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Stochastic comparisons and bounds for conditional distributions by using copula properties

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Abstract: We prove that different conditional distributions can be represented as distorted distributions. These representations are used to obtain stochastic comparisons and bounds for them based on properties of the underlying copula. These properties can be used to explain the meaning of mathematical properties of copulas connecting them with dependence concepts. Some applications and illustrative examples are provided as well.

Keywords: Conditional distributions, distorted distributions, copula, stochastic orders, bounds

MSC: 62K10, 60E15, 90B25

1 Introduction

Conditional distributions can be used to incorporate information to a given random vector. They can also be used to construct multivariate models (see [3]). Different conditioning events can be considered in different practical cases. For example, if we have the random vector (X, Y), then we could consider the conditional distributions: $(Y|X > x)$, $(Y|X \leq x)$ and $(Y|X = x)$ which have different meanings (applications) in practice.

The distorted distribution of a baseline distribution function F is defined by $F_q = q(F)$ where $q : [0, 1] \to [0, 1]$ is a continuous increasing function such that $q(0) = 0$ and $q(1) = 1$. Distorted distributions appeared in the theory of choice under risk (see [36]) to model the uncertainty in the distribution of the variable under study and have applications in a variety of contexts. In reliability, e.g., they can be used to represent order statistics, coherent systems or proportional hazard rate models (see, e.g., [21]); in risk theory, they are used to define premium principles (see [14, 31, 35]); in the context of Bayesian analysis, they are used to define classes of prior distributions (see [2]). They were extended in [22] to build distortions of n distribution functions. These representations can be used to perform stochastic comparisons (see [18–23, 33]) and to provide bounds for distributions and expectations (see [16, 17, 25]).

In the present paper we prove that conditional distributions can be represented as distorted distributions. The distortion function depends on the underlying copula. These representations are used to obtain stochastic comparisons and bounds and to explain the meaning of mathematical properties of copulas connecting them with some dependence concepts. Some illustrative examples are provided as well.

The rest of the article is organized as follows. The different representations of conditional distributions as distorted distributions are given in Section 2. The resulting stochastic comparisons and bounds are given in Sections 3 and 4, respectively. The connections with dependence properties are included in Section 3. Ap-

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plications in risk theory and economics are given in Section 5. Some illustrative examples are included in Section 6 and the conclusions in Section 7.

Throughout the paper, we say that a function g is increasing (resp. decreasing) if $g(x) \le g(y)$ ($\ge$) for all $x \le y$. Whenever we use a derivative, an expectation or a conditional distribution, we are assuming tacitly that they exist.

2 Representations for conditional distributions

First, we give the formal definition of distorted distributions.

Definition 1. *We say that F_q is a distorted distribution of a distribution function F if*

$$F_q(t) = q(F(t)) \tag{2.1}$$

for all t, where q is a distortion function, that is, $q : [0, 1] \to [0, 1]$ is a continuous increasing function such that $q(0) = 0$ and $q(1) = 1$.

Note that the properties of the distortion function q imply that F_q is a proper distribution function for any distribution function F. Also note that q is a restriction of a continuous distribution function with support included in the interval $[0, 1]$. As a consequence we have a similar relationship between the respective reliability functions

$$\bar{F}_q(t) = \bar{q}(\bar{F}(t)), \tag{2.2}$$

where $\bar{F}_q = 1 - F_q$, $\bar{F} = 1 - F$ and $\bar{q}$ is another distortion function (called *dual distortion function*) given by $\bar{q}(u) = 1 - q(1 - u)$. Representations (2.1) and (2.2) are equivalent but sometimes it is better to use (2.2) instead of (2.1) (or vice versa).

Now we can study representations for conditional distributions. Let us consider first the bivariate case. The representations for the general n dimensional case are similar and will be stated later. Thus, let (X, Y) be a bivariate random vector over a probability space. Then, from the copula theory, it is well known (see, e.g., [10, 26]) that its joint distribution function $\boldsymbol{F}$ can be represented as

$$\boldsymbol{F}(x, y) = \Pr(X \le x, Y \le y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \le x)$ and $G(y) = \Pr(Y \le y)$ are the (marginal) distribution functions of X and Y, respectively, and where C is a copula (i.e., C is a restriction of a continuous distribution function with uniform marginals over the interval $(0, 1)$). Alternatively, we might use the joint reliability function $\bar{\boldsymbol{F}}$ which can also be represented as

$$\bar{\boldsymbol{F}}(x, y) = \Pr(X > x, Y > y) = \hat{C}(\bar{F}(x), \bar{G}(y)),$$

where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ are the (marginal) reliability functions of X and Y and where $\hat{C}$ is also a copula, called *survival copula*. The survival copula $\hat{C}$ is determined by the "distributional" copula C (and vice versa) by the following relationship

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

We are going to consider three different conditional distributions. The first one is the distribution of the random variable $(Y|X \le x)$ which can be written as

$$\Pr(Y \le y|X \le x) = \frac{\Pr(X \le x, Y \le y)}{\Pr(X \le x)} = \frac{C(F(x), G(y))}{F(x)} = q_1(G(y)), \tag{2.3}$$

where the distortion function is given by

$$q_1(u) = \frac{C(F(x), u)}{F(x)}$$

for any x such that $F(x) > 0$. Analogously, its reliability function can be written as

$$\Pr(Y > y|X \leq x) = \frac{F(x) - C(F(x), G(y))}{F(x)} = \bar{q}_1(\bar{G}(y)), \tag{2.4}$$

where the dual distortion function is given by

$$\bar{q}_1(u) = 1 - q_1(1-u) = \frac{F(x) - C(F(x), 1-u)}{F(x)} = \frac{u - \hat{C}(\bar{F}(x), u)}{F(x)}.$$

Note that, as a consequence, the function $u - \hat{C}(p, u)$ is increasing in u for any $p \in [0, 1)$ and for any copula $\hat{C}$. Similar representations can be obtained for $(X|Y \leq y)$.

The second option is the random variable $(Y|X > x)$ whose reliability can be written as

$$\Pr(Y > y|X > x) = \frac{\Pr(X > x, Y > y)}{\Pr(X > x)} = \frac{\hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = \bar{q}_2(\bar{G}(y)), \tag{2.5}$$

where the dual distortion function is given by

$$\bar{q}_2(u) = \frac{\hat{C}(\bar{F}(x), u)}{\bar{F}(x)}$$

for any x such that $\bar{F}(x) > 0$. Analogously, its distribution function can be written as

$$\Pr(Y \leq y|X > x) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), \bar{G}(y))}{\bar{F}(x)} = q_2(G(y)), \tag{2.6}$$

where the distortion function is given by

$$q_2(u) = \frac{\bar{F}(x) - \hat{C}(\bar{F}(x), 1-u)}{\bar{F}(x)} = \frac{u - C(F(x), u)}{\bar{F}(x)}.$$

Note that, as a consequence, we obtain again that the function $u - C(p, u)$ is increasing in u for any $p \in [0, 1)$ and for any copula C. Similar representations can be obtained for $(X|Y > y)$.

In the third option we consider the usual conditional distribution, that is, the distribution of the random variable $(Y|X = x)$. In this case we need to assume that the random vector has an absolutely continuous joint distribution. Throughout the paper, we assume this as long as we consider this case. Then the (one) joint probability density function (pdf) $\boldsymbol{f}$ of (X, Y) can be written as

$$\boldsymbol{f}(x, y) =_{a.e.} f(x)g(y)\partial_1\partial_2 C(F(x), G(y)), \tag{2.7}$$

where $=_{a.e.}$ denotes equality almost everywhere, $f(x) = \int_{-\infty}^{+\infty} \boldsymbol{f}(x, z)dz =_{a.e.} F'(x)$ and $g(y) = \int_{-\infty}^{+\infty} \boldsymbol{f}(z, y)dz =_{a.e.} G'(y)$ are the (some) marginal density functions of X and Y and where $\partial_i C(u, v)$ denotes the partial derivative with respect to the ith variable of C, for $i = 1, 2$. We use these marginal pdf throughout the paper. Hence we have the following representation.

Proposition 2.1. *For almost surely all x, a version of a distribution function of $(Y|X = x)$ can be written as*

$$\Pr(Y \leq y|X = x) = q_3(G(y)), \tag{2.8}$$

where q_3 is a distortion function given by $q_3(u) = \partial_1 C(F(x), u)$ for $u \in (0, 1)$.

Proof. For a fixed x such that

$$f(x) = \int_{-\infty}^{+\infty} \boldsymbol{f}(x, z)dz > 0,$$

from (2.7), a version of the distribution function of the random variable Y conditioned on the set $X = x$ can be written as

$$\begin{aligned}\Pr(Y \leq y|X = x) &= \int_{-\infty}^{y} \frac{\boldsymbol{f}(x,z)}{f(x)} dz \\ &= \int_{-\infty}^{y} g(z)\partial_2\partial_1 C(F(x), G(z)) dz \\ &= \partial_1 C(F(x), G(y)) - \lim_{u\to 0^+} \partial_1 C(F(x), u).\end{aligned}$$

In particular, we have

$$1 = \lim_{y\to\infty} \Pr(Y \leq y|X = x) = \lim_{u\to 1^-} \partial_1 C(F(x), u) - \lim_{u\to 0^+} \partial_1 C(F(x), u). \tag{2.9}$$

Moreover, we know from Theorem 2.2.7 in [26, p. 13] that $\partial_1 C(F(x), u)$ is an increasing function of u and that $0 \leq \partial_1 C(F(x), u) \leq 1$, which, together with (2.9), implies

$$\lim_{u\to 1^-} \partial_1 C(F(x), u) = 1 \quad \text{and} \quad \lim_{u\to 0^+} \partial_1 C(F(x), u) = 0.$$

Therefore, q_3 is a distortion function and (2.8) holds. □

The joint pdf of (X, Y) can also be written as

$$\boldsymbol{f}(x, y) =_{a.e.} f(x)g(y)\partial_1\partial_2 \hat{C}(\bar{F}(x), \bar{G}(y)).$$

Then, under the assumptions of the preceding proposition, the reliability function of the conditional random variable $(Y|X = x)$ can be written as

$$\Pr(Y > y|X = x) = \bar{q}_3(\bar{G}(y)), \tag{2.10}$$

where the dual distortion function $\bar{q}_3$ is given by

$$\bar{q}_3(u) = 1 - \partial_1 C(F(x), 1 - u) = \partial_1 \hat{C}(\bar{F}(x), u).$$

Similar representations can be obtained for a version of $(X|Y = y)$ when $g(y) = \int_{-\infty}^{+\infty} \boldsymbol{f}(z, y)dz > 0$.

Open Problem 1. *One anonymous reviewer note that the property*

$$\lim_{v\to 0^+} \partial_1 C(u, v) = 0 \textit{ for } 0 < u < 1,$$

(obtained in the proof of the preceding proposition) is not true for all the copulas providing the following counterexample. Let $C(u, v) = uv/(u + v - uv)$ for $(u, v) \in [0, 1]^2$ be a Clayton copula and let us consider the copula D defined by

$$D(u, v) = \begin{cases} (v - C(1 - 2u, v))/2, & \textit{for } 0 \leq u \leq 1/2, 0 \leq v \leq 1 \\ (v + C(2u - 1, v))/2, & \textit{for } 1/2 < u \leq 1, 0 \leq v \leq 1. \end{cases}$$

The copula D is twice differentiable and $\partial_1 D(1/2, v) = 1$ for all $0 < v < 1$. Hence the limit when v tends to 0^+ is 1 not 0. The explanation is the following. A straightforward calculation shows that the pdf d of D satisfies $d(1/2, v) = 0$ for all $0 < v < 1$. Hence the first marginal at $1/2$ is $d_1(1/2) = \int_0^1 d(1/2, v)dv = 0$. So we cannot apply Proposition 2.1 to $(V|U = 1/2)$. The survival copula of D gives the counterexample to the second limit

$$\lim_{v\to 1^-} \partial_1 C(u, v) = 1 \textit{ for } 0 < u < 1.$$

There are other conditional distributions that can also be represented as distorted distributions which will not be studied in this paper (since the results are similar). For example, the distribution function of the random variable $(Y|X \le x, Y \le y)$ can be written as

$$\Pr(Y \le z|X \le x, Y \le y) = \frac{\Pr(X \le x, Y \le z)}{\Pr(X \le x, Y \le y)} = \frac{C(F(x), G(z))}{C(F(x), G(y))} = q_4(G_y(z))$$

for $z \le y$, where $G_y(z) = G(z)/G(y)$ is the distribution function of $(Y|Y \le y)$ and where the distortion function is given by

$$q_4(u) = \frac{C(F(x), uG(y))}{C(F(x), G(y))}$$

for any x, y such that $C(F(x), G(y)) > 0$. Similar representations can be obtained for the random variables $(Y|X > x, Y > y)$, $(Y|X \le x, Y > y)$ and $(Y|X > x, Y \le y)$. The case $x = y = t$ is examined in [9] given some connections with positive dependence properties. Other cases are studied in [8, 18, 20, 24].

In the general case the joint distribution function of the random vector $(X_1, \dots, X_n)$ can be represented as

$$\boldsymbol{F}(x_1, \dots, x_n) = \Pr(X_1 \le x_1, \dots, X_n \le x_n) = C(F_1(x_1), \dots, F_n(x_n)),$$

where F_i is the (marginal) distribution of X_i for $i = 1, \dots, n$. Alternatively, we might use the joint reliability function represented as

$$\bar{\boldsymbol{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)),$$

where $\bar{F}_i$ is the (marginal) reliability of X_i for $i = 1, \dots, n$. The survival copula $\hat{C}$ is determined by the "distributional" copula C and vice versa. Proceeding as in the bivariate case, we can obtain the following representations for the different conditional distributions. In the first case, we have

$$\Pr(X_n \le x_n|X_1 \le x_1, \dots, X_{n-1} \le x_{n-1}) = \frac{C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n(x_n))}{C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1)} = q_1(F_n(x_n)), \tag{2.11}$$

where the distortion function is given by

$$q_1(u) = \frac{C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), u)}{C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1)}$$

whenever $C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1) > 0$.

In the second case, we get

$$\Pr(X_n > x_n|X_1 > x_1, \dots, X_{n-1} > x_{n-1}) = \frac{\hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_{n-1}(x_{n-1}), \bar{F}_n(x_n))}{\hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_{n-1}(x_{n-1}), 1)} = q_2(\bar{F}_n(x_n)), \tag{2.12}$$

where the distortion function is given by

$$q_2(u) = \frac{\hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_{n-1}(x_{n-1}), u)}{\hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_{n-1}(x_{n-1}), 1)}$$

whenever $\hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_{n-1}(x_{n-1}), 1) > 0$.

Finally, in the third option, we obtain

$$\begin{aligned}\Pr(X_n \le x_n|X_1 = x_1, \dots, X_{n-1} = x_{n-1}) &= \frac{\partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n(x_n))}{\partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1)} \\ &= q_3(F_n(x_n)),\end{aligned} \tag{2.13}$$

where the distortion function is given by

$$q_3(u) = \frac{\partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), u)}{\partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1)}$$

for any $x_1, \dots, x_{n-1}$ such that $\partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), 1) > 0$ and

$$\lim_{u \to 0^+} \partial_1 \dots \partial_{n-1} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), u) = 0.$$

Similar expressions can be obtained from the survival copula $\hat{C}$.

3 Stochastic comparisons and dependence properties

We are going to study the following stochastic orders. Their basic properties can be seen in [28]. Let X and Y be two random variables having distribution functions F and G, and reliability (survival) functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. If these distributions are absolutely continuous, f and g represent their respective probability density functions. Then we say that X is smaller than Y:

- in the *usual stochastic order* (denoted by $X \leq_{ST} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all t;
- in the *hazard rate order* (denoted by $X \leq_{HR} Y$) if the ratio $\bar{G}(t)/\bar{F}(t)$ is increasing;
- in the *reversed hazard rate order* (denoted by $X \leq_{RHR} Y$) if the ratio $G(t)/F(t)$ is increasing;
- in the *likelihood ratio order* (denoted by $X \leq_{LR} Y$) if the ratio $g(t)/f(t)$ is increasing;
- in the *mean residual life order* (denoted by $X \leq_{MRL} Y$) if, and only if,

$$E(X - t|\, X > t) \leq E(Y - t|\, Y > t)$$

  for all t such that these conditional expectations exist.
- in the *increasing convex order* (denoted by $X \leq_{ICX} Y$) if

$$\int_x^\infty \bar{F}(t)\, dt \leq \int_x^\infty \bar{G}(t)\, dt \text{ for all } x.$$

Some of these stochastic orders may be expressed in terms of comparisons of the residual lifetimes or inactivity times of the corresponding random variables. Thus:

- $X \leq_{HR} Y$ if, and only if, $(X - t|\, X > t) \leq_{ST} (Y - t|\, Y > t)$ for all t;
- $X \leq_{RHR} Y$ if, and only if, $(t - X|\, X \leq t) \geq_{ST} (t - Y|\, Y \leq t)$ for all t;
- $X \leq_{LR} Y$ if, and only if, $(X|\, a \leq X \leq b) \leq_{ST} (Y|\, a \leq X \leq b)$ for all $a \leq b$;
- $X \leq_{MRL} Y$ if, and only if, $(X - t|\, X > t) \leq_{ICX} (Y - t|\, Y > t)$ for all t.

The very well known relationships among these stochastic orders are summarized in Table 1. The reverse implications do not hold.

Table 1: Relationships among univariate stochastic orders.

$X \leq_{LR} Y$	$\Rightarrow$	$X \leq_{HR} Y$	$\Rightarrow$	$X \leq_{MRL} Y$		
$\Downarrow$		$\Downarrow$		$\Downarrow$		
$X \leq_{RHR} Y$	$\Rightarrow$	$X \leq_{ST} Y$	$\Rightarrow$	$X \leq_{ICX} Y$	$\Rightarrow$	$E(X) \leq E(Y)$

Ordering properties for distorted distributions were obtained in [21, 23] (see also [18, 19, 22]). For completeness we include some of them in the following proposition.

Proposition 3.1. *Let X_1 and X_2 be two random variables with distribution functions $F_{q_1} = q_1(F)$ and $F_{q_2} = q_2(F)$ obtained as distorted distributions from the same distribution function F and from the distortion functions q_1 and q_2, respectively. Let $\bar{q}_1$ and $\bar{q}_2$ be the respective dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ for all F if and only if $q_1 \geq q_2$ (or $\bar{q}_1 \leq \bar{q}_2$) in $(0, 1)$.
(ii) $X_1 \leq_{HR} X_2$ for all F if and only if $\bar{q}_2/\bar{q}_1$ is decreasing in $(0, 1)$.
(iii) $X_1 \leq_{RHR} X_2$ for all F if and only if q_2/q_1 is increasing in $(0, 1)$.
(iv) $X_1 \leq_{LR} X_2$ for all absolutely continuous F if and only if q_2'/q_1' is increasing (or $\bar{q}_2'/\bar{q}_1'$ is decreasing) in $(0, 1)$.
(v) $X_1 \leq_{MRL} X_2$ for all F such that $E(X_1) \leq E(X_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$ (i.e., it is decreasing in $(0, u_0)$ and increasing in $(u_0, 1)$ for a $u_0 \in (0, 1]$).

Clearly, these results can be applied to compare conditional distributions by using the representations obtained in the preceding section. Note that to apply the condition in (iv) for the LR order we need to assume that both derivatives exist. Here we have several options. For example, we can compare the conditional distributions with the marginal distributions. Thus, for $(Y|X \le x)$, we have the following ordering properties from (2.3), (2.4) and Proposition 3.1.

Proposition 3.2. *Let (X, Y) be a random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*
(i) $Y \ge_{ST} (Y|X \le x)$ ($\le_{ST}$) for all G if and only if $C(F(x), u) \ge uF(x)$ ($\le$) for all $u \in (0, 1)$.
(ii) $Y \ge_{ST} (Y|X \le x)$ ($\le_{ST}$) for all G if and only if $\hat{C}(\bar{F}(x), u) \ge u\bar{F}(x)$ ($\le$) for all $u \in (0, 1)$.
(iii) $Y \ge_{HR} (Y|X \le x)$ ($\le_{HR}$) for all G if and only if $\hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in u in the interval $(0, 1)$.
(iv) $Y \ge_{RHR} (Y|X \le x)$ ($\le_{RHR}$) for all G if and only if $C(F(x), u)/u$ is decreasing (increasing) in u in the interval $(0, 1)$.
(v) $Y \ge_{LR} (Y|X \le x)$ ($\le_{LR}$) for all G if and only if $C(F(x), u)$ is concave (convex) in u in the interval $(0, 1)$.
(vi) $Y \ge_{LR} (Y|X \le x)$ ($\le_{LR}$) for all G if and only if $\hat{C}(\bar{F}(x), u)$ is concave (convex) in u in the interval $(0, 1)$.
(vii) $Y \ge_{MRL} (Y|X \le x)$ ($\le_{MRL}$) for all G if $\hat{C}(\bar{F}(x), u)/u$ is bathtub (upside-down bathtub) in u in the interval $(0, 1)$ and $E(Y) \ge E(Y|X \le x)$ ($\le$).

The positive (negative) quadrant dependent, PQD (NQD), property of (X, Y) can be characterized by $C \ge \Pi$ ($\le$), where $\Pi(u, v) = uv$ is the product copula (see [9] or [26, p. 188]). It can also be characterized by $\hat{C} \ge \Pi$ ($\le$). Hence, from Proposition 3.2, (*i*) or (*ii*), the PQD (NQD) property is equivalent to $Y \ge_{ST} (Y|X \le x)$ ($\le_{ST}$) for all F, G (a well known result, see, e.g., [26, p. 191]). Moreover, from Table 1, all the properties given in the preceding proposition can be seen as positive (negative) dependence conditions and all of them (except that for the MRL order) imply PQD (NQD). Hence, under these conditions, Spearman's rho and Kendall's tau coefficients are nonnegative (nonpositive), see [10, p. 62].

Analogously, from (2.5), (2.6) and Proposition 3.1, we have the following ordering properties for the conditional random variable $(Y|X > x)$. Note that they are equivalent to the properties given in the preceding proposition and so they can also be seen as positive (negative) dependence properties. This equivalence is due to the following representation

$$G(y) = F(x)\Pr(Y \le y|X \le x) + \bar{F}(x)\Pr(Y \le y|X > x)$$

valid for any F such that $0 < F(x) < 1$. Note that the marginal distribution of Y is a mixture of the conditional distributions of $(Y|X \le x)$ and $(Y|X > x)$ with weights $F(x)$ and $\bar{F}(x)$.

Proposition 3.3. *Let (X, Y) be a random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*
(i) $Y \le_{ST} (Y|X > x)$ ($\ge_{ST}$) for all G if and only if $C(F(x), u) \ge uF(x)$ ($\le$) for all $u \in (0, 1)$.
(ii) $Y \le_{ST} (Y|X > x)$ ($\ge_{ST}$) for all G if and only if $\hat{C}(\bar{F}(x), u) \ge u\bar{F}(x)$ ($\le$) for all $u \in (0, 1)$.
(iii) $Y \le_{HR} (Y|X > x)$ ($\ge_{HR}$) for all G if and only if $\hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in u in the interval $(0, 1)$.
(iv) $Y \le_{RHR} (Y|X > x)$ ($\ge_{RHR}$) for all G if and only if $C(F(x), u)/u$ is decreasing (increasing) in u in the interval $(0, 1)$.
(v) $Y \le_{LR} (Y|X > x)$ ($\ge_{LR}$) for all G if and only if $C(F(x), u)$ is concave (convex) in u in the interval $(0, 1)$.
(v) $Y \le_{LR} (Y|X > x)$ ($\ge_{LR}$) for all G if and only if $\hat{C}(\bar{F}(x), u)$ is concave (convex) in u in the interval $(0, 1)$.
(vi) $Y \le_{MRL} (Y|X > x)$ ($\ge_{MRL}$) for all G if $\hat{C}(\bar{F}(x), u)/u$ is bathtub (upside-down bathtub) in u in the interval $(0, 1)$ and $E(Y) \le E(Y|X > x)$ ($\ge$).

By replacing in the preceding proposition x with the quantile $F^{-1}(p)$ for a $p \in (0, 1)$, we obtain the results given in [34] for the ST, HR and RHR orders.

Finally, in the case of an absolutely continuous joint distribution, from (2.8), (2.10) and Proposition 3.1, we obtain the following ordering properties for $(Y|X = x)$.

Proposition 3.4. *Let (X, Y) be a random vector with absolutely continuous copulas C and $\hat{C}$, absolutely continuous marginal distributions F and G and probability density functions $f(x) = \int_{-\infty}^{+\infty} \boldsymbol{f}(x, z)dz$ and $g(y) = \int_{-\infty}^{+\infty} \boldsymbol{f}(z, y)dz$. Let x such that $f(x) > 0$. Then:*

(i) $Y \leq_{ST} (Y|X = x)$ ($\geq_{ST}$) for all G if and only if $\partial_1 C(F(x), u) \leq u$ ($\geq$) for all $u \in (0, 1)$.

(ii) $Y \leq_{ST} (Y|X = x)$ ($\geq_{ST}$) for all G if and only if $\partial_1 \hat{C}(\bar{F}(x), u) \geq u$ ($\leq$) for all $u \in (0, 1)$.

(iii) $Y \leq_{HR} (Y|X = x)$ ($\geq_{HR}$) for all G if and only if $\partial_1 \hat{C}(\bar{F}(x), u)/u$ is decreasing (increasing) in u in the interval $(0, 1)$.

(iv) $Y \leq_{RHR} (Y|X = x)$ ($\geq_{RHR}$) for all G if and only if $\partial_1 C(F(x), u)/u$ is increasing (decreasing) in u in the interval $(0, 1)$.

(v) $Y \leq_{LR} (Y|X = x)$ ($\geq_{LR}$) for all G if and only if $\partial_1 C(F(x), u)$ is convex (concave) in u in the interval $(0, 1)$.

(vi) $Y \leq_{LR} (Y|X = x)$ ($\geq_{LR}$) for all G if and only if $\partial_1 \hat{C}(\bar{F}(x), u)$ is concave (convex) in u in the interval $(0, 1)$.

(vii) $Y \leq_{MRL} (Y|X = x)$ ($\geq_{MRL}$) for all G if $\partial_1 \hat{C}(\bar{F}(x), u)/u$ is bathtub (upside-down bathtub) in u in the interval $(0, 1)$ and $E(Y) \leq E(Y|X = x)$ ($\geq$).

Note that, by using a similar procedure, we can also compare a conditional distribution with a different conditional distribution or with a similar one at a different point x. For example, $(Y|X = x) \leq_{ST} (Y|X > x)$ holds for all F, G such that $\bar{F}(x) > 0$ and $f(x) > 0$ if, and only if,

$$\bar{F}(x)\partial_1 \hat{C}(\bar{F}(x), u) \leq \hat{C}(\bar{F}(x), u) \tag{3.1}$$

for all $u \in (0, 1)$. Hence (3.1) implies $E(Y|X = x) \leq E(Y|X > x)$. If X and Y represent the lifetimes of two units, then the expected value of the second one is greater when the first one is alive at time x than when it fails at time x. Analogously, from (2.3) and Proposition 3.1, $(Y|X \leq x_1) \leq_{ST} (Y|X \leq x_2)$ holds for all G if, and only if,

$$\frac{C(F(x_1), v)}{F(x_1)} \geq \frac{C(F(x_2), v)}{F(x_2)} \tag{3.2}$$

for all $v \in (0, 1)$.

The random vector (X, Y) is said to be *Left Tail Decreasing* in Y, shortly written as LTD($Y|X$), if $\Pr(Y \leq y|X \leq x)$ is decreasing in x for all y, that is, if $(Y|X \leq x)$ is ST-increasing in x. Hence, from (3.2), it is LTD($Y|X$) for all F, G if and only if $C(u, v)/u$ is decreasing in u for all $v \in [0, 1]$. This property was given in [9] and [26, p. 192]. The analogous negative dependence property is defined as follows: (X, Y) is said to be *Left Tail Increasing* in Y, shortly written as LTI($Y|X$), if $\Pr(Y \leq y|X \leq x)$ is increasing in x for all y, that is, if $(Y|X \leq x)$ is ST-decreasing in x. Then (X, Y) is LTI($Y|X$) for all F, G if and only if $C(u, v)/u$ is increasing in u for all $v \in [0, 1]$. Similar properties can be obtained for the other orders. They can be used to define other tail dependence properties which only depend on copula properties. The respective conditions are given in the following proposition. It is well known that LTD($Y|X$) (resp. LTI($Y|X$)) implies the PQD (NQD) property of (X, Y) (see [26, p. 192]). Hence, from Table 1, all the positive (negative) dependence conditions given in the following proposition (except that for the MRL order) imply LTD($Y|X$) (resp. LTI($Y|X$)) and PQD (NQD). Hence, under these conditions, Spearman's rho and Kendall's tau coefficients are nonnegative (nonpositive). Therefore, all the properties given in this proposition can also be seen as positive (negative) dependence properties.

Proposition 3.5. *Let (X, Y) be a random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*

(i) $(Y|X \leq x)$ is ST-increasing (decreasing) in x for all F, G if and only if $C(u, v)/u$ is decreasing (increasing) in u for all $v \in (0, 1)$.

(ii) $(Y|X \leq x)$ is ST-increasing (decreasing) in x for all F, G if and only if $(v - \hat{C}(1 - u, v))/u$ is increasing (decreasing) in u for all $v \in (0, 1)$.

(iii) $(Y|X \leq x)$ is HR-increasing (decreasing) in x for all F, G if and only if $(v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(iv) $(Y|X \leq x)$ is RHR-increasing (decreasing) in x for all F, G if and only if $C(u_2, v)/C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(v) $(Y|X \leq x)$ is LR-increasing (decreasing) in x for all F, G if and only if $\partial_2 C(u_2, v)/\partial_2 C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.

(vi) $(Y|X \le x)$ *is MRL-increasing (decreasing) in* x *for all* F, G *if and only if* $(v - \hat{C}(u_2, v))/(v - \hat{C}(u_1, v))$ *is bathtub (upside-down bathtub) in* v *for all* $0 < u_1 \le u_2 < 1$ *and* $E(Y|X \le x)$ *is increasing (decreasing) in* x.

Analogously, from (2.5) and Proposition 3.1, $(Y|X > x_1) \le_{ST} (Y|X > x_2)$ holds for all G if, and only if,

$$\frac{\hat{C}(\bar{F}(x_1), v)}{\bar{F}(x_1)} \le \frac{\hat{C}(\bar{F}(x_2), v)}{\bar{F}(x_2)} \tag{3.3}$$

for all $v \in (0, 1)$. The random vector (X, Y) is said to be *Right Tail Increasing* in Y, shortly written as RTI$(Y|X)$, if $\Pr(Y > y|X > x)$ is increasing in x for all y, that is, if $(Y|X > x)$ is ST-increasing in x. Hence, from (3.3), it is RTI$(Y|X)$ for all F, G if and only if $\hat{C}(u, v)/u$ is decreasing in u for all $v \in [0, 1]$. The analogous negative dependence property is defined in a similar way: (X, Y) is said to be *Right Tail Decreasing* in Y, shortly written as RTD$(Y|X)$, if $\Pr(Y > y|X > x)$ is decreasing in x for all y, that is, if $(Y|X \le x)$ is ST-decreasing in x. Then (X, Y) is RTD$(Y|X)$ for all F, G if and only if $\hat{C}(u, v)/u$ is increasing in u for all $v \in [0, 1]$. Similar properties can be obtained for the other orders. They can be used to define other right tail dependence properties which only depend on copula properties. The respective conditions are given in the following proposition. Note that they can be obtained by replacing C with $\hat{C}$ in the preceding proposition. It is well known that RTI$(Y|X)$ (resp. RTD$(Y|X)$) implies the PQD (NQD) property of (X, Y). Hence, from Table 1, all the positive (negative) dependence conditions given in the following proposition (except that for the MRL order) imply RTI$(Y|X)$ (resp. RTD$(Y|X)$) and PQD (NQD). Hence, under these conditions, Spearman's rho and Kendall's tau coefficients are nonnegative (nonpositive).

Proposition 3.6. *Let* (X, Y) *be a random vector with copulas* C *and* $\hat{C}$ *and marginal distributions* F *and* G. *Then:*

(i) $(Y|X > x)$ *is ST-increasing (decreasing) in* x *for all* F, G *if and only if* $\hat{C}(u, v)/u$ *is decreasing (increasing) in* u *for all* $v \in (0, 1)$.

(ii) $(Y|X > x)$ *is ST-increasing (decreasing) in* x *for all* F, G *if and only if* $(v - C(1 - u, v))/u$ *is increasing (decreasing) in* u *for all* $v \in (0, 1)$.

(iii) $(Y|X > x)$ *is HR-increasing (decreasing) in* x *for all* F, G *if and only if* $\hat{C}(u_2, v)/\hat{C}(u_1, v)$ *is increasing (decreasing) in* v *for all* $0 < u_1 \le u_2 < 1$.

(iv) $(Y|X > x)$ *is RHR-increasing (decreasing) in* x *for all* F, G *if and only if* $(v - C(u_2, v))/(v - C(u_1, v))$ *is increasing (decreasing) in* v *for all* $0 < u_1 \le u_2 < 1$.

(v) $(Y|X > x)$ *is LR-increasing (decreasing) in* x *for all* F, G *if and only if* $\partial_2\hat{C}(u_2, v)/\partial_2\hat{C}(u_1, v)$ *is increasing (decreasing) in* v *for all* $0 < u_1 \le u_2 < 1$.

(vi) $(Y|X > x)$ *is MRL-increasing (decreasing) in* x *for all* F, G *if and only if* $\hat{C}(u_2, v)/\hat{C}(u_1, v)$ *is upside-down bathtub (bathtub) in* v *for all* $0 < u_1 \le u_2 < 1$ *and* $E(Y|X > x)$ *is increasing (decreasing) in* x.

Note that the conditions of the preceding proposition can be related with those in Proposition 3.3. For example, if $(Y|X > x)$ is ST-increasing (decreasing) in x, then $Y =_{ST} \lim_{x\to-\infty}(Y|X > x) \le_{ST} (Y|X > x)$ $(\ge_{ST})$, obtaining (i) of Proposition 3.3. We can do the same with the other orderings in the preceding proposition and with the properties in Proposition 3.5.

The representations for $(Y|X = x)$ can also be used to study the following dependence property defined in [26, p. 196]: Y is *Stochastically Increasing (Decreasing)* in X, shortly written as SI$(Y|X)$ (SD$(Y|X)$), if $(Y|X = x)$ is ST-increasing (decreasing) in x. A random vector (X, Y) is said to be positively (negative) dependent through stochastic ordering, shortly written as PDS (NDS), if it is both SI$(Y|X)$ and SI$(X|Y)$ (SD$(Y|X)$ and SD$(X|Y)$), see [5]. Hence, from representation (2.8), (X, Y) is PDS (NDS) if and only if $C(u, v)$ is concave (convex) in u and v (Theorem 5.2.11 in [26, p. 197]). Analogously, from representation (2.10), (X, Y) is PDS (NDS) if and only if $\hat{C}(u, v)$ is concave (convex) in u and v. We can define similar dependence properties by using the other orderings. The results for $(Y|X = x)$ are given in the following proposition. The results for $(X|Y = y)$ are similar. Again note that these dependence properties only depend on the underlying copula and that all of them (except that for the MRL) imply the SI$(Y|X)$ (SD$(Y|X)$) property.

Proposition 3.7. *Let (X, Y) be a random vector with absolutely continuous copulas C and $\hat{C}$, absolutely continuous marginal distributions F and G and probability density functions $f(x) = \int_{-\infty}^{+\infty} \boldsymbol{f}(x, z)dz$ and $g(y) = \int_{-\infty}^{+\infty} \boldsymbol{f}(z, y)dz$. Let $S_1 = \{x : f(x) > 0\}$. Then:*
(i) $(Y|X = x)$ is ST-increasing (decreasing) in x in the set S_1 for all F, G if and only if $C(u, v)$ is concave (convex) in u for all $v \in (0, 1)$.
(ii) $(Y|X = x)$ is ST-increasing (decreasing) in x in the set S_1 for all F, G if and only if $\hat{C}(u, v)$ is concave (convex) in u for all $v \in (0, 1)$.
(iii) $(Y|X = x)$ is HR-increasing (decreasing) in x in the set S_1 for all F, G if and only if $\partial_1 \hat{C}(u_2, v)/\partial_1 \hat{C}(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.
(iv) $(Y|X = x)$ is RHR-increasing (decreasing) in x in the set S_1 for all F, G if and only if $\partial_1 C(u_2, v)/\partial_1 C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.
(v) $(Y|X = x)$ is LR-increasing (decreasing) in x in the set S_1 for all F, G if and only if $\partial_{1,2} C(u_2, v)/\partial_{1,2} C(u_1, v)$ is increasing (decreasing) in v for all $0 < u_1 \leq u_2 < 1$.
(vi) $(Y|X = x)$ is MRL-increasing (decreasing) in x in the set S_1 for all F, G if and only if $\partial_1 \hat{C}(u_2, v)/\partial_1 \hat{C}(u_1, v)$ is upside-down bathtub (bathtub) in v for all $0 < u_1 \leq u_2 < 1$ and $E(Y|X = x)$ is increasing (decreasing) in x.

From the preceding results some surprising connections can be stated between some positive (or negative) dependence properties. For example, it is easy to prove that the following conditions are equivalent:
(i) (X, Y) is $LTD(X|Y)$.
(ii) $Y \geq_{RHR} (Y|X \leq x)$ for all x.
(iii) $Y \leq_{RHR} (Y|X > x)$ for all x.
(iv) $(Y|X \leq x) \leq_{RHR} (Y|X > x)$ for all x.
(v) $C(u, v)/v$ is decreasing in v for all u.
The equivalence between (i) and (iv) corresponds with the result in Lemma 2.1 (i) of [7]. Analogously, we can also prove that the following conditions are equivalent:
(i) (X, Y) is $RTI(X|Y)$.
(ii) $Y \geq_{HR} (Y|X \leq x)$ for all x.
(iii) $Y \leq_{HR} (Y|X > x)$ for all x.
(iv) $(Y|X \leq x) \leq_{HR} (Y|X > x)$ for all x.
(v) $\hat{C}(u, v)/v$ is decreasing in v for all u.
The equivalence between (i) and (iv) corresponds with the result in Lemma 2.1 (ii) of [7]. Analogously, for the SI notion, we have that the following conditions are equivalent:
(i) (X, Y) is $SI(X|Y)$.
(ii) $Y \geq_{LR} (Y|X \leq x)$ for all x.
(iii) $Y \leq_{LR} (Y|X > x)$ for all x.
(iv) $(Y|X \leq x) \leq_{LR} (Y|X > x)$ for all x.
(v) $C(u, v)$ is concave in v for all u.
(vi) $\hat{C}(u, v)$ is concave in v for all u.
Analogous equivalences can be stated for the respective negative dependence notions.

We can go further by taking into account the following two positive dependence properties (see, e.g., [26, p. 198]): X and Y are *Left Corner Set Decreasing* (shortly written as $LCSD$) if $\Pr(X \leq x_1, Y \leq y_1|X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2 for all x_1 and y_1. Similarly, X and Y are *Right Corner Set Increasing* (shortly written as $RCSI$) if $\Pr(X > x_1, Y > y_1|X > x_2, Y > y_2)$ is increasing in x_2 and y_2 for all x_1 and y_1. It is shown in Corollary 5.2.17 of [26, p. 200] that $LCSD$ holds if and only if C is *totally positive of order two* (TP_2), which means that $C(u_1, v_1)\, C(u_2, v_2) \geq C(u_1, v_2)\, C(u_2, v_1)$ for all $0 < u_1 \leq u_2 < 1$ and $0 < v_1 \leq v_2 < 1$. Similarly, it is shown that $RCSI$ holds if and only if $\hat{C}$ is TP_2. Since C is TP_2 if and only if $C(u_2, v)/C(u_1, v)$ is increasing in v for all $0 < u_1 \leq u_2 < 1$, it follows from Proposition 3.5 (iv) that the following conditions are equivalent:
(i) $LCSD$.
(ii) $(Y|X \leq x)$ is RHR-increasing.
(iii) C is TP_2.

By using a similar argument, it follows from Proposition 3.6 (*iii*) that the following conditions are equivalent:
(i) *RCSI*.
(ii) $(Y|X > x)$ is HR-increasing.
(iii) $\hat{C}$ is TP_2.
Similar results can be obtained for the analogous negative dependence properties *Left Corner Set Increasing* (*LCSI*) and *Right Corner Set Decreasing* (*RCSD*).

The dependence properties given in Proposition 3.7 can be shortly written as $SI_{ORD}(Y|X)$ and $SI_{ORD}(X|Y)$ replacing *ORD* with the respective orderings. For example, $SI_{HR}(Y|X)$ means that $(Y|X = x)$ is HR-increasing in x. The negative dependence properties can be written as $SD_{ORD}(Y|X)$ and $SD_{ORD}(X|Y)$. From Proposition 3.7 (*iv*), we have the following equivalences:
(i) $SI_{RHR}\,(X|Y)$.
(ii) $(Y|X \le x)$ is LR-increasing.
(iii) $\partial_2 C$ is TP_2.
Moreover, from Proposition 3.7 (*iii*), the following properties are also equivalent:
(i) $SI_{HR}\,(X|Y)$.
(ii) $(Y|X > x)$ is LR-increasing.
(iii) $\partial_2 \hat{C}$ is TP_2.
The preceding equivalences can be used to prove the following relationships from the ordering relationships given in Table 1. Some of them are well known (see, Figure 5.8 in [26, p. 200]).

$SI_{LR}(X\|Y)$	$\Rightarrow$	$SI_{RHR}(X\|Y)$	$\Rightarrow$	$LCSD$
$\Downarrow$		$\Downarrow$		$\Downarrow$
$SI_{HR}(X\|Y)$	$\Rightarrow$	$SI_{ST}(X\|Y)$	$\Rightarrow$	$LTD(X\|Y)$
$\Downarrow$		$\Downarrow$		$\Downarrow$
$RCSI$	$\Rightarrow$	$RTI(X\|Y)$	$\Rightarrow$	PQD

Analogous relationships hold for $(Y|X)$ and for the respective negative dependence properties. Moreover note that the copulas given in Exercises 5.30, 5.32 and 5.33 of [26, p. 204-205] can be used to prove that $Y \ge_{HR} (Y|X \le x)$ (or $Y \ge_{RHR} (Y|X \le x)$) is not equivalent to $Y \ge_{LR} (Y|X \le x)$ and that the conditions in Proposition 3.5 (*i*) and (*iv*) (or in Proposition 3.6 (*i*) and (*iii*)) are not equivalent.

Similar results to those given in this section can be stated for the general n-dimensional case by using Proposition 3.1 and representations (2.11), (2.12) and (2.13) obtained in the preceding section. They can be used to define the respective dependence multivariate notions.

4 Bounds

The representations given in Section 2 can also be used to obtain bounds for conditional distribution (or reliability) functions and conditional expectations. The results are based on the corresponding results for distorted distributions obtained in [6, 13, 25]. Again we just state the results for the bidimensional case. The results for the general case can be stated in a similar way. For the first conditional distribution we obtain the following bounds.

Proposition 4.1. *Let (X, Y) be a random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*

$$G(y) \inf_{u\in(0,1]} \frac{C(F(x),u)}{uF(x)} \le \Pr(Y \le y|X \le x) \le G(y) \sup_{u\in(0,1]} \frac{C(F(x),u)}{uF(x)} \tag{4.1}$$

and

$$\bar{G}(y) \inf_{u\in(0,1]} \frac{u - \hat{C}(\bar{F}(x),u)}{uF(x)} \le \Pr(Y > y|X \le x) \le \bar{G}(y) \sup_{u\in(0,1]} \frac{u - \hat{C}(\bar{F}(x),u)}{uF(x)} \tag{4.2}$$

whenever $F(x) > 0$. *The bounds are sharp.*

Proof. The bounds in (4.1) trivially hold when $G(y) = 0$ and $F(x) > 0$. For x, y such that $F(x)G(y) > 0$, from (2.3), we have

$$\frac{\Pr(Y \le y|X \le x)}{G(y)} = \frac{C(F(x), G(y))}{G(y)F(x)} \le \sup_{u\in(0,1]} \frac{C(F(x), u)}{uF(x)}$$

and we obtain the upper bound in (4.1). The lower bound is obtained in a similar way.

To show that the upper bound in (4.1) can be attained we consider two cases. If the supremum is attained at $u_0 \in (0, 1]$, we consider the distribution function

$$G(y) = \begin{cases} 0 & \text{for } y < 0 \\ u_0 & \text{for } 0 \le y < 1 \\ 1 & \text{for } y \ge 1 \end{cases}$$

and then, for $0 \le y < 1$, we have

$$\frac{\Pr(Y \le y|X \le x)}{G(y)} = \frac{C(F(x), u_0)}{u_0 F(x)} = \sup_{u\in(0,1]} \frac{C(F(x), u)}{uF(x)}.$$

If the supremum is attained when $u \to 0^+$, then it is attained in the limit when $n \to \infty$ with the following distributions

$$G_n(y) = \begin{cases} 0 & \text{for } y < 0 \\ u_n & \text{for } 0 \le y < 1 \\ 1 & \text{for } y \ge 1 \end{cases}$$

where $u_n \to 0^+$ when $n \to \infty$. The proofs for the other bounds are similar. □

As an immediate consequence we have the following bounds for the conditional expectations when the random vector is nonnegative.

Corollary 4.2. *Let* (X, Y) *be a nonnegative random vector with copulas* C *and* $\hat{C}$ *and marginal distributions* F *and* G. *Then:*

$$E(Y) \inf_{u\in(0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)} \le E(Y|X \le x) \le E(Y) \sup_{u\in(0,1]} \frac{u - \hat{C}(\bar{F}(x), u)}{uF(x)}$$

whenever $F(x) > 0$. *The bounds are sharp.*

The proof is an immediate consequence of (4.2) and the representation of the mean for nonnegative random variables as

$$E(Y|X \le x) = \int_0^\infty \Pr(Y > y|X \le x)dy.$$

If Y can take negative values, similar bounds can be obtained by using (4.1) and (4.2) and the representation of the mean as

$$E(Y|X \le x) = -\int_{-\infty}^0 \Pr(Y \le y|X \le x)dy + \int_0^\infty \Pr(Y > y|X \le x)dy.$$

Analogously, for the other conditional distributions, we have the following results. The proofs are similar.

Proposition 4.3. *Let* (X, Y) *be a random vector with copulas* C *and* $\hat{C}$ *and marginal distributions* F *and* G. *Then:*

$$G(y) \inf_{u\in(0,1]} \frac{u - C(F(x), u)}{u\bar{F}(x)} \le \Pr(Y \le y|X > x) \le G(y) \sup_{u\in(0,1]} \frac{u - C(F(x), u)}{u\bar{F}(x)}$$

and

$$\bar{G}(y) \inf_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)} \le \Pr(Y > y|X > x) \le \bar{G}(y) \sup_{u\in(0,1]} \frac{\hat{C}(\bar{F}(x), u)}{u\bar{F}(x)}$$

whenever $\bar{F}(x) > 0$. *The bounds are sharp.*

Corollary 4.4. *Let (X, Y) be a nonnegative random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*

$$E(Y)\inf_{u\in(0,1]}\frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)}\le E(Y|X>x)\le E(Y)\sup_{u\in(0,1]}\frac{\hat{C}(\bar{F}(x),u)}{u\bar{F}(x)}$$

whenever $\bar{F}(x)>0$. The bounds are sharp.

Proposition 4.5. *Let (X, Y) be a random vector with absolutely continuous copulas C and $\hat{C}$, absolutely continuous marginal distributions F and G and marginal density functions $f(x) = \int_{-\infty}^{+\infty}\boldsymbol{f}(x,z)dz$ and $g(y) = \int_{-\infty}^{+\infty}\boldsymbol{f}(z,y)dz$. Then there exists a version of the conditional probability such that:*

$$G(y)\inf_{u\in(0,1]}\frac{\partial_1 C(F(x),u)}{u}\le \Pr(Y\le y|X=x)\le G(y)\sup_{u\in(0,1]}\frac{\partial_1 C(F(x),u)}{u}$$

and

$$\bar{G}(y)\inf_{u\in(0,1]}\frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}\le \Pr(Y> y|X=x)\le \bar{G}(y)\sup_{u\in(0,1]}\frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}$$

whenever $f(x)>0$. The bounds are sharp.

Corollary 4.6. *Under the assumptions of the preceding proposition we have*

$$E(Y)\inf_{u\in(0,1]}\frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}\le E(Y|X=x)\le E(Y)\sup_{u\in(0,1]}\frac{\partial_1 \hat{C}(\bar{F}(x),u)}{u}$$

whenever $f(x)>0$ (and these expectations and partial derivative exist). The bounds are sharp.

Note that we have obtained bounds for the regression curve $E(Y|X = x)$ which are distribution-free with respect to the distribution of Y. Some illustrative examples are given in the following section.

Finally we use the technique developed in [13] (see also [6]) to obtain bounds in terms of the Gini mean difference dispersion measure defined by

$$\Delta_G = 2\int_{-\infty}^{\infty} G(y)(1-G(y))dy.$$

Thus we obtain the following bounds.

Proposition 4.7. *Let (X, Y) a be nonnegative random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*

$$\inf_{u\in(0,1)}\frac{u\bar{F}(x)-\hat{C}(\bar{F}(x),u)}{2u(1-u)F(x)}\le\frac{E(Y|X\le x)-E(Y)}{\Delta_G}\le\sup_{u\in(0,1)}\frac{u\bar{F}(x)-\hat{C}(\bar{F}(x),u)}{2u(1-u)F(x)} \tag{4.3}$$

whenever $F(x)>0$, $\Delta_G>0$ and $0=\inf\{y: G(y)>0\}$. The bounds are sharp.

Proof. Let $\beta=\sup\{y:\bar{G}(y)>0\}$. Then, from (2.4), we have

$$\begin{aligned}E(Y|X\le x)-E(Y) &= \int_0^{\infty}\frac{\bar{G}(y)-\hat{C}(\bar{F}(x),\bar{G}(y))}{F(x)}-\bar{G}(y)dy\\ &= \int_0^{\beta}\frac{\bar{G}(y)\bar{F}(x)-\hat{C}(\bar{F}(x),\bar{G}(y))}{2F(x)\bar{G}(y)(1-\bar{G}(y))}2\bar{G}(y)(1-\bar{G}(y))dy\\ &\le \sup_{u\in(0,1)}\frac{u\bar{F}(x)-\hat{C}(\bar{F}(x),u)}{2u(1-u)F(x)}\int_0^{\beta}2G(y)(1-G(y))dy\\ &\le \Delta_G\sup_{u\in(0,1)}\frac{u\bar{F}(x)-\hat{C}(\bar{F}(x),u)}{2u(1-u)F(x)}\end{aligned}$$

and we obtain the upper bound in (4.3). The lower bound is obtained in a similar way. To prove that they are sharp we proceed as in Proposition 4.1. □

Analogously, for the other conditional expectations, we have the following bounds.

Proposition 4.8. *Let (X, Y) be a nonnegative random vector with copulas C and $\hat{C}$ and marginal distributions F and G. Then:*

$$\inf_{u\in(0,1)} \frac{\hat{C}(\bar{F}(x), u) - u\bar{F}(x)}{2u(1-u)\bar{F}(x)} \leq \frac{E(Y|X > x) - E(Y)}{\Delta_G} \leq \sup_{u\in(0,1)} \frac{\hat{C}(\bar{F}(x), u) - u\bar{F}(x)}{2u(1-u)\bar{F}(x)}$$

whenever $F(x) > 0$, $\Delta_G > 0$ and $0 = \inf\{y : G(y) > 0\}$. The bounds are sharp.

Proposition 4.9. *Let (X, Y) be a nonnegative random vector with absolutely continuous copulas C and $\hat{C}$, absolutely continuous marginal distributions F and G and marginal density functions f and g. Then there exists a version of the conditional expected value such that:*

$$\inf_{u\in(0,1)} \frac{\partial_1\hat{C}(\bar{F}(x), u) - u}{2u(1-u)} \leq \frac{E(Y|X = x) - E(Y)}{\Delta_G} \leq \sup_{u\in(0,1)} \frac{\partial_1\hat{C}(\bar{F}(x), u) - u}{2u(1-u)}$$

whenever $f(x) = \int_{-\infty}^{+\infty} \boldsymbol{f}(x, z)dz > 0$, $\Delta_G > 0$ and $0 = \inf\{y : G(y) > 0\}$. The bounds are sharp.

5 Applications

5.1 Applications in Risk Theory

Let (X, Y) be a random vector describing losses of a portfolio of risks, with copula C and marginal distribution functions F and G. In portfolio risk theory, conditional distributions of the form $(Y|X \leq x)$, $(Y|X > x)$ and $(Y|X = x)$ are specially interesting when $x = \mathrm{VaR}_\alpha[X]$, where

$$\mathrm{VaR}_\alpha[X] = F^{-1}(\alpha) = \sup\{x : F(x) \leq \alpha\},\ \alpha \in (0, 1),$$

is the value-at-risk (VaR) of X at level α (or the α-quantile of X), the benchmark risk measure in today's financial world. It is well-known that $X \leq_{ST} Y$ if and only if $\mathrm{VaR}_\alpha[X] \leq \mathrm{VaR}_\alpha[Y]$ for all $\alpha \in (0, 1)$. In this framework, conditional distributions of the form $(Y|X > \mathrm{VaR}_\alpha[X])$, $(Y|X \leq \mathrm{VaR}_\alpha[X])$ and $(Y|X = \mathrm{VaR}_\alpha[X])$, for some $\alpha \in (0, 1)$, describe the risk of one component (or even the aggregate risk of the portfolio) given that the other component is under stress. Risk measures associated to these conditional distributions are called co-risk measures (the prefix "co" stands for conditional, comovement or contagion). They are used to assess the systemic risk, which is related to the risk that the failure or loss of a component spreads to another one or even to the system. There are basically two approaches to adjust VaR to dependence between X and Y. The two notions appear in the literature under the name Conditional Value-at-Risk (CoVaR). The first one, introduced in [11], is based on the stress scenario $\{X \leq \mathrm{VaR}_\alpha[X]\}$ (which, depending on the context, sometimes takes the form $\{X > \mathrm{VaR}_\alpha[X]\}$). The second one was introduced in [1] based on the stress scenario $\{X = \mathrm{VaR}_\alpha[X]\}$. The following definition follows the notation of [15].

Definition 2. *For $\alpha, \beta \in (0, 1)$, we define:*
(a) $CoVaR^{\leq}_{\alpha,\beta}[Y|X] = VaR_\beta(Y|X \leq VaR_\alpha[X])$,
(b) $CoVaR^{>}_{\alpha,\beta}[Y|X] = VaR_\beta(Y|X > VaR_\alpha[X])$,
(c) $CoVaR^{=}_{\alpha,\beta}[Y|X] = VaR_\beta(Y|X = VaR_\alpha[X])$.

In words: CoVaR in (a) (respectively, (b), (c)) is the Value-at-Risk at level β of the conditional distribution of Y given $\{X \leq \mathrm{VaR}_\alpha[X]\}$ (respectively, $\{X > \mathrm{VaR}_\alpha[X]\}$, $\{X = \mathrm{VaR}_\alpha[X]\}$). Due to the increasing interest in the study of the systemic risk, some papers related to CoVaR have recently appeared in the literature. For example, [4] and [12] provide a directory of CoVaR values (of type a) for different families of copulas and [30] studies the consistency of CoVaRs (of type b) and their contributions to risk with respect to different stochastic orderings under different positive dependence assumptions. Now we give a result on the consistency of CoVaR of type (c) with respect to the stochastic order of the marginals. Whenever we use $\text{CoVaR}^{=}_{\alpha,\beta}[Y|X]$, we assume that the (a version) conditional random variable $(Y|X = \text{VaR}_\alpha[X])$ exists.

Proposition 5.1. *Let (X, Y) and (X', Y') be two random vectors with the same copula C. Then $Y \le_{ST} Y'$ implies*

$$CoVaR^{=}_{\alpha,\beta}[Y|X] \le CoVaR^{=}_{\alpha,\beta}\left[Y'|X'\right], \ \forall \alpha, \beta \in (0,1). \tag{5.1}$$

Proof. Let G and G' be the marginal distributions of Y and Y' respectively and let $\alpha \in (0,1)$. Then we know from (2.10) that

$$\Pr(Y > y|X = \text{VaR}_\alpha[X]) = h(\bar{G}(y))$$

and

$$\Pr(Y' > y|X' = \text{VaR}_\alpha\left[X'\right]) = h(\bar{G}'(y))$$

where $h(u) = 1 - \partial_1 C(\alpha, 1-u)$. Clearly, the condition $\bar{G}(y) \le \bar{G}'(y)$ holds for all y implies $h(\bar{G}(y)) \le h(\bar{G}'(y))$ for all y, that is,

$$(Y|X = \text{VaR}_\alpha[X]) \le_{ST} (Y'|X' = \text{VaR}_\alpha\left[X'\right]) \ \forall \alpha \in (0,1)$$

which is the same as (5.1). □

Another important univariate risk measure (which is more sensitive than VaR to the losses in the tail of the distribution) is the expected shortfall (ES) given by

$$\text{ES}_\alpha[X] = \frac{1}{1-\alpha}\int_\alpha^1 \text{VaR}_t[X]\,dt, \ \alpha \in (0,1).$$

It is well-known (see, for example, Lemma 2.1 in [32]) that

$$X \le_{ICX} Y \text{ if and only if } \text{ES}_\alpha[X] \le \text{ES}_\alpha[Y] \ \forall \alpha \in (0,1). \tag{5.2}$$

In [15], ES is adjusted to dependence between X and Y defining CoES as follows.

Definition 3. *For $\alpha, \beta \in (0,1)$, we define:*
(a) $CoES^{\le}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 CoVaR^{\le}_{\alpha,t}[Y|X]\,dt$,
(b) $CoES^{>}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 CoVaR^{>}_{\alpha,t}[Y|X]\,dt$,
(c) $CoES^{=}_{\alpha,\beta}[Y|X] = \frac{1}{1-\beta}\int_\beta^1 CoVaR^{=}_{\alpha,t}[Y|X]\,dt$.

In words: CoES in (a) (respectively, (b), (c)) is the Expected Shartfall at level β of the conditional distribution of Y given $\{X \le \text{VaR}_\alpha[X]\}$ (respectively, $\{X > \text{VaR}_\alpha[X]\}$, $\{X = \text{VaR}_\alpha[X]\}$). [15] and [30] study the consistency of CoES of types (a) and (b) with respect to copula dependence parameters in the case where the marginals are stochastically ordered. Now we make a similar study in terms of CoES of type (c).

Proposition 5.2. *Let (X, Y) and (X', Y') be two random vectors with the same copula C such that $\partial_1 C(\alpha, u)$ is convex in u. Then $Y \le_{ICX} Y'$ implies*

$$CoES^{=}_{\alpha,\beta}[Y|X] \le CoES^{=}_{\alpha,\beta}\left[Y'|X'\right], \ \forall \alpha, \beta \in (0,1). \tag{5.3}$$

Proof. We have seen in the proof of Proposition 5.1 that $(Y|X = \text{VaR}_\alpha[X])$ and $(Y'|X' = \text{VaR}_\alpha\left[X'\right])$ are random variables with survival functions $h(\bar{G}(y))$ and $h(\bar{G}'(y))$, respectively, where h is given by $h(u) = 1-\partial_1 C(\alpha, 1-u)$. From the assumptions, $h(u)$ is a concave distortion. Then it follows from $Y \le_{ICX} Y'$ and Theorem 2.6 (v) in [21] that

$$(Y|X = \text{VaR}_\alpha[X]) \le_{ICX} (Y'|X' = \text{VaR}_\alpha\left[X'\right]) \ \forall \alpha \in (0,1).$$

From this inequality, by using (5.2), we obtain (5.3). □

5.2 An application in Economics

The variance $Var(X)$ and the Gini mean difference Δ_G can be defined as special cases of a covariance (see, e.g., [39]):

$$Var(Y) = Cov(Y, Y), \ \ \Delta_G = 4Cov(Y, G(Y)).$$

Given a random vector (X, Y) with marginal distributions F and G, both measures can be generalized to study dependence properties between X and Y. On one hand, we have the covariance $Cov(X, Y)$ and its standarized version, the Pearson's correlation coefficient, given by

$$\rho = \frac{Cov(X, Y)}{\left(Var(X)Var(Y)\right)^{1/2}},$$

a very popular symmetric measure of dependence. On the other hand, we have two covariance-equivalents associated with Δ_G given by $Cov(X, G(Y))$ and $Cov\left(Y, F(X)\right)$ and the corresponding asymmetric Gini correlation coefficients given by

$$\Gamma_{XY} = \frac{Cov(X, G(Y))}{Cov(X, F(X))}, \quad \Gamma_{YX} = \frac{Cov(Y, F(X))}{Cov(Y, G(Y))}.$$

Many properties and applications of Gini correlation coefficients can be found in [37–39]. For some connections with stochastic dominance see [29]. The Gini correlation coefficients are related to the absolute concentration curve (see [37]), a tool used in the field of income distributions to describe the impact of taxes on income distributions. It is defined as follows.

Definition 4. *The absolute curve of concentration of Y with respect to X, denoted by $A_{Y\circ X}$, is defined by*

$$A_{Y\circ X}(p) = \int_{-\infty}^{F^{-1}(p)} m(x)dF(x), \ \ 0 \le p \le 1,$$

where $m(x) = E\left(Y \mid X = x\right)$.

It is well-known (see [38]), that

$$Cov\left(Y, F(X)\right) = \int_0^1 [pE(Y) - A_{Y\circ X}(p)]\, dp.$$

Consequently,

$$\Gamma_{YX} = \frac{4\int_0^1 [pE(Y) - A_{Y\circ X}(p)]\, dp}{\Delta_G}.$$

Observing that $A_{Y\circ X}(p) = pE\left(Y \mid X \le F^{-1}(p)\right)$, we can write

$$\Gamma_{YX} = \frac{4\int_0^1 p\left[E(Y) - E\left(Y \mid X \le F^{-1}(p)\right)\right] dp}{\Delta_G}.$$

Now, from Proposition 4.7 we obtain, after a straightforward manipulation, the following bounds for Γ_{YX}, which only depend on the copula of the vector:

$$\inf_{u\in(0,1)} \frac{\int_0^1 \left[\widehat{C}(1-p, u) - u(1-p)\right] dp}{2u(1-u)} \le \Gamma_{YX} \le \sup_{u\in(0,1)} \frac{\int_0^1 \left[\widehat{C}(1-p, u) - u(1-p)\right] dp}{2u(1-u)}$$

Bounds for Γ_{XY} can be obtained similarly. Even more, using that

$$p\left[E(Y) - E\left(Y \mid X \le F^{-1}(p)\right)\right] = (1-p)\left[E\left(Y \mid X > F^{-1}(p)\right) - E(Y)\right], \ p \in (0, 1),$$

we see that Γ_{YX} can alternatively be written as

$$\Gamma_{YX} = \frac{4\int_0^1 (1-p)\left[E\left(Y \mid X > F^{-1}(p)\right) - E(Y)\right] dp}{\Delta_G}$$

and we may obtain bounds for Γ_{YX} using Proposition 4.8 (rather than Proposition 4.7).

6 Examples

In this section we study ordering properties and bounds for conditional distributions with specific dependence models (copulas). In the first example, we study the results for a specific Clayton-Oakes copula.

Example 1. *Let us consider a random vector (X, Y) with the following Clayton-Oakes (distributional) copula:*

$$C(u, v) = \frac{uv}{u + v - uv}$$

for $0 \le u, v \le 1$. *Then the distortion function of* $(Y|X \le x)$ *is*

$$q_1(u) = \frac{u}{F(x) + u - uF(x)}.$$

From Proposition 3.2, (i), we have $Y \ge_{ST} (Y|X \le x)$ *since* $C(F(x), u) \ge uF(x)$ *for all* $0 < u, F(x) < 1$. *Even more, as*

$$C(F(x), u) = \frac{uF(x)}{F(x) + u - uF(x)}$$

is concave in u in the interval $(0, 1)$, *from Proposition 3.2,* (v), *we have* $Y \ge_{LR} (Y|X \le x)$ *for all* F, G *and* x *such that* $F(x) > 0$. *In a similar way, from Proposition 3.3,* (v), *we can prove that* $Y \le_{LR} (Y|X > x)$ *for all* F, G *and* x *such that* $\bar{F}(x) > 0$. *Even more, from Propositions 3.5, 3.6 and 3.7,* (v), *we can prove that* $(Y|X \le x)$, $(Y|X > x)$ *and* $(Y|X = x)$ *are LR-increasing in* x. *Hence we have a positive dependence. Spearman's rho and Kendall's tau are given by* $\rho_S = -39 + 4\pi^2 \cong 0.478417$ *and* $\tau = 1/3$.

Analogously, we obtain

$$q_3(u) = \partial_1 C(F(x), u) = \left(\frac{u}{F(x) + u - uF(x)}\right)^2 = (q_1(u))^2.$$

Hence, as $q_3(u)/q_1(u) = q_1(u)$ *is increasing, we have from Proposition 3.1, (iii), that* $(Y|X = x) \ge_{RHR} (Y|X \le x)$ *for all* F, G *and* x *such that* $F(x), f(x) > 0$. *Even more, as* $q_3'(u)/q_1'(u) = 2q_1(u)$ *is increasing, we have from Proposition 3.1, (iv), that* $(Y|X = x) \ge_{LR} (Y|X \le x)$ *for all absolutely continuous distributions* F, G *and* x *such that* $F(x), f(x) > 0$. *From Proposition 3.4,* (iii), *we have* $Y \le_{HR} (Y|X = x)$ *when* $F(x) \ge 1/2$ *and* $f(x) > 0$. *However, they are not ST ordered when* $0 < F(x) < 1/2$.

Moreover, from Section 4, we can obtain bounds for the conditional distribution (reliability) functions and expectations. Thus, from Proposition 4.1, for $F(x) > 0$, *we have*

$$G(y) \le \Pr(Y \le y|X \le x) \le \frac{1}{F(x)} G(y)$$

and

$$0 \le \Pr(Y > y|X \le x) \le \bar{G}(y).$$

Then $E(Y|X \le x) \le E(Y)$. *Note that, from the first expression, a lower bound for* $\Pr(Y > y|X \le x)$ *is*

$$1 - \frac{1}{F(x)} G(y) \le \Pr(Y > y|X \le x).$$

Analogously, from Proposition 4.5, for $f(x) > 0$, *we have*

$$0 \le \Pr(Y \le y|X = x) \le G(y)$$

when $F(x) \ge 1/2$ *and*

$$0 \le \Pr(Y \le y|X = x) \le \frac{1}{4F(x)\bar{F}(x)} G(y)$$

when $F(x) < 1/2$.

For the reliability functions, we have

$$\bar{G}(y) \le \Pr(Y > y|X = x) \le 2F(x)\bar{G}(y)$$

when $F(x) \geq 2/3$ *and*

$$\min(1, 2F(x))\bar{G}(y) \leq \Pr(Y > y|X = x) \leq \frac{(2 - F(x))^2}{4 - 4F(x)}\bar{G}(y)$$

when $F(x) < 2/3$. *Then for nonnegative random vectors we have*

$$E(Y) \leq E(Y|X = x) \leq 2F(x)E(Y)$$

when $F(x) \geq 2/3$ *and*

$$\min(1, 2F(x))E(Y) \leq E(Y|X = x) \leq \frac{(2 - F(x))^2}{4 - 4F(x)}E(Y)$$

when $F(x) < 2/3$.

In the following example we study the results for all the Farlie-Gumbel-Morgenstern (FGM) bidimensional copulas which include positive and negative (weak) dependence.

Example 2. *Let us consider the random vector* (X, Y) *with the following FGM distributional copula:*

$$C(u, v) = uv[1 + \theta(1 - u)(1 - v)]$$

for $0 \leq u, v \leq 1$ *and* $-1 \leq \theta \leq 1$. *In this case, the survival copula coincides with the distributional copula, that is,* $\hat{C}(u, v) = C(u, v)$ *for all* $0 \leq u, v \leq 1$.

The function

$$C(F(x), u) = uF(x)[1 + \theta(1 - u)\bar{F}(x)]$$

is concave (convex) in u in the interval $(0, 1)$ *for* $\theta > 0$ *(*$\theta < 0$*). Hence from Proposition 3.2,* (v), *we have* $Y \geq_{LR} (Y|X \leq x)$ *(*$\leq_{LR}$*) for all* $\theta > 0$ *(*$\theta < 0$*) and for all* F, G *and x such that* $F(x) > 0$. *In a similar way, from Proposition 3.3,* (v), *we can prove that* $Y \leq_{LR} (Y|X > x)$ *(*$\geq_{LR}$*) for all* $\theta > 0$ *(*$\theta < 0$*) and for all* F, G *and x such that* $\bar{F}(x) > 0$. *Of course, if* $\theta = 0$ *(product copula), we have* $Y =_{ST} (Y|X \leq x) =_{ST} (Y|X > x)$ *(since they are independent). Analogously, the function*

$$\partial_1 C(F(x), u) = u(1 + \theta - 2\theta F(x)) + u^2\theta(2F(x) - 1)$$

is convex (concave) in u if $\theta(2F(x)-1) > 0$ *(*$\theta(2F(x)-1) < 0$*). Then we have* $Y \geq_{LR} (Y|X = x)$ *(*$\leq_{LR}$*) if* $\theta(2F(x)-1) > 0$ *(*$\theta(2F(x) - 1) < 0$*), for all absolutely continuous distributions* F, G *and for all x such that* $f(x) > 0$. *In a similar way, it can be proved that* $(Y|X \leq x)$, $(Y|X > x)$ *and* $(Y|X = x)$ *are LR-increasing (decreasing) in x when* $\theta > 0$ *(*$\theta < 0$*). From Proposition 4.1, we obtain the following bounds*

$$G(y) \leq \Pr(Y \leq y|X \leq x) \leq (1 + \theta\bar{F}(x))G(y)$$

for $\theta \geq 0$ *and*

$$(1 + \theta\bar{F}(x))G(y) \leq \Pr(Y \leq y|X \leq x) \leq G(y)$$

for $\theta \leq 0$. *For the reliability functions we have*

$$(1 - \theta\bar{F}(x))\bar{G}(y) \leq \Pr(Y > y|X \leq x) \leq \bar{G}(y)$$

for $\theta \geq 0$ *and*

$$\bar{G}(y) \leq \Pr(Y > y|X \leq x) \leq (1 - \theta\bar{F}(x))\bar{G}(y)$$

for $\theta \leq 0$. *Finally, for the expectations we have*

$$\min(1, 1 - \theta\bar{F}(x))E(Y) \leq E(Y|X \leq x) \leq \max(1, 1 - \theta\bar{F}(x))E(Y).$$

Sometimes we can study the properties obtained above in families of copulas. Let us see an example where we consider strict Archimedean copulas as in Exercice 5.34 of Nelsen [26, p. 205] (see also [7]).

Example 3. *If the copula C is a strict Archimedean copula then it can be written as*

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v)),$$

where ϕ is a strict generator (see, e.g., [26, p. 112]), that is, ϕ is a strictly decreasing continuous and convex function from $[0,1]$ *to* $[0,\infty]$ *such that* $\phi(0) = \infty$ *and* $\phi(1) = 0$. *Then*

$$\phi(C(u,v)) = \phi(u) + \phi(v)$$

and, if we assume that ϕ is differentiable, we have

$$\partial_1 C(u,v) = \frac{\phi'(u)}{\phi'(C(u,v))}$$

whenever $\phi'(C(u,v)) < 0$.

From Proposition 3.5 (i) *we know that* $(Y|X \le x)$ *is ST-increasing (decreasing) in x, i.e.,* (X, Y) *is LTD*$(Y|X)$ *(LTI*$(Y|X)$*), if and only if, the function* $g(u) = C(u,v)/u$ *is decreasing in* $(0,1)$ *for all* $v \in (0,1)$. *Differentiating, we have*

$$g'(u) =_{sign} u\partial_1 C(u,v) - C(u,v) =_{sign} C(u,v)\phi'(C(u,v)) - u\phi'(u).$$

Moreover, we know that $u = C(u,1) \ge C(u,v)$. *Hence if* $u\phi'(u)$ *is increasing (decreasing) in* $(0,1)$, *then* $g'(u) \le 0$ $(\ge)$ *and* $(Y|X \le x)$ *is ST-increasing (decreasing) in x for all F, G. Moreover,* $(X|Y \le y)$ *is also ST-increasing (decreasing) in y for all F, G. Similar properties can be obtained for* $(Y|X > x)$ *by using Proposition 3.6,* (i) *when the survival copula is a strict Archimedean copula.*

Therefore it is easy to study if $(Y|X \le x)$ *is ST-increasing (decreasing) in x for a strict Archimedean copula with a given generator (some of them can be seen in, e.g., Table 4.1 of [26, p. 116]). For example, if* $\phi(u) = -\ln(u)$, *then* $u\phi'(u) = -1$ *is constant and we have that* $(Y|X \le x)$ *is ST-constant as expected since this generator leads to the product copula. In a similar way the Clayton-Oakes family of copulas is obtained from* $\phi(u) = (u^{-\theta} - 1)/\theta$ *which is a strict generator when* $\theta > 0$ *(see line 1 in Table 4.1 of [26, p. 116]). Hence* $u\phi'(u) = -u^{-\theta}$ *is increasing and then* $(Y|X \le x)$ *is ST-increasing in x for all F, G. Analogously, the Ali-Mikhail-Hak family of copulas is obtained from* $\phi(u) = \ln(1-\theta(1-u)) - \ln(u)$ *for* $\theta \in [-1,1)$ *(see line 3 in Table 4.1 of [26, p. 116]). Hence*

$$u\phi'(u) = -1 + \frac{u\theta}{1-\theta+\theta u}$$

which is increasing (decreasing) when $\theta \ge 0$ $(\theta \le 0)$ *and then* $(Y|X \le x)$ *is ST-increasing (decreasing) in x for all F, G when* $\theta \ge 0$ $(\theta \le 0)$. *Here the product copula is obtained with* $\theta = 0$. *Finally, the Gumbel family of copulas is obtained from* $\phi(u) = (-\ln(u))^{\theta}$ *for* $\theta \ge 1$ *(see line 4 in Table 4.1 of [26, p. 116]). Hence* $u\phi'(u) = -\theta(-\ln u)^{\theta-1}$ *is increasing and then* $(Y|X \le x)$ *is ST-increasing in x for all F, G.*

We can study other orders in a similar way. For example, from Proposition 3.5, (v), *we have that* $(Y|X \le x)$ *is LR-increasing (decreasing) in x for all F, G if and only if* $-\phi'(C(u,v))$ *is* RR_2 *(*TP_2*), that is, reverse regular of order two (totally positive of order two); see, e.g., [26, p. 199].*

We can determine the monotonicity of $(Y|X = x)$ *in a similar way from Proposition 3.7. Capéraà and Genest [7] proved that* (X, Y) *is PDS if and only if* $-\ln(-\varphi'(u))$ *is convex where* $\varphi(u) = \phi^{-1}(u)$ *is the inverse function of* ϕ. *From Proposition 3.7* (i) *we have that* (X, Y) *is PDS (NDS) if and only if* $\phi'(C(u,v))/\phi'(u)$ *is increasing (decreasing) in u. From Proposition 3.7* (ii) *the same property holds for the survival copula. For example, for the Clayton-Oakes family we obtain* $\phi'(u) = -u^{-\theta-1}$ *and*

$$\frac{\phi'(C(u,v))}{\phi'(u)} = \left[\frac{C(u,v)}{u}\right]^{-\theta-1}$$

which is increasing in u (from the first part of this example and $\theta > 0$*). So* (X, Y) *is PDS. Analogously, for the Gumbel family we have* $\phi'(u) = -\theta u^{-1}(-\ln u)^{\theta-1}$ *and*

$$\frac{\phi'(C(u,v))}{\phi'(u)} = \frac{u}{C(u,v)}\left[\frac{-\ln C(u,v)}{-\ln u}\right]^{\theta-1} = \frac{u}{C(u,v)}\left[1 + \left(\frac{-\ln v}{-\ln u}\right)^{\theta}\right]^{(\theta-1)/\theta}$$

which is increasing in u (from the first part of this example and $\theta \ge 1$*). So* (X, Y) *is PDS.*

The upper bound in (4.1) is 1 for many copulas. The following example shows that the upper bound can be greater than 1 for some copulas.

Example 4. *We consider (X, Y) with distributional copula obtained in Example 2.1 of [27] with $m = 1$ and $n = 2$:*

$$C(u,v) = \begin{cases} 3uv & \text{for } 0 \le u, v \le 1/3 \\ uv + \frac{(1-u)(1-v)}{2} & \text{for } 1/3 < u, v \le 1 \\ \min(u,v) & \text{otherwise} \end{cases}$$

for $0 \le u, v \le 1$. Then the survival copula is

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v) = \begin{cases} \frac{3uv}{2} & \text{for } 0 \le u, v \le 2/3 \\ 2 - 2u - 2v + 3uv & \text{for } 2/3 \le u, v \le 1 \\ \min(u,v) & \text{otherwise.} \end{cases}$$

As $C(F(x), u) \ge uF(x)$ for $0 < F(x) < 1$, from Proposition 3.2, (i), we have $Y \ge_{ST} (Y|X \le x)$. Even more, by Proposition 3.2, (ii), we get $Y \ge_{HR} (Y|X \le x)$. Also, from Proposition 3.3, (ii), we have $Y \le_{HR} (Y|X > x)$.

From Proposition 4.1, for $0 < F(x) \le 1/3$, we have

$$G(y) \le \Pr(Y \le y|X \le x) \le 3G(y)$$

and for $1/3 < F(x) < 1$, we have

$$G(y) \le \Pr(Y \le y|X \le x) \le \frac{1}{F(x)} G(y).$$

Analogously, by Proposition 4.1, for $0 < \bar{F}(x) \le 2/3$, we have

$$\frac{2 - 3\bar{F}(x)}{2} \bar{G}(y) \le \Pr(Y > y|X \le x) \le \bar{G}(y)$$

and for $2/3 < \bar{F}(x) < 1$, we have

$$0 \le \Pr(Y > y|X \le x) \le \bar{G}(y).$$

Then, for the conditional expectations, we have

$$\max\left(0, \frac{2 - 3\bar{F}(x)}{2}\right) E(Y) \le E(Y|X \le x) \le E(Y).$$

Moreover, we have

$$(Y|X \le x_1) =_{ST} (Y|X \le q_{1/3}) \le_{LR} (Y|X \le x_2) \le_{LR} (Y|X \le x_3) \le_{LR} Y$$

for $x_1 \le q_{1/3} \le x_2 \le x_3$ and $F(q_{1/3}) = 1/3$.

7 Conclusions

The copula representation is a successful way to model the dependence in a random vector. Many dependence concepts are defined in terms of (different) conditional distributions. In the present paper we have obtained copula representations for the different conditional distributions. These representations are based on the concept of distorted distributions. Hence we can apply the results for distorted distributions (obtained recently) to compare conditional distributions or to obtain bounds for them. These comparisons and bounds only depend on the underlying copula. The comparison results can be used to characterize well known dependence concepts (as PQD/NQD, LTD/LTI, RTI/RTD and SI/SD) by using the usual stochastic order. They can also be used to define similar dependence concepts (which only depend on the copula) based on other orders. The bounds can be used to get bounds for the different regression curves $E(Y|X = x)$, $E(Y|X \le x)$ and $E(Y|X > x)$. We focus the paper on the bivariate case, but the results can be extended to the multivariate case (with $n > 2$) by using the representations included in Section 2.

The procedures and results obtained here are a starting point to develop and study new dependence concepts based on mathematical properties of copulas. These concepts should be studied for particular copulas and for families of copulas built using particular methods. In particular, we should find more examples in which the dependence concepts defined in Propositions 3.2-3.7 do not coincide.

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References

[1] Adrian T. and M. K. Brunnermeier (2016). CoVaR. *Am. Econ. Rev. 106*(7), 1705–1741.

[2] Arias-Nicolás J. P., F. Ruggeri, and A. Suárez-Llorens (2016). New classes of priors based on stochastic orders and distortion functions. *Bayesian Anal. 11*(4), 1107–1136.

[3] Arnold B. C., E. Castillo, and J. M. Sarabia (1999). *Conditional Specification of Statistical Models*. Springer-Verlag, New York.

[4] Bernardi M., F. Durante, and P. Jaworski (2017). CoVaR of families of copulas. *Statist. Probab. Lett. 120*, 8–17.

[5] Block H. W., T. H. Savits, and M. Shaked (1985). A concept of negative dependence using stochastic ordering. *Statist. Probab. Lett. 3*(2), 81–86.

[6] Burkschat M. and J. Navarro (2018). Stochastic comparisons of systems based on sequential order statistics via properties of distorted distributions. *Probab. Eng. Inform. Sci. 32*(2), 246–274.

[7] Capéraà P. and C. Genest (1993). Spearman's ρ is larger than Kendall's τ for positively dependent random variables. *J. Nonparametr. Stat. 2*(2), 183–194.

[8] Durante F. and R. Foschi (2014). Dependence of exchangeable residual lifetimes subject to failure. *Appl. Math. Comput. 235*, 502–511.

[9] Durante F., R. Foschi, and F. Spizzichino (2008). Threshold copulas and positive dependence. *Statist. Probab. Lett. 78*(17), 2902–2909.

[10] Durante F. and C. Sempi (2016). *Principles of Copula Theory.* CRC Press, Boca Raton FL.

[11] Girardi G. and A. T. Ergün (2013). Systemic risk measurement: multivariate GARCH estimation of CoVaR. *J. Bank. Financ. 37*(8), 3169–3180.

[12] Jaworski P. (2017). On Conditional Value at Risk (CoVaR) for tail-dependent copulas. *Depend. Model. 5*, 1–19.

[13] Kozyra P. M. and T. Rychlik (2017). Sharp bounds on the expectations of L-statistics expressed in the Gini mean difference units. *Comm. Statist. Theory Methods 46*(6), 2921–2941.

[14] López-Díaz M., M.A. Sordo, and A. Suárez-Llorens (2012). On the L_p-metric between a probability distribution and its distortion. *Insurance Math. Econom. 51*(2), 257–264.

[15] Mainik G. and E. Schaanning (2014). On dependence consistency of CoVaR and some other systemic risk measures. *Stat. Risk Model. 31*(1), 49–77.

[16] Miziuła P. and J. Navarro (2017a). Sharp bounds for the reliability of systems and mixtures with ordered components. *Naval Res. Logist. 64*(2), 108–116.

[17] Miziuła P. and J. Navarro (2017b). Bounds for the reliability functions of coherent systems with heterogeneous components. *Appl. Stoch. Models Bus. Ind. 34*(2), 158–174.

[18] Navarro J. (2018). Distribution-free comparisons of residual lifetimes of coherent systems based on copula properties. *Statist. Papers 59*(2), 781–800.

[19] Navarro J. and Y. del Águila (2017). Stochastic comparisons of distorted distributions, coherent systems and mixtures with ordered components. *Metrika 80*(6-8), 627–648.

[20] Navarro J. and F. Durante (2017). Copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components. *J. Multivariate Anal. 158*, 87–102.

[21] Navarro J., Y. del Águila, M. A. Sordo, and A. Suárez-Llorens (2013). Stochastic ordering properties for systems with dependent identically distributed components. *Appl. Stoch. Models Bus. Ind. 29*(3), 264–278.

[22] Navarro J., Y. del Águila, M. A. Sordo, and A. Suárez-Llorens (2016). Preservation of stochastic orders under the formation of generalized distorted distributions. *Methodol. Comput. Appl. Probab. 18*(2), 529–545.

[23] Navarro J. and M. C. Gomis (2016). Comparisons in the mean residual life order of coherent systems with identically distributed components. *Appl. Stoch. Models Bus. Ind. 32*(1), 33–47.

[24] Navarro J., M. Longobardi, and F. Pellerey (2017). Comparison results for inactivity times of k-out-of-n and general coherent systems with dependent components. *TEST 26*(4), 822–846.

[25] Navarro J and T. Rychlik (2010). Comparisons and bounds for expected lifetimes of reliability systems. *Eur. J. Oper. Res. 207*(1), 309–317.
[26] Nelsen R.B. (2006). *An Introduction to Copulas*. Second edition. Springer, New York.
[27] Rodríguez-Lallena J. A. and M. Úbeda-Flores (2004). A new class of bivariate copulas. *Statist. Probab. Lett. 66*(3), 315–325.
[28] Shaked M. and J. G. Shanthikumar (2007). *Stochastic Orders*. Springer, New York.
[29] Shaked M., M.A. Sordo, and A. Suárez-Llorens (2012). Global dependence stochastic orders. *Methodol. Comput. Appl. Probab. 14*(3), 617–648.
[30] Sordo M. A., A. J. Bello, and A. Suárez-Llorens (2018). Stochastic orders and co-risk measures under positive dependence. *Insurance Math. Econom. 78*, 105–113.
[31] Sordo M. A., A. Castaño-Martínez, and G. Pigueiras (2016). A family of premium principles based on mixtures of TVaRs. *Insurance Math. Econom. 70*, 397–405.
[32] Sordo M. A. and H. M. Ramos (2007). Characterizations of stochastic orders by L-functionals. *Statist. Papers 48*(2), 249–263.
[33] Sordo M. A. and A. Suárez-Llorens (2011). Stochastic comparisons of distorted variability measures. *Insurance Math. Econom. 49*(1), 11–17.
[34] Sordo M. A., A. Suárez-Llorens, and A. J. Bello (2015). Comparison of conditional distributions in portfolios of dependent risks. *Insurance Math. Econom. 61*, 62–69.
[35] Wang S. (1996). Premium calculation by transforming the layer premium density. *Astin Bull. 26*(1), 71–92.
[36] Yaari M. E. (1987). The dual theory of choice under risk. *Econometrica 55*(1), 95–115.
[37] Yitzhaki S and I. Olkin (1991). Concentration indices and concentration curves. In K. Mosler and M. Scarsini (Eds.), *Stochastic Orders and Decision Under Risk*, pp. 380–392. Inst. Math. Statist., Hayward CA.
[38] Yitzhaki S. (2003). Gini's mean difference: a superior measure of variability for non-normal distributions. *Metron LXI*(2), 285–316.
[39] Yitzhaki S. and E. Schechtman (2013). *The Gini Methodology: A Primer on a Statistical Methodology*. Springer-Verlag, New York.