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# Quadratic transformation of multivariate aggregation functions

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**Abstract:** In this work, we prove that quadratic transformations of aggregation functions must come from quadratic aggregation functions. We also show that this is different from quadratic transformations of (multivariate) semi-copulas and quasi-copulas. In the latter case, those two classes are actually the same and consists of convex combinations of the identity map and another fixed quadratic transformation. In other words, it is a convex set with two extreme points. This result is different from the bivariate case in which the two classes are different and both are convex with four extreme points.

**Keywords:** quadratic transformation, quadratic construction, semi-copula, quasi-copula, aggregation function

MSC: 62H86, 26E60

### 1 Introduction

We are currently living in the age of information. With a large amount of data, a representative of raw data is needed to get inside information. For numerical data, an aggregation function is usually used for such representatives. Examples of aggregation functions include average function, minimum, and maximum. Which of these functions are appropriate as a representative will depend on the situation. Also, these examples are not always appropriate, for instance, in the present of a very large or small outlier. Thus, other aggregation functions are also required. Over the years, several constructions of aggregation functions have been proposed in the literature including constructions of special types of aggregation functions such as semi-copulas and quasi-copulas [1–3, 5, 6, 8, 9].

We are interested in the method of transformations, that is, a method that transforms aggregation functions into other aggregation functions. Such methods have been studied by several others [4, 7, 8, 11, 12]. However, most of these studies are only applied to bivariate aggregation functions. It is therefore interesting to investigate transformations of multivariate aggregation functions. Specifically, we will investigate quadratic transformations in this work. Quadratic transformations of bivariate functions have been studied in [4, 7, 8]. Here, we show that such ideas can be extended to multivariate cases. The difference is that there are fewer quadratic transformations in this case. Also, the class of quadratic transformations of quasi-copulas coincide with the class of quadratic transformations of semi-copulas in this case.

In the next section, we will review related terminologies used throughout this work. We will also review quadratic transformations of bivariate functions done by Kolesarova et al [8]. Characterization of quadratic transformations of multivariate functions will be described in Section 3.

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# 2 Basic concepts and terminologies

Henceforth, let  $\mathbb{I}$  denote the unit interval and  $\mathbb{R}$  denote the set of all real numbers. A vector in  $\mathbb{R}^k$  will be denoted by  $\vec{x} = (x_1, \dots, x_k)$ . In other words,  $x_i$  is the  $i^{th}$  component of the vector  $\vec{x}$ . The standard basis in  $\mathbb{R}^k$  will be denoted by  $\vec{e}_1, \dots, \vec{e}_k$ , that is,  $\vec{e}_i = (e_{i1}, \dots, e_{ik})$  where  $e_{ii} = 1$  when i = j and  $e_{ij} = 0$  otherwise. Denote also,  $\vec{0} = (0, ..., 0)$ ,  $\vec{1} = (1, ..., 1)$ , and  $\vec{e}_{-i} = \vec{1} - \vec{e}_i$ .

An *aggregation function* is simply a function  $A: \mathbb{I}^k \to \mathbb{I}$  nondecreasing in each of its arguments such that  $A(\vec{0}) = 0$  and  $A(\vec{1}) = 1$ . Here,  $A(\vec{x})$  can be thought of as a representative of numerical data  $\vec{x} = (x_1, \dots, x_k)$ . We assume, in this case, that all data lies in the unit interval. Aggregation functions for data lying in other intervals can be defined analogously. Examples of aggregation functions include the following functions:

mean 
$$(x_1, ..., x_k) = \frac{1}{k} \sum_{i=1}^k x_i;$$

$$L(x_1, ..., x_k) = \begin{cases} x_i, & x_j = 1 \ \forall j \neq i; \\ 0, & \text{otherwise}; \end{cases}$$

$$W(x_1, ..., x_k) = \max \left(0, \sum_{i=1}^k x_i - k + 1\right);$$

$$M(x_1, ..., x_k) = \min(x_1, ..., x_k).$$

The class of *k*-variate aggregation functions will be denoted by  $A_k$ .

A *semi-copula* is an aggregation function  $S: \mathbb{I}^k \to \mathbb{I}$  satisfying the following condition:

$$S\left(\vec{e}_{-i} + x\vec{e}_i\right) = x$$

for all  $x \in \mathbb{I}$  and all i = 1, 2, ..., k. Equivalently, an aggregation function S is a semi-copula if and only if  $L \leq S \leq M$ .

A semi-copula which is  $L^1$ -Lipschitz is called a quasi-copula. In other words, a *quasi-copula* is a semicopula  $Q: \mathbb{I}^k \to \mathbb{I}$  such that

$$|Q(\vec{x}) - Q(\vec{y})| \le \sum_{i=1}^{k} |x_i - y_i|$$

for all  $\vec{x}, \vec{y} \in \mathbb{I}^k$ . This implies a quasi-copula is differentiable almost everywhere and that  $0 \le \partial_i Q \le 1$  a.e.

The class of k-variate semi-copulas will be denoted by  $S_k$  while the class of k-variate quasi-copulas will be denoted by  $\Omega_k$ . Note that  $\Omega_k \subseteq S_k \subseteq A_k$  and these three classes are different. In fact, mean  $\notin S_k$  while  $L \notin \mathcal{Q}_k$ . Also, both  $W, M \in \mathcal{Q}_k$  and  $W \leq Q \leq M$  for all  $Q \in \mathcal{Q}_k$ .

Given a (k+1)-variate function  $P: \mathbb{I}^{k+1} \to \mathbb{R}$ , we may define a transformation  $\tau_P$  by letting

$$\tau_P(A)(\vec{x}) = P(\vec{x}, A(\vec{x}))$$

for any  $\vec{x} \in \mathbb{I}^k$  and any k-variate function  $A : \mathbb{I}^k \to \mathbb{R}$ . The question is whether  $\tau_P(\mathfrak{F}) \subseteq \mathfrak{F}$  when  $\mathfrak{F}$  is either  $A_k$ ,  $S_k$ , or  $Q_k$ . Several works has been done for k = 2, see [4, 7, 8, 11, 12]. For quadratic polynomial P, the following result is known.

**Theorem 2.1.** [8] Let P be a quadratic polynomial.

- If P is linear, then  $\tau_P(A_2) \subseteq A_2$  if and only if P(x, y, z) = ax + by + cz where  $a, b, c \in \mathbb{I}$  with a + b + c = 1.
- $\tau_P(S_2) \subseteq S_2$  if and only if  $P(x,y,z) = cz^2 + dxy cz(x+y) + (1+c-d)z$  where  $d \in \mathbb{I}$  and  $d-1 \le c \le 1$  $\min(d, 1 - d)$ .
- $\tau_P(Q_2) \subseteq Q_2$  if and only if  $P(x, y, z) = cz^2 + dxy cz(x + y) + (1 + c d)z$  where  $d \in \mathbb{I}$  and  $d 1 \le c \le d$ .

It can be seen from the theorem that  $\tau_P(S_2) \subseteq S_2$  implies  $\tau_P(Q_2) \subseteq Q_2$  but the converse statement does not hold. In the next section, we will extend this result to the multivariate case. Moreover, we will show that  $\tau_P(S_k) \subseteq S_k$  if and only if  $\tau_P(Q_k) \subseteq Q_k$  for all k > 2. Therefore, the situation for the multivariate case is actually different from that of the bivariate case.

## 3 Main Results

Henceforth, the quadratic polynomial  $P : \mathbb{I}^{k+1} \to \mathbb{R}$  will be written as

$$P(\vec{x},z) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} x_i x_j + az^2 + z \sum_{i=1}^{k} b_i x_i + cz + \sum_{i=1}^{k} c_i x_i + d$$
 (1)

where a,  $a_{ij}$ ,  $b_i$ , c,  $c_i$ , d are real coefficients. The goal is to characterize these coefficients so that either  $\tau_P(A_k) \subseteq A_k, \tau_P(S_k) \subseteq S_k$ , or  $\tau_P(Q_k) \subseteq Q_k$ .

First, we will characterize *P* such that  $\tau_P(A_k) \subseteq A_k$ .

**Theorem 3.1.** Let  $P: \mathbb{I}^{k+1} \to \mathbb{R}$  be a quadratic polynomial as defined in equation (1). For any  $k \ge 2$ ,  $\tau_P(A_k) \subseteq A_k$  if and only if P is a aggregation function.

*Proof.* Assume *P* is a quadratic aggregation function. Then *P* is nondecreasing which implies  $\tau_P(A)$  is nondecreasing whenever *A* itself is nondecreasing. Now

$$\tau_{P}(A)\left(\vec{0}\right) = P\left(\vec{0}, A\left(\vec{0}\right)\right) = P\left(\vec{0}, 0\right) = 0$$

and

$$\tau_{P}(A)\left(\vec{1}\right) = P\left(\vec{1}, A\left(\vec{1}\right)\right) = P\left(\vec{1}, 1\right) = 1$$

whenever  $A(\vec{0}) = 0$  and  $A(\vec{1}) = 1$ . Therefore,  $\tau_P(A_k) \subseteq A_k$ .

On the contrary, assume  $\tau_P(\mathcal{A}_k) \subseteq \mathcal{A}_k$ . Using the same arguments, we have  $P\left(\vec{0}\right) = 0$  and  $P\left(\vec{1}\right) = 1$ . Define

$$A_{i,a,z,\epsilon}\left(\vec{x}\right) = \begin{cases} \left(\frac{z}{a-\epsilon}\right)x_i, & x_i < a - \epsilon; \\ z, & a - \epsilon \le x_i \le a + \epsilon; \\ 1 - \left(\frac{1-z}{1-a-\epsilon}\right)(1-x_i), & x_i > a + \epsilon. \end{cases}$$

Then  $A_{i,a,z,\epsilon}$  is an aggregation function whenever  $0 < a - \epsilon < a + \epsilon < 1$  and  $z \in \mathbb{I}$ . Moreover,  $\partial_i A_{i,a,z,\epsilon} \left( \vec{x} \right) = 0$  whenever  $\vec{x} \in \mathbb{I}^k$  is such that  $x_i = a$ . Since  $\tau_P \left( A_{i,a,z,\epsilon} \right)$  is nondecreasing and differentiable when  $x_i = a$ , we must have  $\partial_i \tau_P \left( A_{i,a,z,\epsilon} \right) \left( \vec{x} \right) \ge 0$  whenever  $\vec{x} \in \mathbb{I}^k$  is such that  $x_i = a$ . Now,

$$\partial_{i}\tau_{P}\left(A_{i,a,z,\epsilon}\right)\left(\vec{x}\right) = \partial_{i}P\left(\vec{x},z\right) + \partial_{k+1}P\left(\vec{x},z\right)\partial_{i}A_{i,a,z,\epsilon}\left(\vec{x}\right) = \partial_{i}P\left(\vec{x},z\right)$$

for such point. Therefore, we can conclude that  $\partial_i P\left(\vec{x},z\right) \ge 0$  whenever  $0 < x_i < 1$  and 0 < z < 1. Since P is a polynomial function, we can conclude that  $\partial_i P\left(\vec{x},z\right) \ge 0$  for all  $\left(\vec{x},z\right) \in \mathbb{I}^{k+1}$  and all  $i=1,\ldots,k$ . Therefore, P is nondecreasing with respect to the first k variables.

To see that *P* is nondecreasing with respect to the last variable, fix 0 < a < 1 and 0 < z < 1. For each c > 1, let  $p = p_c(a, z) = \frac{\ln z - \ln c}{\ln a}$  and

$$B_{i,a,z,c}(\vec{x}) = \min(cx_i^p, 1)$$
.

Then  $B_{i,a,z,c}$  is an aggregation function with  $B_{i,a,z,c}\left(\vec{x}\right)=z$  and  $\partial_i B_{i,a,z,c}\left(\vec{x}\right)=pcx_i^{p-1}=\frac{pz}{a}$  whenever  $\vec{x}\in\mathbb{I}^k$  is such that  $x_i=a$ . Now,

$$\begin{split} &0 \leq \partial_{i} \tau_{P} \left(B_{i,a,z,c}\right) \left(\vec{x}\right) \\ &= \partial_{i} P\left(\vec{x},z\right) + \partial_{k+1} P\left(\vec{x},z\right) \partial_{i} B_{i,a,z,c} \left(\vec{x}\right) \\ &= \partial_{i} P\left(\vec{x},z\right) + \partial_{k+1} P\left(\vec{x},z\right) \left(\frac{pz}{a}\right). \end{split}$$

Therefore,

$$\partial_{k+1}P\left(\vec{x},z\right) \geq -\frac{a}{pz}\partial_{i}P\left(\vec{x},z\right)$$

whenever  $\vec{x} \in \mathbb{I}^k$  is such that  $x_i = a$ . Let  $c \to \infty$ , then  $p \to \infty$  and hence  $\partial_{k+1} P\left(\vec{x}, z\right) \ge 0$  as desired. Since a and z are arbitrary, we are done.

The characterization of quadratic aggregation functions has been done in [10]. Thus, we immediately have the following result.

**Corollary 3.2.** Let  $P: \mathbb{F}^{k+1} \to \mathbb{R}$  be a quadratic polynomial as defined in equation (1) for some  $k \geq 2$ . Then  $\tau_P(A_k) \subseteq A_k$  if and only if P is a convex combination of  $\pi_{ij}$  and  $\zeta_{ij}$  where

$$\pi_{ij}(\vec{u}) = u_i u_j,$$
  
$$\zeta_{ij}(\vec{u}) = u_i + u_j - u_i u_j$$

for all  $\vec{u} = (\vec{x}, z) \in \mathbb{I}^{k+1}$ . In particular, such polynomials P forms a convex set with exactly  $(k+1)^2 + k + 1$  extreme points.

Proof. Follows directly from Theorem 2 and Corollary 3 in [10].

As an illustration, consider  $P: \mathbb{I}^4 \to \mathbb{R}$  defined by  $P = \frac{1}{2}(\pi_{12} + \pi_{34})$  Then P is a quadratic aggregation function which immediately implies  $A = \tau_P$  (mean) must be an aggregation function too. Direct computation yields

$$A(x, y, z) = \frac{1}{2}xy + \frac{1}{6}xz + \frac{1}{6}yz + \frac{1}{6}z^2$$

for all  $x, y, z \in \mathbb{I}$ .

From Theorem 3.1, we might guess that  $\tau_P(\mathcal{F}) \subseteq \mathcal{F}$  if and only if  $P \in \mathcal{F}$  when  $\mathcal{F}$  is either  $\mathcal{A}_k$ ,  $\mathcal{S}_k$ , or  $\mathcal{Q}_k$ . This guess is, of course, incorrect even in the case k = 2. To characterize P for the remaining cases, we will first consider the possible form of P. Note that if either  $\tau_P(\mathcal{S}_k) \subseteq \mathcal{S}_k$  or  $\tau_P(\mathcal{Q}_k) \subseteq \mathcal{Q}_k$  hold, then we would have  $\tau_P(M) \in \mathcal{S}_k$ .

**Lemma 3.3.** Let  $P : \mathbb{I}^{k+1} \to \mathbb{R}$  be a quadratic polynomial as defined in equation (1). For any k > 2,  $\tau_P(M) \in \mathbb{S}_k$  if and only if P is in the form

$$P(\vec{x}, z) = az^2 - az \sum_{i=1}^{k} x_i + (ak - a + 1)z$$

for some  $-\frac{1}{k-1} \le a \le 0$ .

*Proof.* Assume that  $\tau_P(M) \in S_k$ . For any  $x \in \mathbb{I}$ , we would have

$$0 \le \tau_P(M) (x \vec{e}_i) \le \tau_P(M) (\vec{e}_{-i}) = 0$$

by choosing  $j \neq i$ . Since  $M(x\vec{e}_i) = 0$ , we have

$$\tau_P(M)(x\vec{e}_i) = a_{ii}x^2 + c_ix + d$$

for all  $x \in \mathbb{I}$  and hence  $a_{ii} = c_i = d = 0$ .

Similarly, we can use the fact that k > 2 to show that  $M(x\vec{e}_i + y\vec{e}_i) = 0$  and

$$2a_{ij}xy = \tau_P(M)\left(x\vec{e}_i + y\vec{e}_i\right) = 0$$

for all  $x, y \in \mathbb{I}$ . Thus,  $a_{ij} = 0$ . Hence, P must take the form

$$P(\vec{x},z) = az^2 + z \sum_{i=1}^k b_i x_i + cz.$$

Now, we use the fact that  $M\left(\vec{e}_{-i} + x\vec{e}_{i}\right) = \tau_{P}\left(M\right)\left(\vec{e}_{-i} + x\vec{e}_{i}\right) = x$  to obtain

$$x = \tau_P(M) \left( \vec{e}_{-i} + x \vec{e}_i \right)$$
$$= (a + b_i) x^2 + \left( \sum_{j \neq i}^k b_j + c \right) x$$

which is only possible when  $a + b_i = 0$  and  $\sum_{j \neq i}^k b_j + c = 1$ . Therefore,  $b_i = -a$  and  $c = 1 - \sum_{j \neq i}^k b_j = 1 + (k-1)a$ . Hence, P must be given in the above form. To see that  $-\frac{1}{k-1} \le a \le 0$ , we use the fact that  $\tau_P(M)$  is nondecreasing. Consider

$$f(x) = \tau_P(M)(x, ..., x)$$
  
=  $ax^2 - akx^2 + (ak - a + 1)x$ .

Since  $\tau_P(M)$  is nondecreasing, so is f. Now,  $f'(x) = 2a(1-k)x + (ak-a+1) \ge 0$  for all  $x \in \mathbb{I}$ . Substitute x = 0 yields  $a \ge -\frac{1}{k-1}$ . Similarly, consider

$$g(x) = \tau_P(M)\left(x, \frac{1}{2}, \dots, \frac{1}{2}\right)$$
$$= \frac{1}{4}a - \frac{1}{2}a\left(x + \frac{1}{2}(k-1)\right) + \frac{1}{2}(ak - a + 1)$$

where  $x \ge \frac{1}{2}$ . Then g must also be nondecreasing on the interval  $\left[\frac{1}{2}, 1\right]$ . Now,  $g'(x) = -\frac{1}{2}a \ge 0$  for all  $x \in \left[\frac{1}{2}, 1\right]$ . Therefore,  $a \le 0$ .

On the contrary, if *P* is given by the above formula, then

$$\tau_{P}(M)(\vec{e}_{-i} + x\vec{e}_{i}) = aM^{2}(\vec{e}_{-i} + x\vec{e}_{i}) - aM(\vec{e}_{-i} + x\vec{e}_{i})(k-1+x) + (ak-a+1)M(\vec{e}_{-i} + x\vec{e}_{i})$$

$$= ax^{2} - ax(k-1+x) + (ak-a+1)x$$

$$= x.$$

Therefore,  $\tau_P(M) \in S_k$ .

Remark 3.4. If we define  $P_0\left(\vec{x},z\right)=z$  and  $P_1\left(\vec{x},z\right)=-\frac{1}{k-1}z^2+\frac{1}{k-1}z\sum_{i=1}^k x_i=-\frac{1}{k-1}z^2+\frac{k}{k-1}z\cdot \text{mean}\left(\vec{x}\right)$ , then the above lemma states that  $\tau_P\left(M\right)\in \mathbb{S}_k$  if and only if P is a convex combination of  $P_0$  and  $P_1$ , that is,  $P=P_t=tP_1+(1-t)P_0$  for some  $t\in \mathbb{I}$ . Also, we always have  $\tau_t=\tau_{P_t}=t\tau_{P_1}+(1-t)\tau_{P_0}$ . Clearly,  $\tau_0$  is the identity map and hence  $\tau_0\left(\mathbb{S}_k\right)=\mathbb{S}_k$  and  $\tau_0\left(\mathbb{Q}_k\right)=\mathbb{Q}_k$ . If we can show that  $\tau_1\left(\mathbb{S}_k\right)\subseteq \mathbb{S}_k$ , we can use the fact that  $\mathbb{S}_k$  is a convex set to conclude that  $\tau_t\left(\mathbb{S}_k\right)\subseteq \mathbb{S}_k$  for all  $t\in \mathbb{I}$ . Similar conclusion also holds when we replace  $\mathbb{S}_k$  with  $\mathbb{Q}_k$ . Note that  $P_1$  is not an aggregation function. Thus,  $\tau_1$  is not a transformation of aggregation functions.

Also, for k = 2,  $\{\tau_t | t \in \mathbb{I}\}$  is a proper subset of all quadratic transformations preserving semi-copulas (quasi-copulas).

**Theorem 3.5.** Let  $P: \mathbb{I}^{k+1} \to \mathbb{R}$  be a quadratic polynomial. For any k > 2,  $\tau_P(S_k) \subseteq S_k$  if and only if  $\tau_P = \tau_t$ , that is,  $P = P_t$  for some  $t \in \mathbb{I}$ .

*Proof.* If  $\tau_P(S_k) \subseteq S_k$ , then  $\tau_P(M) \in S_k$  which implies  $P = P_t$  for some  $t \in \mathbb{I}$  by the previous lemma. For the converse, it is sufficient to show that  $\tau_1(S)$  is nondecreasing since  $\tau_1(S)\left(\vec{e}_{-i} + x\vec{e}_i\right) = x$  for all  $x \in \mathbb{I}$  via direct computation.

For any function F, denote  $\Delta_{i,\epsilon}F\left(\vec{x}\right)=F\left(\vec{x}+\epsilon\vec{e}_{i}\right)-F\left(\vec{x}\right)$ . Then

$$(k-1)\Delta_{i,\epsilon}\tau_{1}(S)(\vec{x}) = -S^{2}(\vec{x} + \epsilon\vec{e}_{i}) + S(\vec{x} + \epsilon\vec{e}_{i})\left(\sum_{ij=1}^{k} x_{j} + \epsilon\right) + S^{2}(\vec{x}) - S(\vec{x})\sum_{j=1}^{k} x_{j}$$

$$\geq S^{2}(\vec{x}) - S^{2}(\vec{x} + \epsilon\vec{e}_{i}) + S(\vec{x} + \epsilon\vec{e}_{i})\left(\sum_{j=1}^{k} x_{j} + \epsilon\right) - S(\vec{x})\left(\sum_{j=1}^{k} x_{j} + \epsilon\right)$$

$$= \Delta_{i,\epsilon}S(\vec{x})\left(\sum_{j=1}^{k} x_{j} + \epsilon - S(\vec{x} + \epsilon\vec{e}_{i}) - S(\vec{x})\right)$$

$$\geq \Delta_{i,\epsilon}S(\vec{x})\left(\sum_{j=1}^{k} x_{j} + \epsilon - M(\vec{x} + \epsilon\vec{e}_{i}) - M(\vec{x})\right)$$

$$\geq \Delta_{i,\epsilon} S(\vec{x}) \left( \sum_{j=1}^{k} x_j + \epsilon - 2M(\vec{x} + \epsilon \vec{e}_i) \right).$$

Since *S* is nondecreasing,  $\Delta_{i,\epsilon}S\left(\vec{x}\right) \ge 0$ . Since  $k \ge 2$ ,  $\sum_{j=1}^{k} x_j + \epsilon \ge 2M\left(\vec{x} + \epsilon\vec{e}_i\right)$ . Therefore,  $\Delta_{i,\epsilon}\tau_1\left(S\right) \ge 0$  which implies  $\tau_1\left(S\right)$  is nondecreasing as desired.

**Theorem 3.6.** Let  $P: \mathbb{I}^{k+1} \to \mathbb{R}$  be a quadratic polynomial as defined in equation (1) for some k > 2 Then  $\tau_P(Q_k) \subseteq Q_k$  if and only if  $\tau_P = \tau_t$ , that is,  $P = P_t$  for some  $t \in \mathbb{I}$ .

*Proof.* Following the same arguments, it is sufficient to show that  $\tau_1(Q_k) \subseteq Q_k$ . Since  $Q_k \subseteq S_k$ , we already have  $\tau_1(Q_k) \subseteq S_k$ . Thus, it remains to show that  $\partial_i \tau_1(Q) \le 1$  a.e. for any  $Q \in Q_k$ .

Since  $P_1$  is a polynomial and  $\partial_i Q$  exists a.e.,  $\partial_i \tau_1(Q)$  must exist a.e. too. Recall that  $Q(\vec{x}) \ge W(\vec{x}) = \max(0, \sum_{i=1}^k x_i - k + 1)$ . Direct computation yields

$$\begin{split} \partial_{i}\tau_{1}\left(Q\right)\left(\vec{x}\right) &= -\frac{2}{k-1}Q\left(\vec{x}\right)\partial_{i}Q\left(\vec{x}\right) + \frac{1}{k-1}Q\left(\vec{x}\right) + \frac{1}{k-1}\partial_{i}Q\left(\vec{x}\right)\sum_{i=1}^{k}x_{i} \\ &= \frac{1-\partial_{i}Q\left(\vec{x}\right)}{k-1}Q\left(\vec{x}\right) + \frac{1}{k-1}\partial_{i}Q\left(\vec{x}\right)\left(\sum_{i=1}^{k}x_{i} - Q\left(\vec{x}\right)\right) \\ &\leq \frac{1-\partial_{i}Q\left(\vec{x}\right)}{k-1}Q\left(\vec{x}\right) + \frac{1}{k-1}\partial_{i}Q\left(\vec{x}\right)\left(\sum_{i=1}^{k}x_{i} - W\left(\vec{x}\right)\right) \\ &\leq 1-\partial_{i}Q\left(\vec{x}\right) + \partial_{i}Q\left(\vec{x}\right) \\ &\leq 1 \end{split}$$

a.e. as desired.

Note that a quadratic function  $p_{\vec{x}}(z) = P_1(\vec{x},z) = \frac{1}{k-1}z\left(\sum_{i=1}^k x_i - z\right)$  only has two fixed points which are z = 0 and  $z = \sum_{i=1}^k x_i - k + 1$ . Thus,  $\tau_1$  must transform any semi-copula (quasi-copula) to a different semi-copula (quasi-copula) excepted only when that semi-copula is either L or W. In other words, L and L are the only fixed points of  $t_1$ . Similar conclusion also holds for  $t_1$  when  $t \neq 0$ .

We end this section with an example of quasi-copula construction using quadratic transformations.

**Example 3.7.** Consider a function  $C_{\theta}: \mathbb{I}^k \to \mathbb{R}$  defined by

$$C_{\theta}\left(\vec{x}\right) = \prod_{i=1}^{k} x_i + \theta \prod_{i=1}^{k} x_i \left(1 - x_i\right)$$

for all  $\vec{x} \in \mathbb{I}^k$ . It can be easily proved that  $C_{\theta}$  is a semi-copula if and only if it is a quasi-copula if and only if  $-1 \le \theta \le 1$ . Using the transformation  $\tau_1$ , another family

$$Q_{\theta} = \tau_1(C_{\theta}) = \frac{1}{k-1} \left( kC_{\theta} \cdot \text{mean} - C_{\theta}^2 \right)$$

of quasi-copulas can be constructed. Simplification of  $Q_{\theta}$  is also possible. For example,

$$Q_0(\vec{x}) = \frac{1}{k-1} \sum_{i=1}^k \left( x_i^2 \prod_{i \neq i} x_i \right) - \frac{1}{k-1} \prod_{i=1}^k x_i^2$$

for all  $\vec{x} \in \mathbb{I}^k$ .

## 4 Conclusion and Discussion

In this work, we propose a construction of multivariate aggregation functions via quadratic transformations of such functions. A characterization has been made to ensure which quadratic polynomials can be used to define such transformations. It turns out that the structure of quadratic transformations of general aggregation functions is quite complicated. In the bivariate case, such quadratic transformations form a convex set with 12 extreme points. Moreover, the number of extreme points is roughly the square of the dimension in general. On the contrary, quadratic transformations of multivariate semi-copulas and quasi-copulas are very simple. Both classes are the same, which are convex sets with only two extreme points. Moreover, one of the extreme points is the identity transformation. Therefore, there is only one non-trivial quadratic transformation. This situation differs from the bivariate case in several ways. First, in the bivariate case, such transformations form a convex set with four extreme points. Second, these two classes are different in the bivariate case with, surprisingly, the class of quadratic transformations of bivariate semi-copulas is a subclass of the class of quadratic transformations of bivariate quasi-copulas.

Other subclasses of aggregation functions can also be considered. For example, we could try to characterize quadratic transformations of conjunctive aggregation functions and disjunctive aggregation functions. It is also interesting to see whether these transformations overlapped. For instance, whether a quadratic transformation of conjunctive aggregation functions is also a transformation of disjunctive aggregation functions. More generally, whether a transformation of some subclass is also a transformation of its dual class. We hope that these questions can be answer in the future.

During the investigation, we also found that the form of quadratic transformation  $\tau_P$  can be found by considering  $\tau_P(M)$  alone. It is interesting to see whether this situation can be generalized. For example, when P is a rational linear function as studied in [4]. Characterizing functions P with this property is also interesting.

Characterization of polynomial transformations  $\tau_P$  is also an open problem. The existence of such transformation is guaranteed by compositions of quadratic transformations. For aggregation functions, we may use the exact same proof to conclude that *P* itself is an aggregation function. However, the characterization of polynomial aggregation functions is also an open problem. For semi-copulas and quasi-copulas, the situation would be more bizarre because we would not be able to guess the form of such transformations. At most, we are able to confirm that these classes would be convex with non-linear boundary. Thus, they would have infinitely many extreme points.

All these questions require further study.

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