)}, \dots, S_{n_2}^{(2)} \leq h_{t_{n_2}^{(2)}}^{(2)})|A_t].$$

The two claim subvectors being partially Schur-constant for any dimension, we can reason exactly as for Proposition 4.1 to obtain $\phi_{\text{or}}(t|A_t)$. After elimination of the conditioning on A_t, we directly deduce the results (44)–(46). □

Recall that $E_{(n_1,n_2)}(t)$ in (44) is indeed computable because $(S^{(1)}_{n_1}, S^{(2)}_{n_2})$ has density (A8), and as stated previously, $(T^{(j)}_1, \ldots, T^{(j)}_{n_j})$, $j = 1, 2$, are two independent vectors, each distributed as $(t\ U^{(j)}_{1:n_j}, \ldots, t\ U^{(j)}_{n_j:n_j})$.

Let us examine the particular case where premiums accumulate at a constant rate $c^{(j)}$, from an initial capital $v^{(j)}$. This situation therefore resembles that discussed in Corollary 4.2.

Corollary 4.4. *If, in addition, $h^{(j)}_t = v^{(j)} + c^{(j)}t$, $t \geq 0$, then the formula (44) applies with $E_{(n_1,n_2)}(t)$ which simplifies to the expectation, taken only with respect to $(S^{(1)}_{n_1}, S^{(2)}_{n_2})$ and $(T^{(1)}_{n_1}, T^{(2)}_{n_2})$, defined by*

$$E_{(n_1,n_2)}(t) = E\Bigg\{\prod_{j=1}^{2}[(v^{(j)} + c^{(j)}T^{(j)}_{n_j})^{n_j-2}(v^{(j)} + c^{(j)}T^{(j)}_{n_j}/n_j)/(S^{(j)}_{n_j})^{n_j-1}] \\ \mathbf{1}(T^{(1)}_{n_1}, T^{(2)}_{n_2} < t,\ S^{(1)}_{n_1} \leq v^{(1)} + c^{(1)}T^{(1)}_{n_1}, S^{(2)}_{n_2} \leq v^{(2)} + c^{(2)}T^{(2)}_{n_2})\Bigg\}. \tag{47}$$

Proof. We first condition on the numbers (n_1, n_2) of claims up to t. Given the form of $h^{(j)}_t$, the two polynomials involved in (45) become

$$G_{n_j-1}(0|\{v^{(j)} + c^{(j)}T^{(j)}_i, 1 \leq i \leq n_j - 1\}), \quad j = 1, 2. \tag{48}$$

Of course, each claim arrival time $T^{(j)}_i$ is given by

$$T^{(j)}_i = Y^{(j)}_1 + \ldots + Y^{(j)}_i, \quad i \geq 1, \tag{49}$$

where $Y^{(j)}_i$ represents the time between the $(i-1)$-th and i-th claim arrivals. By construction, the two vectors $(Y^{(j)}_1, \ldots, Y^{(j)}_{n_j})$ are conditionally independent, each of them being composed of exchangeable random variables.

Now, by using (4), we can rewrite the polynomial G_{n_j-1} of (48) as follows:

$$\begin{aligned} &(c^{(j)})^{n_j-1}\ G_{n_j-1}(-v^{(j)}/c^{(j)}|\{T^{(j)}_i, 1 \leq i \leq n_j - 1\}) \\ &= (c^{(j)})^{n_j-1}\ G_{n_j-1}(-v^{(j)}/c^{(j)} - T^{(j)}_{n_j}|\{T^{(j)}_i - T^{(j)}_{n_j}, 1 \leq i \leq n_j - 1\}) \\ &= (-c^{(j)})^{n_j-1}\ G_{n_j-1}(v^{(j)}/c^{(j)} + T^{(j)}_{n_j}|\{T^{(j)}_{n_j} - T^{(j)}_i, 1 \leq i \leq n_j - 1\}). \end{aligned} \tag{50}$$

From (49) (with $T^{(j)}_0 \equiv 0$), we have

$$T^{(j)}_{n_j} - T^{(j)}_i = Y^{(j)}_{i+1} + \ldots + Y^{(j)}_{n_j}, \quad 0 \leq i \leq n_j - 1,$$

and because of the exchangeability of $(Y^{(j)}_1, \ldots, Y^{(j)}_{n_j})$,

$$T^{(j)}_{n_j} - T^{(j)}_i \text{ is distributed as } Y^{(j)}_1 + \ldots + Y^{(j)}_{n_j-i} \equiv W^{(j)}_{n_j-i}, \quad 0 \leq i \leq n_j - 1, \tag{51}$$

where we have adopted the same notation as in Proposition 2.6. By combining (50) and (51), we then obtain for the expectation (45)

$$\begin{aligned} E_{(n_1,n_2)}(t) = &(n_1 - 1)!(n_2 - 1)!(c^{(1)})^{n_1-1}(c^{(2)})^{n_2-1}E\{[1/(S^{(1)}_{n_1})^{n_1-1}(S^{(2)}_{n_2})^{n_2-1}] \\ &\times G_{n_1-1}(v^{(1)}/c^{(1)} + W^{(1)}_{n_1}|\{W^{(1)}_{n_1-1}, \ldots, W^{(1)}_1\})\ G_{n_2-1}(v^{(2)}/c^{(2)} + W^{(2)}_{n_2}|\{W^{(2)}_{n_2-1}, \ldots, W^{(2)}_1\}) \\ &\mathbf{1}(W^{(1)}_{n_1}, W^{(2)}_{n_2} < t,\ S^{(1)}_{n_1} \leq v^{(1)} + c^{(1)}W^{(1)}_{n_1},\ S^{(2)}_{n_2} \leq v^{(2)} + c^{(2)}W^{(2)}_{n_2})\}. \end{aligned} \tag{52}$$

Finally, we also condition on the event $(W^{(1)}_{n_1}, W^{(2)}_{n_2}) = (t^{(1)}_{n_1}, t^{(2)}_{n_2})$. By the tower rule, the expectation $E\{\ldots\}$ in (52) becomes

$$\begin{aligned} &E\{[1/(S^{(1)}_{n_1})^{n_1-1}(S^{(2)}_{n_2})^{n_2-1}]\ \mathbf{1}(t^{(1)}_{n_1}, t^{(2)}_{n_2} < t,\ S^{(1)}_{n_1} \leq v^{(1)} + c^{(1)}t^{(1)}_{n_1},\ S^{(2)}_{n_2} \leq v^{(2)} + c^{(2)}t^{(2)}_{n_2}) \\ &E[\prod_{j=1}^{2} G_{n_j-1}(v^{(j)}/c^{(j)} + t^{(j)}_{n_j}|\{W^{(j)}_{n_j-1}, \ldots, W^{(j)}_1\})|(W^{(1)}_{n_1}, W^{(2)}_{n_2}) = (t^{(1)}_{n_1}, t^{(2)}_{n_2})]\}. \end{aligned} \tag{53}$$

Since the vectors $(Y_1^{(j)}, \ldots, Y_{n_j}^{(j)})$ are independent and each exchangeable, the identity (9) of Proposition 2.6 applies to both terms inside the conditional expectation of (53). This implies (in obvious notation) that

$$E\left[\prod_{j=1}^{2} G_{n_j-1}(\ldots)|(t_{n_1}^{(1)}, t_{n_2}^{(2)})\right] = \prod_{j=1}^{2} \frac{(v^{(j)}/c^{(j)} + t_{n_j}^{(j)})^{n_j-2}}{(n_j-2)!}\left(\frac{v^{(j)}/c^{(j)} + t_{n_j}^{(j)}}{n_j-1} - \frac{t_{n_j}^{(j)}}{n_j}\right). \tag{54}$$

Inserting (53) with (54) into (52) and then removing the conditioning, we deduce the desired formula (47). □

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Appendix

A Partial Schur-constancy

We summarize here the key elements of the notion of partial Schur-constancy introduced in the study by Lefèvre [24]. Consider a random vector partitioned into two groups (for example) of absolutely continuous variables on $\mathbb{R}_+$ of respective sizes n_1 and n_2, denoted

$$[(X_1^{(1)}, \ldots, X_{n_1}^{(1)}), (X_1^{(2)}, \ldots, X_{n_2}^{(2)})]. \tag{A1}$$

Definition A.1. The vector (A1) is partially Schur-constant if there exists a bivariate function $g(x_1, x_2) : \mathbb{R}_+^2 \to (0, 1)$, called generator, such that the joint survival function of (A1) can be expressed in the special form

$$\begin{aligned} &P[(X_1^{(1)} \geq x_{1,1}, \ldots, X_{n_1}^{(1)} \geq x_{1,n_1}), (X_1^{(2)} \geq x_{2,1}, \ldots, X_{n_2}^{(2)} \geq x_{2,n_2})] \\ &\quad = g(x_{1,1} + \ldots + x_{1,n_1}, \ x_{2,1} + \ldots + x_{2,n_2}), \quad \text{for all}\,; x_{j,i_j} \in \mathbb{R}_+, 1 \leq i_j \leq n_j, \ j = 1, 2. \end{aligned} \tag{A2}$$

So, the vector (A1) is partially exchangeable in the sense of de Finetti [13], but with a specific form of dependency. Of course, the partial Schur-constancy implies a simple Schur-constancy for each vector $(X_1^{(j)}, \ldots, X_{n_j}^{(j)})$, $j = 1, 2$. This last property is defined similarly but from a univariate generator $g_j(x_j) : \mathbb{R}_+ \to (0, 1)$ (see, e.g. Nelsen [31], Chi et al. [9] and Lefèvre and Simon [29]). With respect to the bivariate generator $g(x_1, x_2)$, we then have $g_1(x_1) = g(x_1, 0)$ and $g_2(x_2) = g(0, x_2)$.

To actually exist, the generator of the survival function (A2) must satisfy certain conditions. The result below is stated in the study by Lefèvre [24] (see the study by Ressel [39] for a detailed analysis).

Proposition A.2. *A function $g(x_1, x_2)$ may generate a partially Schur-constant vector* (A2)*if and only if it is a (n_1, n_2)-monotone function on $\mathbb{R}_+^2$. In other words, for all $x_1, x_2 \in \mathbb{R}_+$,*

$$(-1)^{k_1+k_2} g^{(k_1,k_2)}(x_1, x_2) \geq 0, \quad 0 \leq k_j \leq n_j, \quad j = 1, 2, \tag{A3}$$

provided that $g(x_1, x_2)$ is sufficiently differentiable.

A family of survival copulas is also defined in the study by Lefèvre [24]. They are called partially Archimedean because they generalize Archimedean copulas by assuming partial exchangeability of the uniform vector involved. It is proved that a partially Schur-constant vector has a partially Archimedean survival copula with the same generator, and vice versa.

Different possible generators are proposed in the study by Lefèvre [24] which are thus (n_1, n_2)-monotone functions (by Proposition A.2 with (A3)). Here are two simple examples, among others.

(a) Consider for $g(x_1, x_2)$ the bivariate exponential Gumbel distribution, i.e.

$$g(x_1, x_2) = e^{-\zeta_1 x_1 - \zeta_2 x_2 - \zeta_3 x_1 x_2}, \quad x_1, x_2 \geq 0, \tag{A4}$$

where $\zeta_1, \zeta_2, \zeta_3$ are positive parameters with $\zeta_3 \leq \zeta_1\zeta_2$.

Then, (A4) is $(1, n)$-monotone if and only if $\zeta_3 \leq (1/n)\zeta_1\zeta_2$. To be $(2, n)$-monotone, a sufficient condition is $\zeta_3 \leq [1 - (1 - 1/n)^{1/2}]\zeta_1\zeta_2$, for example. To be $(3, n)$-monotone, it suffices that $\zeta_3 \leq \{1 - [1 - 1/n(n-1)(n-2)]^{1/3}\}\zeta_1\zeta_2$.

(b) Consider for $g(x_1, x_2)$ the Laplace transform of some random vector (Λ_1, Λ_2), i.e.

$$g(x_1, x_2) = E(e^{-\Lambda_1 x_1 - \Lambda_2 x_2}), \quad x_1, x_2 \geq 0. \tag{A5}$$

Then, (A5) is infinitely monotone in (x_1, x_2). In fact, the multivariate Bernstein-Widder theorem asserts that the reverse implication is always true.

For illustration, when (Λ_1, Λ_2) has a bivariate gamma distribution,

$$g(x_1, x_2) = \frac{1}{(1 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_1 x_2)^{\alpha}}, \quad x_1, x_2 \geq 0,$$

where $\alpha, \zeta_1, \zeta_2, \zeta_3$ are positive parameters satisfying the condition $\zeta_3 \leq \zeta_1\zeta_2$.

As expected, the partial Schur-constancy can be characterized by means of several equivalent representations. The following is easily proved and plays an important role in Section 4. Denote the two partial sums associated with the subvectors in (A1) by

$$S_{i_j}^{(j)} = X_1^{(j)} + \ldots + X_{i_j}^{(j)}, \quad 1 \leq i_j \leq n_j, \quad j = 1, 2.$$

Proposition A.3. *The vector* (A1) *is partially Schur-constant if and only if each subvector* $(X_1^{(j)}/S_{n_j}^{(j)}, \ldots, X_{n_j}^{(j)}/S_{n_j}^{(j)})$ *is independent of the variable* $S_{n_j}^{(j)}$, *and the vector*

$$[(S_1^{(1)}/S_{n_1}^{(1)}, \ldots, S_{n-1}^{(1)}/S_{n_1}^{(1)}), (S_1^{(2)}/S_{n_2}^{(2)}, \ldots, S_{n-1}^{(2)}/S_{n_2}^{(2)})] \tag{A6}$$

is distributed as the vector

$$[(U_{1:n_1-1}^{(1)}, \ldots, U_{n_1-1:n_1-1}^{(1)}), (U_{1:n_2-1}^{(2)}, \ldots, U_{n_2-1:n_2-1}^{(2)})], \tag{A7}$$

where each subvector $(U_{1:n_j-1}^{(j)}, \ldots, U_{n_j-1:n_j-1}^{(j)})$ *corresponds to the order statistics of a sample of* $n_j - 1$ *independent* $(0, 1)$-*uniform random variables, independent of each other and of the vector* $(S_{n_1}^{(1)}, S_{n_2}^{(2)})$.

Moreover, the density function of $(S_{n_1}^{(1)}, S_{n_2}^{(2)})$ *is known explicitly as follows*:

$$f_{n_1,n_2}(s_1, s_2) = g^{(n_1,n_2)}(s_1, s_2)\, \frac{s_1^{n_1-1}}{(n_1-1)!} \frac{s_2^{n_2-1}}{(n_2-1)!}, \quad s_1, s_2 \in \mathbb{R}_+. \tag{A8}$$

Note that the study by Lefèvre [24] briefly discusses an extension of partial Schur-constancy to the modelling of nested and multi-level dependencies.

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