Research Article
Special Issue: 10 years of Dependence Modeling

Wongtawan Chamnan and Songkiat Sumetkijakan*

Characterization of pre-idempotent Copulas

https://doi.org/10.1515/demo-2023-0106
received June 5, 2023; accepted October 4, 2023

Abstract: Copulas C for which $(C^tC)^2 = C^tC$ are called *pre-idempotent* copulas, of which well-studied examples are idempotent copulas and complete dependence copulas. As such, we shall work mainly with the topology induced by the modified Sobolev norm, with respect to which the class $\mathcal{R}$ of pre-idempotent copulas is closed and the class of factorizable copulas is a dense subset of $\mathcal{R}$. Identifying copulas with Markov operators on L^2, the one-to-one correspondence between pre-idempotent copulas and partial isometries is one of our main tools. In the same spirit as Darsow and Olsen's work on idempotent copulas, we obtain an explicit characterization of pre-idempotent copulas, which is split into cases according to the atomicity of its associated σ-algebras, where the nonatomic case gives all factorizable copulas and the totally atomic case yields conjugates of ordinal sums of copies of the product copula.

Keywords: idempotent copulas, pre-idempotent copulas, implicit dependence copulas, factorizable copulas, partial isometries

MSC 2020: 62H05 (primary), 60A10, 28A05, 47B65 (secondary)

1 Introduction

As is well-known, dependence between two continuous random variables can be captured by their copula. Stochastic independence is modeled by the product copula Π. Monotonic dependence is modeled by the Fréchet-Hoeffding bounds $M \coloneqq C_{e,e}$ (comonotonic) or $W \coloneqq C_{e,1-e}$ (countermonotonic) and a general complete dependence copula can be represented by a measure-preserving transformation (MPT) f on $[0, 1]$ and written as $C_{e,f}$ or $C_{f,e}$, where e denotes the identity function. To cite an instance, $C_{e,f}$ is the copula of $[0, 1]$-uniformly distributed random variables U and V for which $V = f(U)$ almost surely, and the mass of its associated measure is concentrated on the graph of f [6,12]. See equation (2.3) for the definition of $C_{f,g}$. Even though every copula is of the form $C_{f,g} = C_{f,e}C_{e,g}$, the Markov product of $C_{f,e}$ and $C_{e,g}$ and relations between the copula and the MPTs f and g are far from simple [5,13,23]. Swapping places of the factors defines a factorizable copula $C_{e,f}C_{g,e}$, whose associated measure has its mass concentrated on $\{(x, y) : f(x) = g(y)\}$ [16,19]. This implies, in particular, that factorizable copulas are singular, yet at the same time gives nice, vivid, and potentially useful interpretations of factorizable copulas. In fact, $C_{e,f}C_{g,e}$ is the copula of some implicitly dependent U and V for which $f(U) = g(V)$ almost surely. See the studies by Panyasakulwong et al. [14,15] for novel investigations on such implicit dependence copulas in which it was shown that the converse is not always true. While complete dependence copulas are at least one-way deterministic, factorizable copulas can be much less deterministic, especially when f and g are highly noninjective. Surprisingly, complete dependence copulas are quite ubiquitous in the theory of copulas [22]. Being more stochastic, factorizable copulas could conceivably play a

*** Corresponding author: Songkiat Sumetkijakan,** Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand, e-mail: songkiat.weng@gmail.com
Wongtawan Chamnan: Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand, e-mail: wongtawan.momild@gmail.com

complementing role to complete dependence copulas. Thus, a systematic study of factorizable copulas must be commenced.

In 2010, Darsow and Olsen [4] provided a characterization of idempotent copulas by splitting them into cases according to the atomicity types. For instance, nonatomic idempotent copulas are precisely the symmetric factorizable copulas $C_{e,f}C_{f,e}$, and every totally atomic idempotent copula is conjugate to an ordinal sum of a countable copies of Π. Even though both M and Π are idempotent, all other complete dependence copulas are not. Moreover, idempotent copulas must be symmetric [4,21] and hence are too limited to model general dependence. However, a novel partial ordering on the idempotent copulas was shown [4] to be equivalent to the inclusion on the classes of their invariant sets. Through some examples, this ordering is related to degrees of dependence. Our goal is to construct and study a class of copulas that accommodates sufficiently many dependence structures, or rather dependence scaffolding in the sense that the class contains all factorizable copulas while still possesses adequate properties to gauge dependence levels. Clearly, for every factorizable copula $C = C_{e,f}C_{g,e}$, the transpose-product C^tC is an idempotent, and so is CC^t. Hence, we shall investigate the class $\mathcal{R}$ of pre-idempotent copulas defined as copulas C for which C^tC is idempotent. The class $\mathcal{R}$ contains not only idempotent copulas and factorizable copulas but also conjugates of ordinal sums of factorizable copulas and copies of Π. Here, C is a conjugate of D if $C = C_{r,e}DC_{e,p}$ for some invertible MPTs r and p. We obtain a characterization of pre-idempotent copulas using tools from the precedent characterization of idempotent copulas in the study by Darsow and Olsen [4] and the fact that the set of pre-idempotent copulas and the set of partially isometric Markov operators are isomorphic. In addition, the atomicity of the associated σ-algebras of copulas defined in the study by Sumetkijakan [19] plays the main role in splitting our consideration into cases. A copula C is in $\mathcal{R}$ if and only if C is conjugate to an ordinal sum $D \coloneqq \oplus_{\mathcal{P}} D_k$ of a factorizable copula and/or copies of Π as follows:
- if C is nonatomic, then $\mathcal{P} = \{[0,1]\}$ and D_1 is factorizable, which implies that C is factorizable;
- if C is totally atomic, then $D_k = \Pi$ for all k; and
- if C is atomic but not totally atomic, then D_1 is factorizable and $D_k = \Pi$ for all $k \geq 2$.

The above characterization and its proof are in Section 3. In the last section, we consider the set of copulas under the strong topology induced by the modified Sobolev norm defined in equation (2.4). It is shown that $\mathcal{R}$ is closed, and the set of factorizable copulas is dense in $\mathcal{R}$.

2 Background knowledge and tools

Throughout the manuscript, let $\mathscr{B} \coloneqq \mathscr{B}(\mathbf{I})$ denote the Borel σ-algebra on $\mathbf{I} \coloneqq [0,1]$, and λ the Lebesgue measure on $\mathbf{I}$. The class of essentially bounded Borel measurable functions on $\mathbf{I}$ is denoted by L^∞. For $1 \leq p < \infty$, L^p denotes the class of p-integrable Borel measurable functions on $\mathbf{I}$, each of which is a Banach space under the usual L^p-norm $\|\cdot\|_p$. Clearly, $L^\infty \subseteq L^p \subseteq L^1$. The indicator function of a Borel set $A \subseteq \mathbf{I}$ is denoted by $\mathbf{1}_A$. The identity function on $\mathbf{I}$ is denoted by e.

Let $p \in [1, \infty)$ and $q \in (1, \infty]$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. For a bounded linear operator S on L^p, its adjoint is the unique bounded linear operator S^* on L^q that satisfies $\int_{\mathbf{I}} S\psi \cdot \xi \mathrm{d}\lambda = \int_{\mathbf{I}} \psi \cdot S^*\xi \mathrm{d}\lambda$ for all $\psi \in L^p$ and $\xi \in L^q$. Recall that $\|S^*\| = \|S\|$. The case $p = 2$ is special in that S^* and S are both linear operators on the same space L^2.

2.1 Markov operators

A linear operator $S : L^p \to L^p$ is called a *Markov operator on L^p* if
(i) $S\mathbf{1}_{\mathbf{I}} = \mathbf{1}_{\mathbf{I}}$;
(ii) $\int_{\mathbf{I}} S\psi \, \mathrm{d}\lambda = \int_{\mathbf{I}} \psi \, \mathrm{d}\lambda$ for $\psi \in L^p$; and
(iii) $S\psi \geq 0$ whenever $\psi \geq 0$.

The class of all such operators is denoted by $\mathcal{M}_p$. By the linearity and positivity of $S \in \mathcal{M}_p$, $|S\psi| \le S|\psi|$ for all $\psi \in L^p$. For $S \in \mathcal{M}_1$, it holds that for $\psi \in L^p$ and $\xi \in L^\infty$,

$$||S\psi||_1 = \int_{\mathbf{I}} |S\psi| \mathrm{d}\lambda \le \int_{\mathbf{I}} S|\psi| \mathrm{d}\lambda = \int_{\mathbf{I}} |\psi| \mathrm{d}\lambda = ||\psi||_1 \tag{2.1}$$

and

$$|S\xi| \le S|\xi| \le S||\xi||_\infty = ||\xi||_\infty S\mathbf{1}_{\mathbf{I}} = ||\xi||_\infty \mathbf{1}_{\mathbf{I}} = ||\xi||_\infty. \tag{2.2}$$

Thus, S is bounded as an operator on L^1 and as an operator on L^∞ with norm 1. For $1 < p < \infty$, by Riesz-Thorin interpolation theorem [7, Theorem 8.23], $||S\psi||_p \le ||\psi||_p$ for all $\psi \in L^p$; hence, every $S \in \mathcal{M}_1$ restricts to a Markov operator on L^p. Since a Markov operator on L^∞ extends uniquely to a Markov operator on L^1, the *adjoint* of $S \in \mathcal{M}_1$ can and usually will be regarded as the adjoint Markov operator $S^* \in \mathcal{M}_1$ (see [7, Theorem 13.2]). Moreover, for $S, T \in \mathcal{M}_1$, $(S^*)^* = S$, $T \circ S \in \mathcal{M}_1$ and $(T \circ S)^* = S^* \circ T^*$. For $1 \le p < \infty$, the *strong operator topology* (SOT) on $\mathcal{M}_p$ is the coarsest topology such that, for each fixed $\psi \in L^p$, the evaluation map $\mathcal{M}_p \to L^p$, $S \mapsto S\psi$ is continuous. The *weak operator topology* (WOT) on $\mathcal{M}_p$ is the coarsest topology such that, for each fixed $\psi \in L^p$ and $\xi \in L^q$, the evaluation map $\mathcal{M}_p \to \mathbb{R}$, $S \mapsto \int_{\mathbf{I}} S\psi \cdot \xi \mathrm{d}\lambda$ is continuous.

The following proposition quoted from [7, Proposition 13.6] says that $\mathcal{M}_1$ and $\mathcal{M}_p$ are homeomorphic. Its proof is a good exercise in functional analysis.

Proposition 2.1. *For $1 \le p \le \infty$, the restriction mapping $\Phi_p : \mathcal{M}_1 \to \mathcal{M}_p$ defined by $\Phi_p(S) := S\,|_{L^p}$ is a bijection. If $p < \infty$ and q is its conjugate exponent, then $\Phi_q(S^*) = \Phi_p(S)^*$ for $S \in \mathcal{M}_1$ and the mapping Φ_p is a homeomorphism for the weak as well as the SOTs.*

For a linear operator T on L^2, recall that T is called an *isometry* if $||T\varphi||_2 = ||\varphi||_2$ for all $\varphi \in L^2$, and it is called a *partial isometry* if it acts isometrically on the orthogonal complement of its kernel, i.e., $||T\varphi||_2 = ||\varphi||_2$ for all $\varphi \in \ker(T)^\perp$. A bounded linear operator P on L^2 is called a *projection* if it is self-adjoint and $P^2 = P$. Let us quote a characterization of partial isometries from the study by Fernández-Polo and Peralta [8] and Garcia et al. [9], which is an important tool in our investigation of pre-idempotent copulas.

Proposition 2.2. *For a bounded linear operator T on L^2, the following are equivalent*:
(*i*) *T is a partial isometry.*
(*ii*) *T^* is a partial isometry.*
(*iii*) *$T = TT^*T$.*
(*iv*) *$T^* = T^*TT^*$.*
(*v*) *T^*T is a projection.*
(*vi*) *TT^* is a projection.*

2.2 Copulas

A function $C : \mathbf{I}^2 \to \mathbf{I}$ is said to be a *copula* if, for $u, v, u', v' \in \mathbf{I}$,
(i) $C(u, 0) = C(0, v) = 0$;
(ii) $C(u, 1) = u$ and $C(1, v) = v$; and
(iii) $C(u', v') - C(u', v) - C(u, v') + C(u, v) \ge 0$ whenever $u \le u'$ and $v \le v'$.

The class of copulas is denoted by $\mathcal{C}$. Every $C \in \mathcal{C}$ induces a unique doubly stochastic measure μ_C via $\mu_C((a, b] \times (c, d]) := C(b, d) - C(b, c) - C(a, d) + C(a, c)$ and the *support* of C is defined as the support of the induced measure μ_C, i.e., $\operatorname{supp}(C) := \mathbf{I}^2 \backslash \cup\{O \subseteq \mathbf{I}^2 : O \text{ is open and } \mu_C(O) = 0\}$. Denote $C^t(u, v) := C(v, u)$ for $u, v \in \mathbf{I}$; and a copula C is said to be *symmetric* if $C^t = C$. The most well-known copulas are the product copula $\Pi(u,v) \coloneqq uv$ and the Fréchet-Hoeffding upper and lower bounds $M(u,v) \coloneqq \min\{u,v\}$ and $W(u,v) \coloneqq \max\{0, u+v-1\}$, respectively. The *Markov product*, also called the *-product, is a binary operation on $\mathcal{C}$, which is defined as follows:

$$(AB)(u,v) \coloneqq \int_0^1 \partial_2 A(u,t)\partial_1 B(t,v)\mathrm{d}t,$$

under which the class $\mathcal{C}$ is a monoid having M as the identity and Π as the zero. A copula C is said to be *left* (respectively, *right*) *invertible* if there is a copula D for which $DC = M$ (respectively, $CD = M$); and D, which must be C^t if exists, is called a *left* (respectively, *right*) *inverse* of C. C is said to be *invertible* if it is both left and right invertible. Note that $(AB)^t = B^t A^t$.

Let $\mathcal{F}$ be the set of all measure-preserving functions on $(\mathbf{I}, \mathscr{B}, \lambda)$. A function $f \in \mathcal{F}$ is said to possess an *essential inverse* $g \in \mathcal{F}$ if $g \circ f = e = f \circ g$ a.e. We denote by $\mathcal{F}_{\mathrm{inv}}$ the set of measure-preserving functions that possess essential inverses. Define the copula $C_{f,g}$ induced by f and g in $\mathcal{F}$ by [6]

$$C_{f,g}(u,v) \coloneqq \lambda(f^{-1}([0,u]) \cap g^{-1}([0,v])) \quad \text{for } u,v \in \mathbf{I}. \tag{2.3}$$

The left (respectively, right) invertible copulas are exactly the copulas $C_{e,f}$ (respectively, $C_{f,e}$) for $f \in \mathcal{F}$. Note that $C_{f,e}C_{e,f} = M$ and that the invertible copulas are exactly the copulas $C_{e,h}$, where $h \in \mathcal{F}_{\mathrm{inv}}$.

From Olsen et al. [13], there is a one-to-one correspondence between $\mathcal{C}$ and $\mathcal{M}_1$ via the isomorphism $C \mapsto T_C$ defined by $(T_C\psi)(x) \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \partial_2 C(x,t)\psi(t)\mathrm{d}t$ for $x \in \mathbf{I}$. In particular, $T_{CD} = T_C \circ T_D$ for $C, D \in \mathcal{C}$. Also, the adjoint of T_C is $T_C^* = T_{C^t}$ and hence $T_C^{**} = T_C$. For brevity, we denote $T_{f,g} \coloneqq T_{C_{f,g}}$. From the study by Darsow and Olsen [4], for every $f \in \mathcal{F}$ and $\phi \in L^1$, $T_{e,f}(\phi) = \phi \circ f$ and $T_{f,e}(\phi \circ f) = \phi$. If $T^* \in \mathcal{M}_1$ is the adjoint of $T \in \mathcal{M}_1$, then, by Proposition 2.1, $(T|_{L^2})^* = (\Phi_2(T))^* = \Phi_2(T^*) = T^*|_{L^2}$, and so we usually write the corresponding L^2-Markov operators of C and its adjoint as T_C and T_C^*, respectively.

In 1992, Darsow et al. [2] showed that, with respect to the uniform distance d_∞, the class $\mathcal{C}$ is closed and compact and the class of invertible copulas is dense in $\mathcal{C}$. Let $\{A_n\}$ and $\{B_n\}$ be sequences in $\mathcal{C}$ converging to copulas A and B, respectively. It holds that $A_nB \to AB$ and $BA_n \to BA$, but it is not always true that $A_nB_n \to AB$.

Let $\|\cdot\|_S$ denote a modified Sobolev norm on span$(\mathcal{C})$ defined by (see [3, the norm $|\cdot|_{1,2}$] and [18])

$$\|C\|_S^2 \coloneqq \int_0^1\int_0^1 |\partial_1 C(u,v)|^2 + |\partial_2 C(u,v)|^2 \mathrm{d}u\mathrm{d}v. \tag{2.4}$$

Let d_2 be the metric on $\mathcal{C}$ induced by the modified Sobolev norm, i.e., $d_2(A,B) \coloneqq \|A-B\|_S$. It follows from symmetry that $A_n \xrightarrow{d_2} A$ if and only if $A_n^t \xrightarrow{d_2} A^t$. It was shown in the study by Darsow and Olsen [3] that the Markov product is jointly continuous with respect to d_2 and that the metric space $(\mathcal{C}, d_2)$ is complete. The metric D_2 introduced in the study by Trutschnig [20] is defined by $D_2^2(A,B) \coloneqq \int_0^1\int_0^1 |\partial_1(A-B)(u,v)|^2\mathrm{d}u\mathrm{d}v$. It can be regarded as an asymmetric version of d_2 via the relation $d_2^2(A,B) = D_2^2(A,B) + D_2^2(A^t,B^t)$. The topology induced by d_2 coincides [20] with the topology on $\mathcal{C}$ induced by the *∂-convergence* studied in the works of Mikusiński and Taylor [10,11]. As such, the corresponding topology on $\mathcal{M}_1$ shall be denoted by $\mathcal{O}_\partial$. Let us denote, respectively, by $\mathcal{O}_w$ and $\mathcal{O}_s$ the WOT and the SOT on $\mathcal{M}_1$. Using the one-to-one correspondence between $\mathcal{C}$ and $\mathcal{M}_1$, the topology $\mathcal{O}_w$ on $\mathcal{M}_1$ corresponds to the topology on $\mathcal{C}$ induced by the metric d_∞ (see [13]), while it was shown in the study by Trutschnig [20] that $\mathcal{O}_s$ corresponds to the topology on $\mathcal{C}$ induced by the metric D_2. So $\mathcal{O}_\partial$ is finer than $\mathcal{O}_s$ and $\mathcal{O}_w$, i.e., $\mathcal{O}_w \subset \mathcal{O}_s \subset \mathcal{O}_\partial$.

2.3 Idempotent copulas

For a sub-σ-algebra $\mathscr{S}$ of $\mathscr{B}$, a set $S \in \mathscr{S}$ is called an *atom* in $\mathscr{S}$ if it has positive measure and there is no subset $E \in \mathscr{S}$ of S having measure strictly between 0 and $\lambda(S)$. $\mathscr{S}$ is said to be *nonatomic* if it has no atom; otherwise, it is called *atomic*. If $\mathscr{S}$ has a (countable) family of essentially disjoint atoms $B_1, B_2, \ldots$ in $\mathscr{S}$ for which the sum of their measures is 1, then it is called *totally atomic*. If $\mathscr{S}$ contains only atoms with measure 1, then it is called

1-atomic. For subclasses $\mathscr{R}$ and $\mathscr{S}$ of $\mathscr{B}$, $\mathscr{S}$ is said to be *essentially equivalent to* $\mathscr{R}$ if, for every $S \in \mathscr{S}$, there exists $R \in \mathscr{R}$ such that $\lambda(S \triangle R) = 0$.[1] Sub-$\sigma$-algebras $\mathscr{R}$ and $\mathscr{S}$ of $\mathscr{B}$ are called *essentially equivalent* and written as $\mathscr{R} \approx \mathscr{S}$ if $\mathscr{S}$ is essentially equivalent to $\mathscr{R}$ and $\mathscr{R}$ is essentially equivalent to $\mathscr{S}$.

Recall that an *idempotent copula* is a copula C such that $C^2 = C$. It was shown in the studies by Darsow and Olsen [4] and Trutschnig [21] that every idempotent copula is symmetric, which yields that $T_C\,|_{L^2}$ is self-adjoint. So the Markov operator on L^2 of an idempotent copula C is a projection. Clearly, a projection Markov operator is the Markov operator of an idempotent copula. Darsow and Olsen [4] gave a characterization of idempotent copulas C by splitting them into cases according to the size of the σ-algebra γ_C of *invariant sets* under C. Recall that a Borel set S is said to be in γ_C if $T_C \mathbf{1}_S = \mathbf{1}_S$. An idempotent copula C is said to be *nonatomic*/*atomic*/*totally atomic* if γ_C is nonatomic/atomic/totally atomic, respectively.

Let $\mathcal{P} = \{(a_k, b_k)\}_{k\in\Lambda}$ be a partition[2] of $\mathbf{I}$ and $\{C_k\}_{k\in\Lambda}$ a collection of copulas, where Λ is either $\mathbb{N}$ or $\{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. Define $\oplus_{\mathcal{P}} C_k$, for $x, y \in \mathbf{I}$ by,

$$\underset{\mathcal{P}}{\oplus} C_k(x,y) := \begin{cases} a_k + (b_k - a_k) C_k\left(\dfrac{x - a_k}{b_k - a_k}, \dfrac{y - a_k}{b_k - a_k}\right) & \text{for } x, y \in (a_k, b_k], \\ M(x,y), & \text{otherwise.} \end{cases}$$

Then, $\oplus_{\mathcal{P}} C_k$ is a copula called the *ordinal sum* of $\{C_k\}$ with respect to the partition $\mathcal{P}$ or the ordinal sum on the partition $\mathcal{P}$ with components C_k. It is straightforward to verify that $(\oplus_{\mathcal{P}} C_k)^t = \oplus_{\mathcal{P}} C_k^t$, $(\oplus_{\mathcal{P}} C_k)(\oplus_{\mathcal{P}} D_k) = \oplus_{\mathcal{P}}(C_k D_k)$, and $\oplus_{\mathcal{P}} C_k = \oplus_{\mathcal{P}} D_k$ if and only if $C_k = D_k$ for all k. Ergo, $\oplus_{\mathcal{P}} C_k$ is idempotent if and only if C_k is idempotent for all k. By [4, Lemma 4.1], a Borel subset Q of (a_k, b_k) is invariant under $\oplus_{\mathcal{P}} C_k$ if and only if $\frac{Q - a_k}{b_k - a_k}$ is invariant under C_k. A partition $\{(a_k, b_k)\}_{k\in\Lambda}$ of $\mathbf{I}$ is called a *special partition* and written as $\{(a_k, a_{k+1})\}_{k\in\Lambda}$ if $b_k = a_{k+1}$ for all $k \in \Lambda$, where $a_{m+1} := b_m$ if $|\Lambda| = m$. A *Borel partition* of $\mathbf{I}$ is a collection $\{S_k\}_{k\in\Lambda}$ consisting of essentially pairwise disjoint Borel sets such that $\lambda(S_k) > 0$ for all k and $\sum_{k\in\Lambda} \lambda(S_k) = 1$. Let us quote theorems from the study by Darsow and Olsen [4], some of which have been rewritten to incorporate necessary details from proofs.

Theorem 2.3. *Let E be an idempotent copula.*
(1) *If E is 1-atomic, then $E = \Pi$.*
(2) *If E is nonatomic, then $E = C_{e,f} C_{f,e}$, where $f \in \mathcal{F}$ such that $\gamma_E \approx f^{-1}(\mathscr{B})$.*

Note that the converses of both statements also hold (*see* [4,19]).

Theorem 2.4. *Let $\mathcal{P} = \{(a_k, a_{k+1})\}$ be a special partition of $\mathbf{I}$ and E a copula.*
(1) *If $E = \oplus_{\mathcal{P}} \Pi$, then E is totally atomic idempotent and $\gamma_E \approx \sigma(\mathcal{P})$.*
(2) *Conversely, if $\mathcal{P}$ is essentially equivalent to γ_E, then*
   (*a*) *E is an ordinal sum with respect to the partition $\mathcal{P}$;*
   (*b*) *if, in addition, $\gamma_E \approx \sigma(\mathcal{P})$, then $E = \oplus_{\mathcal{P}} E_k$ where each component, E_k is 1-atomic copula;*
   (*c*) *if, in addition to the assumption in* (*b*), *E is idempotent, then each E_k is the copula Π.*

Proposition 2.5. *Let $\{S_k\}_{k\in\Lambda}$ be a Borel partition of $\mathbf{I}$. Then, there exists $p \in \mathcal{F}_{\text{inv}}$ and a special partition $\mathcal{P} = \{(a_k, a_{k+1})\}_{k\in\Lambda}$ of $\mathbf{I}$ such that $p^{-1}(S_k) \approx [a_k, a_{k+1})$ for all $k \in \Lambda$. Such a p is called a rearrangement of $\{S_k\}$* (*into $\mathcal{P}$*).

Theorem 2.6. *A copula E is a totally atomic idempotent if and only if $E = C_{p,e}(\oplus_{\mathcal{P}} \Pi) C_{e,p}$ for some $p \in \mathcal{F}_{\text{inv}}$ and special partition $\mathcal{P}$ of $\mathbf{I}$ such that p is a rearrangement of a maximal set of essentially pairwise disjoint atoms $\{S_k\}_{k\in\Lambda}$ in γ_E into $\mathcal{P}$.*

1 Of course, $A \triangle B$ denotes the symmetric difference $(A \backslash B) \cup (B \backslash A)$.
2 $\{(a_k, b_k)\}_{k\in\Lambda}$ is a partition of $\mathbf{I}$ if (a_k, b_k)'s are disjoint and $\sum_{k\in\Lambda}(b_k - a_k) = 1$

Theorem 2.7. *A copula E is an idempotent that is atomic but not totally atomic if and only if $E = C_{p,e}(\oplus_{\mathcal{P}} F_k)C_{e,p}$ for some $p \in \mathcal{F}_{\text{inv}}$, special partition $\mathcal{P}$ of $\mathbf{I}$, and nonatomic idempotent F_1 such that p is a rearrangement of $\{S_k\}_{k\in\Lambda}$ into $\mathcal{P}$, where $F_k = \Pi$ for all $k \geq 2$, $\{S_k\}_{k\geq 2}$ is a maximal set of essentially pairwise disjoint atoms in γ_E, and $S_1 \coloneqq \mathbf{I}\backslash\cup_{k\geq 2} S_k$. Note that $F_1 = C_{e,h}C_{h,e}$ for some $h \in \mathcal{F}$.*

By Corollaries 3.4.1, 4.4.1, and 5.11 in the study by Darsow and Olsen [4], we have

Corollary 2.8. *There is a unique idempotent copula C whose the class γ_C of invariant sets is essentially equivalent to a given sub-σ algebra $\mathscr{S}$ of $\mathscr{B}$.*

2.4 Associated σ-algebras and atomicity of copulas

Following [19], we give some background of associated σ-algebras and various types of the atomicity of copulas. The *associated σ-algebras* of a copula C or its Markov operator $T = T_C$ is defined as follows:

$$\sigma_C = \sigma_T \coloneqq \{S \in \mathscr{B} : T\mathbf{1}_S = \mathbf{1}_R \quad \text{for some } R \in \mathscr{B}\} \text{ and } \sigma_C^* = \sigma_T^* \coloneqq \{R \in \mathscr{B} : T\mathbf{1}_S = \mathbf{1}_R \quad \text{for some } S \in \mathscr{B}\}.$$

Evidently, if C is idempotent, then $\sigma_C = \sigma_C^* = \gamma_C$. Generally, $\sigma_T^* = \sigma_{T^*}$, and hence $\sigma_C^* = \sigma_{C^t}$. In fact, if $T\mathbf{1}_S = \mathbf{1}_R$ then $T^*\mathbf{1}_R = \mathbf{1}_S$. Consequently, σ_C is nonatomic if and only if σ_C^* is nonatomic. This is also valid for all types of atomicity, i.e., σ_C is nonatomic/1-atomic/atomic/totally atomic if and only if σ_C^* is nonatomic/1-atomic/atomic/totally atomic, respectively. Hence, the atomicity of a copula or its Markov operator is defined corresponding to that of its associated σ-algebras.

Since the associated σ-algebras of a Markov operator are defined via indicator functions which are in L^∞, they all coincide no matter which L^p the Markov operator is on, i.e. $\sigma_T = \sigma_{T|_{L^p}}$.

Analogous to the invariant sets, σ_{A_k}'s scale linearly according to $\mathcal{P}$ under taking ordinal sum.

Lemma 2.9. *If $A = \oplus_{\mathcal{P}} A_k$ where $\mathcal{P} = \{(a_k, b_k)\}$, then for all Borel sets $Q, R \subseteq (a_k, b_k)$*

(1) *Q is in σ_A if and only if $\frac{Q - a_k}{b_k - a_k}$ is in σ_{A_k}; and*

(2) *R is in σ_A^* if and only if $\frac{R - a_k}{b_k - a_k}$ is in $\sigma_{A_k}^*$.*

2.5 Nonatomic copulas

We gather some useful results related to nonatomic copulas below. See [4,16,19].

Proposition 2.10. *Let $f, g \in \mathcal{F}$, $C \in \mathcal{C}$, and $\mathscr{S}$ a nonatomic sub-σ-algebra of $\mathscr{B}$.*

(i) $C_{f,f} = M$ and $C_{f,g} = C_{f,e}C_{e,g}$.

(ii) $C_{f,e}C_{g,e} = C_{f\circ g,e}$ and $C_{e,g}C_{e,f} = C_{e,f\circ g}$.

(iii) $\sigma_{C_{e,f}}^* = \sigma_{C_{f,e}} \approx f^{-1}(\mathscr{B})$.

(iv) There exists $h \in \mathcal{F}$ such that $h^{-1}(\mathscr{B}) \approx \mathscr{S}$.

(v) If C is idempotent and $\sigma_C \approx f^{-1}(\mathscr{B})$, then $C = C_{e,f}C_{f,e}$.

(vi) $\sigma_{C_{e,f}C_{g,e}} = \sigma_{C_{g,e}}$ and $\sigma_{C_{e,f}C_{g,e}}^* = \sigma_{C_{e,f}}^*$.

(vii) $\sigma_{C_{f,e}C} = \sigma_C$ and $\sigma_{CC_{e,f}}^* = \sigma_C^*$ provided that $f \in \mathcal{F}_{\text{inv}}$.

(viii) $\sigma_{CC_{e,f}} \approx f(\sigma_C)$ and $\sigma_{C_{f,e}C}^* \approx f(\sigma_C^*)$ provided that $f \in \mathcal{F}_{\text{inv}}$.

(ix) $C_{e,f}C_{f,e} = C_{e,g}C_{g,e}$ if and only if there is $h \in \mathcal{F}_{\text{inv}}$ for which $f = h \circ g$ almost everywhere.

(x) If $f^{-1}(\mathscr{B}) \approx g^{-1}(\mathscr{B})$ then there exists $h \in \mathcal{F}_{\text{inv}}$ such that $f = h \circ g$ almost everywhere.

Given $p \in [1, \infty]$ and a sub-σ-algebra $\mathscr{S}$ of $\mathscr{B}$, we define $L^p(\mathscr{S})$ to be a subclass of L^p consisting only of $\mathscr{S}$-measurable functions.

Theorem 2.11. *For every nonatomic copula C, there exist h and g in $\mathcal{F}$ such that $\sigma_C^* \approx h^{-1}(\mathscr{B})$, $\sigma_C \approx g^{-1}(\mathscr{B})$, and $T_C = T_{e,h}T_{g,e}$ on $L^1(g^{-1}(\mathscr{B}))$.*

A nonatomic copula C is said to be *factorizable* if there exist f and g in $\mathcal{F}$ such that $C = C_{e,f}C_{g,e}$. Given a nonatomic copula C, by Proposition 2.10 (iv), there exist f and g in $\mathcal{F}$ such that $\sigma_C \approx g^{-1}(\mathscr{B})$ and $\sigma_C^* \approx f^{-1}(\mathscr{B})$. The product $C' \coloneqq C_{e,f}C_{f,e}CC_{e,g}C_{g,e}$, which is independent of the choices of f and g, is called the *isoalgebra* copula of C. In fact, C' is factorizable and shares the same set of associated σ-algebras as that of C. The following theorem presents C' as a tool in characterizing factorizable copulas. See [16] for more details.

Theorem 2.12. *A nonatomic copula C is factorizable if and only if C coincides with its isoalgebra copula, i.e. $C = C'$.*

3 Characterization of pre-idempotent copulas

Definition 3.1. Let C be a copula. The *transpose-product* of C is defined as C^tC, and C is *pre-idempotent* if its transpose-product is idempotent, i.e., $C^tCC^tC = C^tC$. Denote $\mathcal{R} \coloneqq \{C \in \mathcal{C} : C^tC \text{ is idempotent}\}$.

Clearly, $\mathcal{R}$ contains all idempotent copulas as well as all factorizable copulas. In fact, if $C = LR$, where L and R^t are left invertible, then $C^tC = (LR)^t(LR) = R^tR$, which is idempotent. By Propositions 2.2 and 2.1,

$$C \in \mathcal{R} \iff C^t \in \mathcal{R} \iff C = CC^tC. \tag{3.1}$$

Note that there are factorizable copulas, which are not idempotent, such as W, the copula A in Example 3.10, and all asymmetric factorizable copulas.

The following lemma shows that the atomicity of a pre-idempotent copula can be verified via that of its transpose-product.

Lemma 3.2. *Let $C \in \mathcal{R}$. Then, $\sigma_C = \sigma_{C^tC}$ and $\sigma_C^* = \sigma_{CC^t}$.*

Proof. Let $C \in \mathcal{R}$ with corresponding L^2-Markov operator T_C. We will show only that $\sigma_C = \sigma_{C^tC}$ because the other identity can be obtained by substituting C^t for C. Let $R \in \sigma_C$ and $S \in \sigma_C^*$ such that $T_C\mathbf{1}_R = \mathbf{1}_S$. So $T_C^*\mathbf{1}_S = \mathbf{1}_R$ and $T_{C^tC}\mathbf{1}_R = T_C^*(T_C\mathbf{1}_R) = T_C^*\mathbf{1}_S = \mathbf{1}_R$. Hence, $R \in \sigma_{C^tC}$. Conversely, let $K \in \sigma_{C^tC}$. Since C^tC is idempotent, $\mathbf{1}_K = T_{C^tC}\mathbf{1}_K = T_C^*(T_C\mathbf{1}_K) = T_C^*g$, where $g \coloneqq T_C\mathbf{1}_K \in L^2$. By properties of Markov operators, $\int_{\mathbf{I}} g\,\mathrm{d}\lambda = \int_{\mathbf{I}} \mathbf{1}_K\,\mathrm{d}\lambda = \lambda(K)$ and $0 \le g \le 1$ a.e. So $0 \le g^2 \le g \le 1$ a.e. Then $\lambda(K) = \langle \mathbf{1}_K, \mathbf{1}_K\rangle = \langle T_C^*g, \mathbf{1}_K\rangle = \langle g, T_C\mathbf{1}_K\rangle = \langle g, g\rangle = \int_{\mathbf{I}} g^2\,\mathrm{d}\lambda \le \int_{\mathbf{I}} g\,\mathrm{d}\lambda = \lambda(K)$, which implies that $g = g^2$ a.e. Hence, $g = \mathbf{1}_E$ for some Borel set E. That is, $K \in \sigma_C$. □

Let us observe here that if $r \in \mathcal{F}_{\text{inv}}$, then $C_{e,r}C_{r,e} = M$ and

$$(C_{r,e}(\oplus_{\mathcal{P}}F_k)C_{e,p})^t(C_{r,e}(\oplus_{\mathcal{P}}F_k)C_{e,p}) = C_{p,e}(\oplus_{\mathcal{P}}F_k^tF_k)C_{e,p}. \tag{3.2}$$

Proposition 3.3. *Let $\mathcal{P} = \{(a_k, a_{k+1})\}_{k\in\Lambda}$ be a special partition of $\mathbf{I}$ and $p, r \in \mathcal{F}_{\text{inv}}$. The following statements hold for $C = C_{r,e}(\oplus_{\mathcal{P}}D_k)C_{e,p}$, where D_k, s are copulas.*

(1) *If $D_k = \Pi$ for all k, then C is a totally atomic pre-idempotent copula.*

(2) *If D_1 is factorizable and $D_k = \Pi$ for $k \ge 2$, then C is a pre-idempotent copula that is atomic but not totally atomic.*

Proof. From equation (3.2), $C^tC = C_{p,e}(\oplus_{\mathcal{P}} D_k^t D_k)C_{e,p}$. If each D_k is either factorizable or Π, then each $D_k^t D_k$, and hence C^tC is idempotent, which implies that C is pre-idempotent. The desired atomicity of C in cases 1 and 2 follows from Lemma 3.2 and Theorems 2.4 and 2.7. □

Lemma 3.4. Π *is the only 1-atomic pre-idempotent copula.*

Proof. Let C be a 1-atomic pre-idempotent. By Lemma 3.2, C^tC and CC^t are 1-atomic idempotent. So $C^tC = CC^t = \Pi$ by Theorem 2.3 (1). To show $C = \Pi$, it suffices to show that for each $u \in \mathbf{I}$, $\partial_1 C(t,u) = u$ a.e. $t \in \mathbf{I}$. Let $u \in \mathbf{I}$. The derivations

$$u^2 = \Pi(u,u) = C^tC(u,u) = \int_0^1 \partial_2 C^t(u,t)\partial_1 C(t,u)\mathrm{d}t = \int_0^1 (\partial_1 C(t,u))^2 \mathrm{d}t \quad \text{and} \tag{3.3}$$

$$\int_0^1 (\partial_1 C(t,u) - u)^2 \mathrm{d}t = \int_0^1 (\partial_1 C(t,u))^2 \mathrm{d}t - \int_0^1 2u\partial_1 C(t,u)\mathrm{d}t + \int_0^1 u^2 \mathrm{d}t = \int_0^1 (\partial_1 C(t,u))^2 \mathrm{d}t - u^2,$$

where the second equality follows from the fact that $\int_0^1 \partial_1 C(t,u)\mathrm{d}t = C(1,u) - C(0,u) = u$, yields $\int_0^1 (\partial_1 C(t,u) - u)^2 \mathrm{d}t = 0$, and the claim follows. □

As mentioned above, every factorizable copula is a nonatomic pre-idempotent. The converse also holds.

Theorem 3.5. *Every nonatomic pre-idempotent copula is factorizable.*

Proof. Let C be a nonatomic pre-idempotent. In light of Theorem 2.11 and Lemma 3.2, there exist $f, g \in \mathcal{F}$ such that $T_C = T_{e,f}T_{g,e}$ on $L^1(g^{-1}(\mathscr{B}))$, $f^{-1}(\mathscr{B}) \approx \sigma_C^* = \sigma_{CC^t}$, and $g^{-1}(\mathscr{B}) \approx \sigma_C = \sigma_{C^tC}$. By Theorem 2.3(2), the transpose-products of C and C^t can be written as $C^tC = C_{e,g}C_{g,e}$ and $CC^t = C_{e,f}C_{f,e}$, respectively. By applying equation (3.1) twice, the isoalgebra copula of C equals $C' := C_{e,f}C_{f,e}CC_{e,g}C_{g,e} = CC^tCC^tC = CC^tC = C$. By Theorem 2.12, C is factorizable. □

Corollary 3.6. *If A is a nonatomic pre-idempotent copula with $\sigma_A \approx g^{-1}(\mathscr{B})$ and $\sigma_A^* \approx f^{-1}(\mathscr{B})$ for some $f, g \in \mathcal{F}$, then there exists an invertible copula S such that $A = C_{e,f}SC_{g,e}$.*

Proof. Since the σ-algebras of idempotents A^tA and AA^t are $\sigma_{A^tA} = \sigma_A \approx g^{-1}(\mathscr{B})$ and $\sigma_{AA^t} = \sigma_A^* \approx f^{-1}(\mathscr{B})$, Theorem 2.3(2) gives $A^tA = C_{e,g}C_{g,e}$ and $AA^t = C_{e,f}C_{f,e}$. By Theorem 3.5, A is factorizable: $A = C_{e,h}C_{k,e}$ for some $h, k \in \mathcal{F}$. Then, $C_{e,k}C_{k,e} = A^tA = C_{e,g}C_{g,e}$ and $C_{e,h}C_{h,e} = AA^t = C_{e,f}C_{f,e}$. So, by Proposition 2.10(ix), there exist $r, p \in \mathcal{F}_{\mathrm{inv}}$ such that $k = r \circ g$ and $h = p \circ f$. Thus, by Proposition 2.10(ii), $A = C_{e,p\circ f}C_{r\circ g,e} = C_{e,f}SC_{g,e}$, where $S := C_{e,p}C_{r,e}$ is invertible. □

The following characterization of factorizable Markov operators is an immediate consequence of Theorem 3.5 and Proposition 2.2.

Corollary 3.7. *For every nonatomic Markov operator T on L^2, T is a partial isometry if and only if T is factorizable.*

Theorem 3.8. *Let $C \in \mathcal{R}$ be atomic, then there exist $p, r \in \mathcal{F}_{\mathrm{inv}}$ and a special partition $\mathcal{P} = \{(a_k, a_{k+1})\}_{k\in\Lambda}$ such that $C = C_{r,e}(\oplus_{\mathcal{P}} D_k)C_{e,p}$, where $D_k = \Pi$ for all $k \geq 2$ (if any). Furthermore, $D_1 = \Pi$ if C is totally atomic; otherwise, D_1 is a factorizable copula.*

Proof. By assumption, C^tC is atomic idempotent with invariant sets $\gamma_{C^tC} = \sigma_C$ and $\gamma_{CC^t} = \sigma_C^*$. Let us first consider the case that C^tC is not totally atomic. Let $\{S_k\}_{k\geq 2}$ be a maximal collection of essentially pairwise disjoint atoms in σ_C, $S_1 := \mathbf{I} \setminus \cup_{k\geq 2} S_k$, and R_k's Borel sets in σ_C^* such that $T_C \mathbf{1}_{S_k} = \mathbf{1}_{R_k}$. Consequently, $\{R_k\}_{k\geq 2}$ is a maximal collection of essentially pairwise disjoint atoms in σ_C^*. By Proposition 2.5 and Theorem 2.7, there exist a special partition $\mathcal{P} = \{(a_k, a_{k+1})\}$ of $\mathbf{I}$, a nonatomic idempotent F_1, and $p, r \in \mathcal{F}_{\text{inv}}$ such that $p^{-1}(S_k) \approx [a_k, a_{k+1}) \approx r^{-1}(R_k)$ for all k and $C^tC = C_{p,e}(\oplus_{\mathcal{P}} F_k)C_{e,p}$, where $F_k = \Pi$ for $k \geq 2$. Put $D \coloneqq C_{e,r}CC_{p,e}$ so that

$$D^tD = C_{e,p}C^tCC_{p,e} = \bigoplus_{\mathcal{P}} F_k, \tag{3.4}$$

which is idempotent. For each $k \in \Lambda$, $T_D\mathbf{1}_{[a_k,a_{k+1})} = T_{e,r}T_CT_{p,e}\mathbf{1}_{p^{-1}(S_k)} = T_{e,r}T_C\mathbf{1}_{S_k} = T_{e,r}\mathbf{1}_{R_k} = \mathbf{1}_{r^{-1}(R_k)} = \mathbf{1}_{[a_k,a_{k+1})}$. By Theorem 2.4, D is an ordinal sum with respect to the partition $\mathcal{P}$, i.e., $D = \oplus_{\mathcal{P}} D_k$. So, by equation (3.4), $\oplus_{\mathcal{P}} D_k^tD_k = D^tD = \oplus_{\mathcal{P}} F_k$. Hence, $D_1^tD_1 = F_1$ and $D_k^tD_k = \Pi$ for all $k \geq 2$. For $k \geq 2$, Lemma 3.4 implies that $D_k = \Pi$. By Theorem 3.5, the nonatomic pre-idempotent D_1 is factorizable. Thus, $C = C_{r,e}DC_{e,p} = C_{r,e}(\oplus_{\mathcal{P}} D_k)C_{e,p}$, where $D_k = \Pi$ for all $k \geq 2$ and D_1 is factorizable.

The case that C is totally atomic can be proved in a similar manner as the previous case with minimal adjustments. In this case, $\{S_k\}$ and $\{R_k\}$ are maximal collections of essentially pairwise disjoint atoms in σ_C and σ_C^*, respectively. Theorem 2.6 is applied instead of Theorem 2.7, thus the copulas F_1 and D_1 in this case become the copula Π. □

For conciseness, only in the following theorem, empty intervals will be permissible in the partition $\mathcal{P}$, i.e., the components D_k in $\oplus_{\mathcal{P}} D_k$ corresponding to empty intervals vanish. This allows us to unify Lemma 3.4 and Theorems 3.5 and 3.8 and conclude our characterization of pre-idempotent copulas.

Theorem 3.9. *For $C \in \mathcal{R}$, there exist invertible measure-preserving functions p and r and a special partition $\mathcal{P} = \{I_k\}_{k\in\mathbb{N}}$ of $\mathbf{I}$ such that $C = C_{r,e}(\oplus_{\mathcal{P}} D_k)C_{e,p}$, where the component D_1 is factorizable, and each component of D_k, $k \geq 2$, is the copula Π.*

If C is nonatomic, then $I_1 = (0,1)$ and $I_k = \varnothing$ for all $k \geq 2$, and the measure-preserving functions r and p can be chosen to be the identity function, hence it is in fact the factorizable copula.

If C is totally atomic, then $I_1 = \varnothing$. That is, the ordinal sum $\oplus_{\mathcal{P}} D_k$ contains only the copula Π as its components. In particular, if C is 1-atomic, then $I_2 = (0,1)$ and all other I_k's are empty, thus $C = \Pi$.

The following example shows that a pre-idempotent with given associated σ-algebras is not necessarily unique.

Example 3.10.

(1) Consider a sub-σ-algebra $\Lambda^{-1}(\mathcal{B})$, where $\Lambda(x) = 2\min\{x, 1-x\}$. Recall [4,19] that $D \coloneqq C_{e,\Lambda}C_{\Lambda,e}$ is a nonatomic idempotent copula for which $\sigma_D = \sigma_D^* \approx \Lambda^{-1}(\mathcal{B})$. Let $A \coloneqq C_{e,\Lambda}C_{e,w}C_{\Lambda,e}$, where $w \in \mathcal{F}_{\text{inv}}$ is defined as $w(x) = 1-x$. Since w is invertible, it follows from Proposition 2.10 (iii, vi, and vii) that $\sigma_A = \sigma_A^* \approx \Lambda^{-1}(\mathcal{B})$. The supports of factorizable copulas D and A are shown in Figure 1.

(2) Let $a_k \coloneqq \frac{k-1}{3}$ and $\mathcal{P} \coloneqq \{(a_k, a_{k+1}), k = 1, 2, 3\}$ be a partition of $\mathbf{I}$. Let $F \coloneqq \oplus_{\mathcal{P}} F_k$ be an ordinal sum on the partition $\mathcal{P}$ with components $F_k \coloneqq \Pi$ for all $k = 1, 2, 3$. Then, F is a totally atomic idempotent copula, hence pre-idempotent, with $\sigma_F = \sigma_F^* \approx \sigma(\mathcal{P})$. Let $\ell \in \mathcal{F}_{\text{inv}}$ map x to $x + 1/3$ on $[1/3, 2/3)$, x to $x - 1/3$ on $[2/3, 1]$,

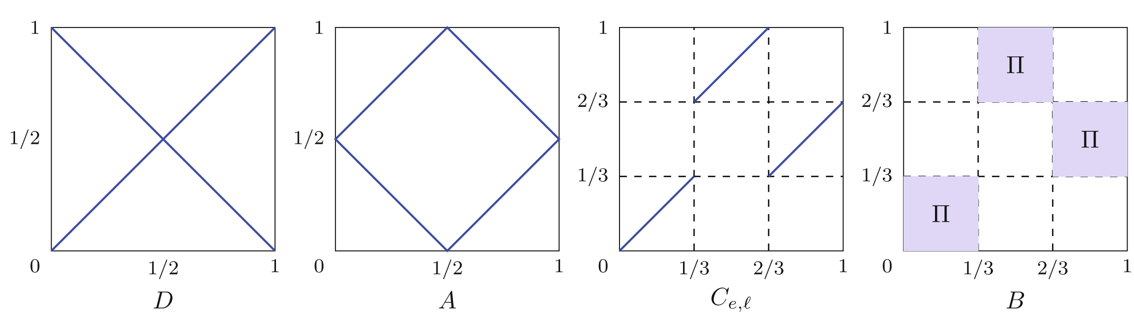

Figure 1: Supports of copulas D, A, $C_{e,\ell}$, and B.

and x to itself elsewhere. Then, $C_{e,\ell}$ is an invertible copula. By direct calculation or using results on shuffles of Min (See [17]), $B \coloneqq C_{e,\ell}F$ is a copula whose support is shown in Figure 1. Consider $B^tB = (F^tC^t_{e,\ell})(C_{e,\ell}F) = F^tF = F$. Thus, $B \in \mathcal{R}$. By Proposition 2.10 (vii and viii), $\sigma_B = \sigma_{C_{e,\ell}F} = \sigma_F$ and $\sigma^*_B = \sigma^*_{C_{e,\ell}F} \approx \ell^{-1}(\sigma^*_F) \approx \sigma^*_F$. But $B \neq F$.

Proposition 3.11. *Let $A, C \in \mathcal{R}$ be such that $\sigma_A = \sigma_C$ and $\sigma^*_A = \sigma^*_C$. Then, there exist $C_1, C_2 \in \mathcal{R}$ such that $\sigma_{C_1} = \sigma^*_{C_1} = \sigma^*_C$, $\sigma^*_{C_2} = \sigma_{C_2} = \sigma_C$, $A = C_1C$, and $A = CC_2$.*

Proof. The case of 1-atomic is clear since A and C are exactly the copula Π by Lemma 3.4.

If A and C are nonatomic sharing the same set of associated σ-algebras, then, by Theorem 3.5, both A and C are factorizable. Let $f, g \in \mathcal{F}$ be such that $C = C_{e,f}C_{g,e}$. Thus, $\sigma_A = \sigma_C \approx g^{-1}(\mathscr{B})$ and $\sigma^*_A = \sigma^*_C \approx f^{-1}(\mathscr{B})$. Corollary 3.6 yields $h \in \mathcal{F}_{\text{inv}}$ such that $A = C_{e,f}C_{e,h}C_{g,e}$. Then, $C_{e,f}C_{e,h}C_{g,e} = (C_{e,f}C_{e,h}C_{f,e})(C_{e,f}C_{g,e}) = C_1C$ and $C_{e,f}C_{e,h}C_{g,e} = (C_{e,f}C_{g,e})(C_{e,g}C_{e,h}C_{g,e}) = CC_2$, where $C_1 \coloneqq C_{e,f}C_{e,h}C_{f,e}$ and $C_2 \coloneqq C_{e,g}C_{e,h}C_{g,e}$ are clearly both in $\mathcal{R}$. By Proposition 2.10 (ii and vi), $\sigma_{C_1} = \sigma^*_{C_1} = \sigma^*_C$ and $\sigma^*_{C_2} = \sigma_{C_2} = \sigma_C$.

For the atomic case, we first suppose C is atomic but not totally atomic. Let $\{S_k\}_{k\geq 2}$ and $\{R_k\}_{k\geq 2}$ be maximal collections of essentially disjoint atoms in σ_C and σ^*_C, respectively, such that $T_C\mathbf{1}_{S_k} = \mathbf{1}_{R_k}$ for all k. Then, $S_1 \coloneqq \mathbf{I}\setminus \cup_{k\geq 2} S_k$ and $R_1 \coloneqq \mathbf{I}\setminus \cup_{k\geq 2} R_k$ have positive and equal measure. Since $\sigma_A = \sigma_C$ and $\sigma^*_A = \sigma^*_C$, there exists a permutation π on the indices of $\{S_k\}_{k\geq 2}$ such that $T_A\mathbf{1}_{S_k} = \mathbf{1}_{R_{\pi(k)}}$. By Theorem 3.8 and its proof, there exist $p_1, p_2, r_1, r_2 \in \mathcal{F}_{\text{inv}}$ and a special partition $\mathcal{P} = \{(a_k, a_{k+1})\}$ such that $p_1^{-1}(S_k) \approx p_2^{-1}\big(S_{\pi^{-1}(k)}\big) \approx [a_k, a_{k+1}) \approx r_2^{-1}(R_{\pi(k)}) \approx r_1^{-1}(R_k)$, and

$$C = C_{r_1,e}(\oplus_{\mathcal{P}}D_k)C_{e,p_1} \quad \text{and} \quad C_{r_2,e}(\oplus_{\mathcal{P}}E_k)C_{e,p_1} = A = C_{r_1,e}(\oplus_{\mathcal{P}}F_k)C_{e,p_2}, \tag{3.5}$$

where $D_k = E_k = F_k = \Pi$ for all $k \geq 2$, and D_1, E_1, and F_1 are factorizable copulas with $\sigma_{D_1} = \sigma_{E_1} = \sigma_{F_1}$ and $\sigma^*_{D_1} = \sigma^*_{E_1} = \sigma^*_{F_1}$. From the nonatomic case, there exist factorizable copulas H_1 and K_1 such that

$$E_1 = H_1D_1, \quad F_1 = D_1K_1, \quad \sigma_{H_1} = \sigma^*_{H_1} = \sigma^*_{D_1}, \quad \text{and} \quad \sigma^*_{K_1} = \sigma_{K_1} = \sigma_{D_1}. \tag{3.6}$$

Setting $H_k = K_k = \Pi$ for all $k \geq 2$, then $(\oplus_{\mathcal{P}}H_k)(\oplus_{\mathcal{P}}D_k) = \oplus_{\mathcal{P}}H_kD_k = \oplus_{\mathcal{P}}E_k$ and $(\oplus_{\mathcal{P}}D_k)(\oplus_{\mathcal{P}}K_k) = \oplus_{\mathcal{P}}D_kK_k = \oplus_{\mathcal{P}}F_k$, which, together with equation (3.5), gives

$$\begin{aligned} A &= C_{r_2,e}(\oplus_{\mathcal{P}}E_k)C_{e,p_1} = C_{r_2,e}(\oplus_{\mathcal{P}}H_k)(\oplus_{\mathcal{P}}D_k)C_{e,p_1} = C_{r_2,e}(\oplus_{\mathcal{P}}H_k)C_{e,r_1}C = C_1C \quad \text{and} \\ A &= C_{r_1,e}(\oplus_{\mathcal{P}}F_k)C_{e,p_2} = C_{r_1,e}(\oplus_{\mathcal{P}}D_k)(\oplus_{\mathcal{P}}K_k)C_{e,p_2} = CC_{p_1,e}(\oplus_{\mathcal{P}}K_k)C_{e,p_2} = CC_2, \end{aligned}$$

where $C_1 = C_{r_2,e}(\oplus_{\mathcal{P}}H_k)C_{e,r_1}$ and $C_2 = C_{p_1,e}(\oplus_{\mathcal{P}}K_k)C_{e,p_2}$. Note that equation (3.6) and Lemma 2.9 imply that $\sigma_{\oplus_{\mathcal{P}}H_k} = \sigma^*_{\oplus_{\mathcal{P}}H_k} = \sigma^*_{\oplus_{\mathcal{P}}D_k} = \sigma^*_{\oplus_{\mathcal{P}}E_k}$ and $\sigma^*_{\oplus_{\mathcal{P}}K_k} = \sigma_{\oplus_{\mathcal{P}}K_k} = \sigma_{\oplus_{\mathcal{P}}D_k} = \sigma_{\oplus_{\mathcal{P}}F_k}$. Consequently, by Proposition 2.10 (vii and viii),

$$\begin{aligned} \sigma_{C_1} &= \sigma_{(\oplus_{\mathcal{P}}H_k)C_{e,r_1}} = r_1(\sigma_{\oplus_{\mathcal{P}}H_k}) = r_1(\sigma^*_{\oplus_{\mathcal{P}}D_k}) = \sigma^*_{C_{r_1,e}(\oplus_{\mathcal{P}}D_k)} = \sigma^*_C \quad \text{and} \\ \sigma^*_{C_1} &= \sigma^*_{C_{r_2,e}(\oplus_{\mathcal{P}}H_k)} = r_2(\sigma^*_{\oplus_{\mathcal{P}}H_k}) = r_2(\sigma^*_{\oplus_{\mathcal{P}}E_k}) = \sigma^*_{C_{r_2,e}(\oplus_{\mathcal{P}}E_k)} = \sigma^*_A = \sigma^*_C. \end{aligned}$$

By a similar argument, $\sigma^*_{C_2} = p_1(\sigma_{\oplus_{\mathcal{P}}D_k}) = \sigma_C$ and $\sigma_{C_2} = p_2(\sigma_{\oplus_{\mathcal{P}}F_k}) = \sigma_A = \sigma_C$.

The case that C is totally atomic follows from a similar proof as in the previous case without the nonatomic part in the first component. That is, the sets $\{S_k\}$ and $\{R_k\}$ are maximal collections of essentially pairwise disjoint atoms in σ_C and σ^*_C, respectively. So all components of the ordinal sum $\oplus_{\mathcal{P}}D_k$, $\oplus_{\mathcal{P}}E_k$, and $\oplus_{\mathcal{P}}F_k$ are Π, and hence C_1 and C_2 are set to be $C_{r_2,e}(\oplus_{\mathcal{P}}D_k)C_{e,r_1}$ and $C_{p_1,e}(\oplus_{\mathcal{P}}D_k)C_{e,p_2}$, respectively. □

4 The closure of the pre-idempotent copulas

As the class of factorizable copulas, denoted by $\mathcal{C}_F$, contains all invertible copulas which form a dense subset of $\mathcal{C}$, both $\mathcal{C}_F$ and $\mathcal{R}$ are dense in $\mathcal{C}$ with respect to the uniform norm. So, in this section, we shall investigate the closure or denseness with respect to d_2 of the class $\mathcal{R}$ and the class $\mathcal{C}_F$. Recall that the Markov product is jointly continuous with respect to d_2.

Theorem 4.1. *$\mathcal{R}$ is closed in $(\mathcal{C}, d_2)$. That is, if $\{C_n\}_{n\geq 1}$ is a sequence in $\mathcal{R}$ converging to a copula C with respect to d_2, then $C \in \mathcal{R}$.*

Proof. By equation (3.1), $C_n = C_n C_n^t C_n$ for all n. Since d_2 is symmetric, we have that $C_n \xrightarrow{d_2} C$ and $C_n^t \xrightarrow{d_2} C^t$, which yield $C_n = C_n C_n^t C_n \xrightarrow{d_2} CC^tC$ by the joint continuity of the Markov product. Thus, $CC^tC = C$. Applying equation (3.1) again gives $C \in \mathcal{R}$. □

Let $\mathcal{C}_N$ denote the class of nonatomic copulas. By Theorem 3.5, $\mathcal{C}_N \cap \mathcal{R} = \mathcal{C}_F$, which is in fact dense in $\mathcal{R}$.

Theorem 4.2. *The closure of $\mathcal{C}_F$ with respect to d_2 is $\mathcal{R}$.*

Proof. From Theorem 4.1, $\overline{\mathcal{C}_F} \subseteq \overline{\mathcal{R}} = \mathcal{R}$. To show that $\mathcal{R} \subseteq \overline{\mathcal{C}_F}$, let $C \in \mathcal{R} \backslash \mathcal{C}_F$. We will find a sequence $\{C_m\}$ in $\mathcal{C}_F$ converging to C with respect to d_2. For each $m \in \mathbb{N}$, let A_m be a patched copula defined as follows:

$$A_m(u, v) \coloneqq \frac{1}{m} \sum_{i=0}^{m-1} M(Q_i(u), v)$$

for $(u, v) \in (\frac{i}{m}, \frac{i+1}{m}] \times [0, 1]$, where Q_i is the uniform distribution on $[\frac{i}{m}, \frac{i+1}{m}]$. A_m is the left invertible copula $C_{e,h}$, where $h(x) = mx \pmod 1$. Then, each $C_m \coloneqq A_m A_m^t$ is factorizable. Theorem 4.4 in the study by Chaidee et al. [1] implies that C_m converges to Π with respect to d_2. This proves the case that $C = \Pi$. Using Theorem 3.8, an atomic copula C in $\mathcal{R}$ can be written as $C = C_{r,e}(\oplus_{\mathcal{P}} D_k)C_{e,p}$, where $D_k = \Pi$ for all $k \geq 2$ and D_1 is either a factorizable copula or the copula Π. For $m \geq 1$, let $F_m \coloneqq C_{r,e}(\oplus_{\mathcal{P}} E_{k,m})C_{e,p}$, where $E_{k,m} = C_m$ for all $k \geq 2$, and $E_{1,m}$ is chosen to be either D_1 whenever D_1 is factorizable, or C_m whenever $D_1 = \Pi$. Then, $\{F_m\}_{m\geq 1}$ is a sequence of factorizable copulas. By direct calculation and using that $C_m \xrightarrow{d_2} \Pi$, we obtain that $\oplus_{\mathcal{P}} E_{k,m} \xrightarrow{d_2} \oplus_{\mathcal{P}} D_k$. Hence, $F_m \xrightarrow{d_2} C_{r,e}(\oplus_{\mathcal{P}} D_k)C_{e,p} = C$. □

Acknowledgements: We are grateful to the anonymous referees for all comments and suggestions, which substantially improved the article. The first author is supported by the Development and Promotion of Science and Technology Talents project scholarship.

Conflict of interest: All authors have no conflicts of interest to declare.

References

[1] Chaidee, N., Santiwipanont, T., & Sumetkijakan, S. (2016). Patched approximations and their convergence. *Communications in Statistics - Theory Methods*, *45*, 2654–2664.

[2] Darsow, W. F., Nguyen, B., & Olsen, E. T. (1992). Copulas and Markov processes. *Illinois Journal of Mathematics*, *36*, 600–642.

[3] Darsow, W. F., & Olsen, E. T. (1995). Norms for copulas. *International Journal of Mathematical Sciences*, *18*, 417–436.

[4] Darsow, W. F., & Olsen, E. T. (2010). Characterization of idempotent 2-copulas. *Note de Matematica*, *30*, 147–177.

[5] de Amo, E., DíazCarrillo, M., & Fernández-Sánchez, J. (2011). Measure-preserving functions and the independence copula. *Mediterranean Journal of Mathematics*, *8*, 431–450.

[6] Durante, F., & Sempi, C. (2015). *Principles of Copula Theory*. Boca Raton: CRC Press.

[7] Eisner, T., Farkas, B., Haase, M., & Nagel, R. (2015). *Operator Theoretic aspect of ergodic theory*. New York: Springer.

[8] Fernández-Polo, F. J., & Peralta, A. M. (2018). Partial isometry: A survey. *Advances in Operator Theory*, *1*, 75–116.

[9] Garcia, S. R., Martin, R. T. W., & Ross, W. T. (2016). Partial orders on partial isometries. *Journal Operator Theory*, *2*, 101–134.

[10] Mikusiński, P., & Taylor, M. D. (2009). Markov operators and n-copulas. *Annales Polonici Mathematici*, *96*, 75–79.

[11] Mikusiński, P., & Taylor, M. D. (2010). Some approximations of n-copulas. *Metrika*, *72*, 385–414.

[12] Nelsen, R. B. (2006). *Introduction to copulas*, (2nd ed.). New York: Springer-Verlag.

[13] Olsen, E. T., Darsow, W. F., & Nguyen, B. (1996). Copulas and Markov operators. In: L. Rüschendorf, B. Schweizer, & M. D. Taylor (Eds.), *Distributions with fixed marginals and related topics* (pp. 244–259). IMS Lecture Notes Monograph Series, 28. Hayward CA: Institute of Mathematical Statistics.

[14] Panyasakulwong, P., Santiwipanont, T., & Sumetkijakan, S. (2021). Generalized factorizability of some implicit dependence copulas. In: *Proceedings of the Annual Pure and Applied Mathematics Conference* (pp. 68–85). Thailand: Chulalongkorn University, .

[15] Panyasakulwong, P., Santiwipanont, T., Sumetkijakan, S., & Yanpaisan, N. (2022). Countably piecewise monotonic surjection implicit dependence copulas. *Fuzzy Sets and Systems*, *448*, 49–64.

[16] Printechapat, T., & Sumetkijakan, S. (2018). Factorizable non-atomic copulas. *Statistics and Probability Letters*, *143*, 86–94.

[17] Ruankong, P., Santiwipanont, T., & Sumetkijakan, S. (2013). Shuffles of copulas and a new measure of dependence. *Journal of Mathematical Analysis and Applications*, *398*, 392–402.

[18] Siburg, K. F., & Stoimenov, P. A. (2010). A measure of mutual complete dependence. *Metrika*, *71*, 239–251.

[19] Sumetkijakan, S. (2017). Non-atomic bivariate copulas and implicitly dependent random variables. *Inference*, *186*, 1–12.

[20] Trutschnig, W. (2011). On a strong metric on the space of copulas and its induced dependence measure. *Journal of Mathematical Analysis and Applications*, *384*, 690–705.

[21] Trutschnig, W. (2013). On Cesáro convergence of iterates of the star product of copulas. *Statistics & Probability Letters*, *83*, 357–365.

[22] Trutschnig, W. (2017). Complete dependence everywhere? In: M. ÚbedaFlores, E. de Amo Artero, F. Durante, & J. Fernández-Sánchez (Eds.), *Copulas and dependence models with applications* (pp. 225–240). Cham: Springer.

[23] Vitale, R. A. (1996). Parametrizing doubly stochastic measures. In: L. Rüschendorf, B. Schweizer, & M. D. Taylor (Eds.), *Distributions with fixed marginals and related topics* (pp. 358–366), IMS Lecture Notes Monograph Series, 28. Hayward CA: Institute of Mathematical Statistics.