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Domination of sample maxima and related extremal dependence measures

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Abstract: For a given d-dimensional distribution function (df) H we introduce the class of dependence measures $\mu(H, Q) = -\mathbb{E}\{\ln H(Z_1, \ldots, Z_d)\}$, where the random vector $(Z_1, \ldots, Z_d)$ has df Q which has the same marginal dfs as H. If both H and Q are max-stable dfs, we show that for a df F in the max-domain of attraction of H, this dependence measure explains the extremal dependence exhibited by F. Moreover, we prove that $\mu(H, Q)$ is the limit of the probability that the maxima of a random sample from F is marginally dominated by some random vector with df in the max-domain of attraction of Q. We show a similar result for the complete domination of the sample maxima which leads to another measure of dependence denoted by $\lambda(Q, H)$. In the literature $\lambda(H, H)$, with H a max-stable df, has been studied in the context of records, multiple maxima, concomitants of order statistics and concurrence probabilities. It turns out that both $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. If H is max-stable we derive useful representations for both $\mu(H, Q)$ and $\lambda(Q, H)$. Our applications include equivalent conditions for H to be a product df and F to have asymptotically independent components.

Keywords: Max-stable distributions, domination of sample maxima, extremal dependence, inf-argmax formula, de Haan representation, records, multiple maxima, concomitants of order statistics, concurrent probabilities

MSC: 60G15, 60G70

1 Introduction

Let H be a d-dimensional distribution function (df) with unit Fréchet marginal dfs $\Phi(x) = e^{-1/x}, x > 0$. We shall assume in the sequel that H is a max-stable df, which in our setup is equivalent to the homogeneity property

$$H^t(x_1, \ldots, x_d) = H(tx_1, \ldots, tx_d), \tag{1.1}$$

for any $t > 0, x_i \in (0, \infty), 1 \le i \le d$, see for instance [2, 9, 25]. The class of max-stable dfs is very large with the two extreme instances

$$H_0(x_1, \ldots, x_d) = \prod_{i=1}^d \Phi(x_i) \quad \text{and} \quad H_\infty(x_1, \ldots, x_d) = \min_{1 \le i \le d} \Phi(x_i),$$

the product df H_0 and the upper df H_∞, respectively. Hereafter $\bar{G} = 1 - G$ stands for the survival function of some univariate df G. It follows easily by the lower Fréchet -Hoeffding bound that

$$(H(nx_1, \ldots, nx_d))^n \quad \ge \quad \left(\max\left(0, 1 - \sum_{i=1}^d \bar{\Phi}(nx_i)\right)\right)^n \ge e^{\liminf_{n\to\infty} n \ln(1 - \sum_{i=1}^d \bar{\Phi}(nx_i))}$$

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$$= \quad H_0(x_1, \dots, x_d), \quad x_i \in (0, \infty), \quad i \le d. \tag{1.2}$$

Indeed, (1.2) is well-known and follows for instance using the Pickands representation of H, see [9, Eq. (4.3.1)] or the inf-argmax formula as shown in Section 4. Consequently, any max-stable df H lies between H_0 and H_∞, i.e.,

$$H_0(x_1, \dots, x_d) \le H(x_1, \dots, x_d) \le H_\infty(x_1, \dots, x_d), \quad x_i \in (0, \infty), 1 \le i \le d. \tag{1.3}$$

From multivariate extreme value theory, see e.g. [2, 4, 9, 25], we know that d-dimensional max-stable dfs H are limiting dfs of the component-wise maxima of d-dimensional independent and identically distributed (iid) random vectors with some df F. In that case, F is said to be in the max-domain of attraction (MDA) of H (abbreviated $F \in MDA(H)$). For simplicity we shall assume throughout in the following that F is a df on $[0, \infty)^d$ with marginal dfs $F_i \in MDA(\Phi)$, $i \le d$ that have norming constants $a_n = n$, $n \in \mathbb{N}$, and thus we have

$$\lim_{n\to\infty} F_i^n(nx) = \Phi(x), \quad x \in \mathbb{R} \tag{1.4}$$

for all $i \le d$, where we set $\Phi(x) = 0$ if $x \le 0$. Consequently, F is in the MDA of some max-stable df H if further

$$\lim_{n\to\infty} \sup_{x_i \in \mathbb{R}, 1 \le i \le d} \Big| F^n(nx_1, \dots, nx_d) - H(x_1, \dots, x_d) \Big| = 0. \tag{1.5}$$

In the special case that F has asymptotically independent marginal dfs, meaning that for $(X_1, \dots, X_d)$ with df F

$$\lim_{n\to\infty} n\mathbb{P}\{X_i > nx_i, X_j > nx_j\} = 0, \quad x_i, x_j \in (0, \infty), \quad \forall i \neq j \le d, \tag{1.6}$$

then $F \in MDA(H_0)$ if simply $F_i \in MDA(\Phi)$, $i \le d$.
In various applications it is important to be able to determine if some max-stable df H resulting from the approximation in (1.5) is equal to H_0, which in the light of multivariate extreme value theory means that the component-wise maxima $\boldsymbol{M}_n := (\max_{1\le i\le n} X_{i1}, \dots, \max_{1\le i\le n} X_{id})$, $n \ge 1$ of a d-dimensional random sample $(X_{i1}, \dots, X_{id})$, $i = 1, \dots, n$ of size n from F has asymptotically independent components.
The strength of dependence of the components of $\boldsymbol{M}_n$, or in other words the extremal dependence manifested in F, in view of the approximation (1.5) can be measured by calculating some appropriate dependence measure for H (when the limiting df H is known).
For any random vector $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with df Q which has the same marginal dfs as H we introduce a class of dependence measure for H indexed by Q given by

$$\mu(H, Q) = -\mathbb{E}\big\{\ln H(Z_1, \dots, Z_d)\big\}. \tag{1.7}$$

In view of (1.3), since $-\ln H_i(Z_i)$ is a unit exponential random variable, we have

$$1 = \max_{1\le i\le d} \mathbb{E}\{-\ln H_i(Z_i)\} \le -\mathbb{E}\Big\{\ln \min_{1\le i\le d} H_i(Z_i)\Big\} \le \mu(H, Q) \le -\mathbb{E}\Big\{\ln \prod_{i=1}^{d} H_i(Z_i)\Big\} = d, \tag{1.8}$$

and, in particular:

$$\mu(H_0, Q) = d, \quad \mu(H_\infty, H_\infty) = 1. \tag{1.9}$$

Clearly, $\mu(H, Q)$ can be defined for any df H and it does not depend on the choice of the marginal dfs of H. In this contribution we shall show that $\mu(H, Q)$ is particularly interesting for H being max-stable.

Next, consider the case that both H and Q are max-stable. It follows that (see Theorem 2.3) for F satisfying (1.5) and $G \in MDA(Q)$

$$\mu(Q, H) = \lim_{n\to\infty} \mu_n(G, F^n), \quad \mu_n(G, F^n) = n \int_{\mathbb{R}^d} [1 - G(x_1, \dots, x_d)]\, dF^n(x_1, \dots, x_d), \tag{1.10}$$

provided that both F and G are continuous. In view of (1.10), we see that $\mu(H, Q)$ relates to F under (1.5).
Let in the following $\boldsymbol{W}$ denote a random vector with df G being independent of $\boldsymbol{M}_n$. We say that $\boldsymbol{W}$ marginally dominates $\boldsymbol{M}_n$, if there exists some $i \le d$ such that $W_i > M_{ni}$. Consequently, assuming further that $\boldsymbol{W}$ is independent of $\boldsymbol{M}_n$ we have

$$\frac{\mu_n(G, F^n)}{n} = \mathbb{P}\{\boldsymbol{W} \text{ marginally dominates } \boldsymbol{M}_n\} =: \underline{\pi}_n.$$

Re-writing (1.10) we have $\lim_{n\to\infty} n\underline{\pi}_n = \mu(H, Q)$ and thus $\mu(H, Q)$ appears naturally in the context of marginal dominance of sample maxima.
Our motivation for introducing $\mu(H, Q)$ comes from results and ideas of A. Gnedin, see [10, 12, 13] where multiple maxima of random samples is investigated. In the turn, the probability of observing a multiple maximum is closely related to the complete domination of sample maxima as we shall explain below.
We say that $\boldsymbol{W}$ completely dominates $\boldsymbol{M}_n$ if $W_i > M_{ni}$ for any $i \le d$. Assuming that F and G are continuous, we have

$$\lambda_n(F^n, G) := n \int_{\mathbb{R}^d} F^n(x_1, \ldots, x_d)\, dG(x_1, \ldots, x_d) = n\mathbb{P}\{\boldsymbol{W} \text{ completely dominates } \boldsymbol{M}_n\} =: n\overline{\pi}_n.$$

If further $F \in MDA(H)$, $G \in MDA(Q)$ we show in Theorem 2.3 below that

$$\lim_{n\to\infty} \lambda_n(G^n, F) = \lambda(Q, H) = \int_{(0,\infty)^d} Q(x_1, \ldots, x_d)\, d\upsilon(x_1, \ldots, x_d), \tag{1.11}$$

where υ denotes the exponent measure of H defined on $E = [0, \infty]^d \setminus (0, \ldots, 0)$, see [9, 25] for more details on the exponent measure. Note in passing that by symmetry $\lim_{n\to\infty} \lambda_n(F^n, G) = \lambda(H, Q)$ follows.

Our notation and definitions of $\overline{\pi}_n$ and $\underline{\pi}_n$ agree with those in [7] for the particular case that $F = G$. Therein the complete and simple records are discussed. If F is continuous and $F = G$ we have that $(n+1)\overline{\pi}_n$ equals

$$\mathbb{P}\Big\{\max_{1\le i\le n+1} X_{ij} = X_{1j}, j = 1, \ldots, d\Big\},$$

which is the probability of observing a multiple maximum, see [8, 11–13, 20, 21]. There are only few contributions that discuss the asymptotics of $\lambda_n(G^n, F)$ for $F \ne G$, see [16, 17, 19].

Since the exponent measure can be defined also for max-id df H, i.e., if H^t is a df for any $t > 0$, then as above $\lambda(Q, H)$ can also be defined for any such df H and any given d-dimensional df Q. We shall show that $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. In particular, for $d = 2$ we have $\mu(H, Q) = 2 - \lambda(Q, H)$, provided that H is a max-id df. In particular, we show how to define $\lambda(Q, H)$ for any H and Q.
For H being a max-id df we also show how to calculate $\mu(H, Q)$ and $\lambda(Q, H)$ by a limiting procedure, which relates to domination of d-dimensional random vectors, see Theorem 2.1 below.

It turns out that both dependence measures $\mu(H, Q)$ and $\lambda(Q, H)$ are very tractable if H is max-stable (note that such H is also max-id df). In particular, we show that $\mu(H, Q)$ is the extremal coefficient of some d-dimensional max-stable df $H^\star$, i.e., $\mu(H, Q) = -\ln H^\star(1, \ldots, 1)$. Moreover, we derive in Theorem 2.5 tractable expressions for $\mu(H, Q)$ and $\lambda(Q, H)$, which are useful for simulations of these dependence measures if the de Haan spectral representation of H is known.

It is of particular interest for multivariate extreme value theory to derive tractable criteria that identify if a max-stable df H is equal to H_0. In our first application we show several equivalent conditions to $H = H_0$.

In view of (1.10) and (1.11) we see that both measures of extremal dependence $\mu(H, Q)$ and $\lambda(Q, H)$ capture the extremal properties of $F \in MDA(H)$. Motivated by the relation between $\mu(H, Q)$ and $\lambda(Q, H)$ we derive in our second application several conditions equivalent to (1.6).

Both $\mu(H, Q)$ and $\lambda(Q, H)$ can be defined for any d-dimensional df H and Q. When H is max-stable, these are dependence measures for H, since independent of the choice of Q, we can determine if $H = H_0$, see Proposition 3.1, statement ii). A simple choice for Q is taking $Q = H$. Alternatively, one can take $Q = H_0$ or $Q = H_\infty$. Independent of the choice of Q we show in Proposition 3.1 that $\mu(H, Q) = 2$ is equivalent to $H = H_0$. In particular, this result shows that $\mu(H, Q)$ is a measure of dependence of H (and not for Q).

A summary of the remainder of the paper follows. In Section 2, we derive the basic properties of both measures of $\mu(H, Q)$ and $\lambda(Q, H)$ if H is a max-id df. More tractable formulas are then derived for H being a max-stable df. Section 3 is dedicated to applications. We present some auxiliary results in Section 4 followed by the proofs of the main results in Section 5.

2 Main Results

In the following H and Q are d-dimensional dfs with unit Fréchet marginals dfs and $\boldsymbol{Z}$ is a random vector with df Q. The second dependence measure $\lambda(Q, H)$ defined in (1.11) is determined in terms of the exponent measure ν of H, under the max-stability assumption on H.
A larger class of multivariate dfs is that of max-id dfs. Recall that H is max-id, if H^t is a df for an $t > 0$. For such dfs the corresponding exponent measure can be constructed, see for example [25], and therefore we can define $\lambda(Q, H)$ as in the Introduction for any H a max-id df and any given df Q. Note that any max-stable df is a max-id df, therefore in the following we shall consider first the general case that H is a max-id df, and then focus on the more tractable case that H is a max-stable df.

2.1 Max-id df H

Our analysis shows that $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. Specifically, if $d = 2$, then $\mu(H, Q) = 2 - \lambda(Q, H)$, provided that H is a max-id df. Such a relationship does not hold for the case $d > 2$. However as we show below it is possible to calculate $\mu(H, Q)$ if we know $\lambda(Q_K, H_K)$ for any non-empty index set $K \subset \{1, \dots, d\}$. A similar result is shown for $\lambda(Q, H)$. In our notation Q_K denotes the marginal df of Q with respect to K and $|K|$ stands for the number of the elements of the index set K. Below μ_n and λ_n are as defined in the Introduction.

Theorem 2.1. *If H is a max-id df, then we have*

$$\mu(H, Q) = \lim_{n\to\infty} \mu_n(H^{1/n}, Q), \quad \lambda(Q, H) = \lim_{n\to\infty} \lambda_n(Q, H^{1/n}). \tag{2.1}$$

Moreover,

$$\mu(H, Q) = d + \sum_{2\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\}, |K|=i} \lambda(Q_K, H_K) \tag{2.2}$$

and

$$\lambda(Q, H) = d + \sum_{2\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\}, |K|=i} \mu(H_K, Q_K). \tag{2.3}$$

Remark 2.2. i) For H a max-stable df and $Q = H$ the claim in (2.3) is shown in [8, Th. 2.2, Eq. (13)].
ii) A direct consequence of (2.3) is that we can define $\lambda(Q, H)$ even if H is not a max-id df by simply using the definition of $\mu(H_K, Q_K)$.
iii) It is clear that $\mu(H, Q) \ge \mu(H_K, Q_K)$ for any non-empty index set $K \subset \{1, \dots, d\}$. Note that (2.1) shows that exactly the opposite relation holds for $\lambda(Q, H)$ when H is a max-id df, namely

$$\lambda(Q, H) \le \lambda(Q_K, H_K).$$

In fact, (2.3) shows that we can calculate both $\mu(H, Q)$ and $\lambda(Q, H)$ by a limit procedure if we assume that H is a max-id df, see for more details (5.1). Although such a limit procedure shows how to interpret these dependence measures in terms of domination of random vectors, it does not give a precise relation with extremal properties of random samples. Therefore in the following we shall restrict our attention to the tractable case that H is a max-stable df.

2.2 Max-stable df H

We show next the relation of $\mu(H, Q)$ and $\lambda(Q, H)$ with the marginal and complete domination of sample maxima mentioned in the Introduction. Recall that in our notation $\bar{F}_i, \bar{G}_i, i \le d$ stand for the marginal survival functions of F and G, respectively.

Theorem 2.3. *If H, Q are max-stable dfs with unit Fréchet marginals and F, G are two d-dimensional continuous dfs such that $\lim_{x\to\infty} \bar{F}_i(x)/\bar{G}_i(x) = c_i \in (0, \infty)$ for $i \le d$ and further $F \in MDA(H)$, $G \in MDA(Q)$, then* (1.10) *and* (1.11) *hold.*

Remark 2.4. The relation $\lim_{n\to\infty} \lambda_n(F^n, F) = \lambda(H, H)$ for $F \in MDA(H)$ is known from works of A. Gnedin, see for instance [12, 13]. Explicit formulas are given in [22] for $d = 2$. See also the recent contributions [7, 8].

In view of [4] (recall H has unit Fréchet marginal dfs) the assumption that H is max-stable implies the following de Haan representation (see e.g. [6, 23])

$$-\ln H(x_1, \ldots, x_d) = \mathbb{E}\Big\{\max_{1\le i\le d} \frac{Y_i}{x_i}\Big\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.4}$$

where Y_j's are non-negative with $\mathbb{E}\{Y_i\} = 1, 1 \le i \le d$. As shown in [18], see also [3, 24] we have the alternative formula

$$-\ln H(x_1, \ldots, x_d) = \sum_{i=1}^{d} \frac{1}{x_i} \Psi_i(x_1, \ldots, x_d), \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.5}$$

where Ψ_i's are non-negative zero-homogeneous, i.e., $\Psi_i(cx_1, \ldots, cx_d) = \Psi_i(x_1, \ldots, x_d)$ for any $c > 0, x_i \in (0, \infty), i \le d$. Moreover, Ψ_i's are bounded by 1, which immediately implies the validity of the lower bound in (1.2).

In the literature $-\ln H(1, \ldots, 1)$ is also referred to as the extremal coefficient of H, denoted by $\theta(H)$, see for example [8].
Our next result gives alternative formulas for $\mu(H, Q)$ and shows that it is the extremal coefficient of the max-stable df $H^\star$ defined by

$$-\ln H^\star(x_1, \ldots, x_d) = \mathbb{E}\Big\{\max_{1\le i\le d} \frac{Y_i}{x_i Z_i}\Big\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.6}$$

with $\boldsymbol{Z}$ being independent of $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$. Note that since

$$\mathbb{E}\{Y_i\} = \mathbb{E}\{1/Z_i\} = 1, \quad i \le d$$

and Y_i/Z_i's are non-negative, then $H^\star$ has unit Fréchet marginal dfs and moreover also $\tilde{H}$ defined by

$$-\ln \tilde{H}(x_1, \ldots, x_d) = \mathbb{E}\Big\{\max_{1\le i\le d} \frac{1}{x_i Z_i}\Big\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.7}$$

is a max-stable df with unit Fréchet marginal dfs.

Theorem 2.5. *If H is a max-stable df with unit Fréchet marginal dfs and de Haan representation* (2.6) *with* $\boldsymbol{Y}$ *being independent of* $\boldsymbol{Z}$ *with df Q which has unit Fréchet marginal dfs, then we have*

$$\mu(H, Q) \quad = \quad \mathbb{E}\Big\{\max_{1\le i\le d} \frac{Y_i}{Z_i}\Big\} = \sum_{i=1}^{d} \mathbb{E}\Big\{\frac{1}{Z_i} \Psi_i(Z_1, \ldots, Z_d)\Big\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.8}$$

and

$$\lambda(Q, H) = \mathbb{E}\Big\{\min_{1\le i\le d} \frac{Y_i}{Z_i}\Big\}. \tag{2.9}$$

Moreover, with H^* defined in (2.6), *we have*

$$\mu(H,Q)=\theta(H^*)\geq\max\Big(\theta(H),\theta(\tilde{H})\Big)\geq 1, \tag{2.10}$$

and

$$\lambda(Q,H)\quad\leq\quad\min\left(\mathbb{E}\Big\{\min_{1\leq i\leq d}Y_i\Big\},\mathbb{E}\Big\{\min_{1\leq i\leq d}\frac{1}{Z_i}\Big\}\right)\leq 1. \tag{2.11}$$

Remark 2.6. i) If $Z_1=\cdots=Z_d=Z$ with Z a unit Fréchet random variable, then the zero-homogeneity of Ψ_i's, (2.5) and (2.8) imply that

$$\mu(H,Q)=\sum_{i=1}^d\Psi_i(1,\ldots,1)\mathbb{E}\Big\{\frac{1}{Z}\Big\}=\sum_{i=1}^d\Psi_i(1,\ldots,1)=\mathbb{E}\{\max_{1\leq i\leq d}Y_i\}=-\ln H(1,\ldots,1)\geq 1. \tag{2.12}$$

Further, by (2.9) we have $\lambda(Q,H)=\mathbb{E}\{\min_{1\leq i\leq d}Y_i\}$.
ii) In view of [8, Th. 2.2] (see also Eq. (6.9) in [22]), for H with de Haan representation (2.6) it holds that

$$\lambda(H,H)=-\mathbb{E}\Big\{\frac{1}{\ln H(Y_1,\ldots,Y_d)}\Big\},$$

which together with (2.10) implies that

$$\mu(H_\infty,H_\infty)=\lambda(H_\infty,H_\infty)=1,$$

and thus the lower bound in (1.8) is sharp. We note in passing that there are numerous papers where $\lambda_n(F^n,F)$ and $\lambda(H,H)$ appear, see [5, 7, 14, 15, 22] and the references therein.
iii) For common max-stable dfs H the spectral random vector $\boldsymbol{Y}$ that defines (2.4) is explicitly known. Consequently, for any given random vector $\boldsymbol{Z}$, using the first expression in (2.8) and (2.9), we can easily evaluate $\mu(H,Q)$ and $\lambda(Q,H)$ by Monte Carlo simulations, respectively.

3 Applications

In multivariate extreme value theory it is important to have conditions that show if a given max-stable df H is equal to H_0. In case $d=2$ it is well-known that $H=H_0$ if and only if $\lambda(H,H)=0$, see [8, Pr. 2.2] or [12, Th. 2]. Consequently, when $d>2$, in view of [9, Th. 4.3.3] we have that $H=H_0$ if and only if

$$\lambda(H_K,H_K)=0, \tag{3.1}$$

for any index set $K\subset\{1,\ldots,d\}$ with two elements. Therefore, in the sequel we consider for simplicity the case $d=2$ discussing some tractable conditions that are equivalent to $H=H_0$ and (1.6).

As in Balkema and Resnick [1], for a given bivariate df H with unit Fréchet margins define $\xi_H:(0,\infty)^2\to[0,1]$ as

$$\xi_H(x_1,x_2)=\lim_{h\to 0}\frac{[B-H((x_1,x_2)+(h,-h))][B-H((x_1,x_2)+(-h,h))]}{A[A+B-H((x_1,x_2)+(h,-h))-H((x_1,x_2)+(-h,h))]},\quad (x_1,x_2)\in(0,\infty)^2, \tag{3.2}$$

where $A=H(x_1,x_2)$ and $B=H(x_1+h,x_2+h)$. If H is a continuous max-id df, then in view of [1] the function ξ_H is non-negative, measurable and bounded by 1, almost everywhere with respect to dH.

Proposition 3.1. *Let H and Q be two bivariate dfs with unit Fréchet marginals. If H is a max-id df, then we have*

$$\lambda(Q,H)\quad=\quad\int_{(0,\infty)^2}[1-\xi_H(x_1,x_2)]\frac{Q(x_1,x_2)}{H(x_1,x_2)}\,dH(x_1,x_2). \tag{3.3}$$

Moreover, if H is a max-stable df, then the following conditions are equivalent:

i) $H = H_0$;
ii) $\theta(H) = -\ln H(1,1) = 2$;
iii) $\mu(H,Q) = 2 - \lambda(Q,H)$;
iv) ξ_H *equals 1 almost everywhere* dH;
v) $\frac{dH^t}{dH} = \frac{t^2 H^t}{H}$ *almost everywhere* dH *for any* $t > 0$.

Remark 3.2. By [12, Th. 2] we have that $\lambda(H,H) = 0$ is equivalent to $H = H_0$ and $\lambda(H,H) = 1$ is equivalent to $H = H_\infty$.
Moreover, statement iii) above holds for any df Q with continuous marginal dfs, and thus $\mu(H,Q)$ and $\lambda(Q,H)$ are both dependence measures for H.

We conclude this section with some equivalent conditions to (1.6).

Proposition 3.3. *Let* F, G *be two continuous bivariate dfs with marginal dfs* $F_i, G_i, i = 1,2$ *satisfying* $\lim_{t\to\infty} \bar{F}_i(t)/\bar{G}_i(t) = 1$. *If further* F_1, F_2 *satisfy* (1.4) *and* (X_1, X_2) *has df* F, *then the following are equivalent:*
i) F *has asymptotically independent components;*
ii) $\lim_{n\to\infty} n\mathbb{P}\{X_1 > n, X_2 > n\} = 0$;
iii) $\lim_{n\to\infty} \lambda_n(G^n, F) = 0$;
iv) $\lim_{n\to\infty} \mu_n(F, G^n) = 2$;
v) $\lim_{n\to\infty} n\mathbb{P}\{G(X_1, X_2) > 1 - 1/n\} = 0$.

Remark 3.4. The equivalence of i) and ii) in Proposition 3.3 is well-known and relates to Takahashi theorem, i.e., it is enough to know that the limiting max-stable df H is a product df at one point, say (1,1). See for more details in the d-dimensional setup [9, p. 452].
Moreover, recall that the assumption $F_i \in MDA(\Phi)$ means that $\lim_{n\to\infty} F_i^n(a_{ni}x) = \Phi(x), x \in \mathbb{R}$ for some norming constants $a_{ni} > 0, n \in \mathbb{N}$. For notational simplicity, in this paper we assume that a_{ni}'s equal n. If this is not the case, then we need to re-formulate statement ii) in Proposition 3.3 as $n \lim_{n\to\infty} n\mathbb{P}\{X_1 > a_{n1}, X_2 > a_{n2}\} = 0$. Note that if $F \in MDA(H)$ with H a max-stable df, then

$$\lim_{n\to\infty} n\mathbb{P}\{X_1 > a_{n1}, X_2 > a_{n2}\} = 2 + \ln H(1,1) = 2 - \theta(H) =: \lambda_F. \tag{3.4}$$

In the literature, λ_F is commonly referred to as the coefficient of upper tail dependence of F, see [9] for more details.

4 Auxiliary Results

Lemma 4.1. *Let* $(V_1, \ldots, V_d)$ *be a random vector with continuous marginal dfs* $H_i, i \le d$. *If further* G *is a* d-*dimensional df with* $G(x_1, \ldots, x_d) < 1$ *for any* $(x_1, \ldots, x_d) \in (0,\infty)^d$ *and the upper endpoint of* $H_i, 1 \le i \le d$ *equals* ∞, *then we have*

$$\lim_{n\to\infty} n\mathbb{E}\{G^{n-1}(V_1, \ldots, V_d)\} \quad = \quad \lim_{n\to\infty} n\mathbb{P}\Big\{G(V_1, \ldots, V_d) > 1 - \frac{1}{n}\Big\} = \kappa \in [0,\infty), \tag{4.1}$$

if either of the limits exists. Further if

$$G(x_1, \ldots, x_d) \quad \le \quad \min_{1\le i\le d} H_i(x_i), \quad (x_1, \ldots, x_d) \in (0,\infty)^d, \tag{4.2}$$

then $\kappa \in [0,1]$.

Proof of Lemma 4.1 The proof of (4.1) follows from [16, Lemma 2.4], see also [12, Pr. 4]. Assuming (4.2), if H denotes the df of $(V_1, \ldots, V_d)$, then we have

$$0 \quad \le \quad n\mathbb{E}\{G^{n-1}(V_1, \ldots, V_d)\} \le n \int_{(0,\infty)^d} \min_{1\le i\le d} H_i^{n-1}(x_i)\, dH(x_1, \ldots, x_d) \le n \int_0^\infty H_1^{n-1}(x_1)\, dH_1(x_1) = 1,$$

establishing the proof. □

Proposition 4.2. *Let $F_n, G_n, n \ge 1$ be two continuous dfs on $[0,\infty)^d$ satisfying*

$$\lim_{n\to\infty} F_n^n(x_1,\dots,x_d) = H(x_1,\dots,x_d), \quad \lim_{n\to\infty} G_n^n(x_1,\dots,x_d) = Q(x_1,\dots,x_d), \quad (x_1,\dots,x_d) \in [0,\infty)^d, \tag{4.3}$$

with H, Q two max-id dfs with unit Fréchet marginal dfs Φ. If for all n large and some $C_1 > 0$, we have

$$G_n^n(x_1,\dots,x_d) \le C_1 \sum_{1\le i\le d} F_{ni}^n(x_i), \quad (x_1,\dots,x_d) \in (0,\infty)^d, \tag{4.4}$$

where F_{ni} is the ith marginal df of F_n, then

$$\lim_{n\to\infty} n \int_{[0,\infty)^d} G_n^n(x_1,\dots,x_d)\, dF_n(x_1,\dots,x_d) = \int_{(0,\infty)^d} Q(x_1,\dots,x_d) d\upsilon(x_1,\dots,x_d), \tag{4.5}$$

where $\upsilon(\cdot)$ is the exponent measure pertaining to H defined on $E := [0,\infty]^d \setminus \{(0,\dots,0)\}$. Furthermore, if for all n large and any $x_1,\dots,x_d$ positive we have

$$1 - G_n(x_1,\dots,x_d) \le C_2 \sum_{1\le i\le d} \bar{F}_{ni}(x_i), \tag{4.6}$$

then

$$\lim_{n\to\infty} n \int_{[0,\infty)^d} [1 - G_n(x_1,\dots,x_d)]\, dF_n^n(x_1,\dots,x_d) = -\int_{(0,\infty)^d} \ln Q(x_1,\dots,x_d)\, dH(x_1,\dots,x_d). \tag{4.7}$$

Proof of Proposition 4.2 For notational simplicity we consider below only the case $d = 2$. From the assumptions,

$$\lim_{n\to\infty} F_n^n(x_{n1},x_{n2}) = H(x_1,x_2), \quad \lim_{n\to\infty} G_n^n(x_{n1},x_{n2}) = Q(x_1,x_2) \tag{4.8}$$

holds for every sequence $(x_{n1},x_{n2}) \to (x_1,x_2) \in (0,\infty)^2$ as $n\to\infty$.

Let υ be the exponent measure of H defined on E, see [25] for details. For any x_0, y_0 positive, since by our assumptions

$$\lim_{n\to\infty} n[1 - F_n(x_1,x_2)] = -\ln H(x_1,x_2)$$

holds locally uniformly for $(x_1,x_2) \in (0,\infty)^2$, using further (4.8) and [18, Lemma 9.3] we obtain

$$\lim_{n\to\infty} \int_{[x_0,\infty)\times[y_0,\infty)} G_n^n(x_1,x_2)\, d(nF_n(x_1,x_2)) = \int_{[x_0,\infty)\times[y_0,\infty)} Q(x_1,x_2)\, d\upsilon(x_1,x_2) =: I(x_0,y_0).$$

Moreover, by (4.4)

$$\begin{aligned} & n \int_{[0,\infty)^2} G_n^n(x_1,x_2)\, dF_n(x_1,x_2) \\ \le\ & nC_1\Big(\int_{[0,x_0]} F_{n1}^{n-1}(x)\, dF_{n1}(x) + \int_{[0,y_0]} F_{n2}^{n-1}(x)\, dF_{n2}(x)\Big) + \int_{[x_0,\infty)\times[y_0,\infty)} G_n^n(x_1,x_2)\, d(nF_n(x_1,x_2)) \\ =\ & C_1(F_{n1}^n(x_0) + F_{n2}^n(y_0)) + \int_{[x_0,\infty)\times[y_0,\infty)} G_n^n(x_1,x_2)\, d(nF_n(x_1,x_2)) \\ \to\ & C_1(e^{-1/x_0} + e^{-1/y_0}) + \int_{[x_0,\infty)\times[y_0,\infty)} Q(x_1,x_2)\, d\upsilon(x_1,x_2), \quad n\to\infty \end{aligned}$$

$$\to \int_{(0,\infty)^2} Q(x_1,x_2)\,d\upsilon(x_1,x_2), \quad x_0 \downarrow 0, y_0 \downarrow 0,$$

where the equality above is a consequence of the assumption that F_n, G_n have continuous marginal dfs. Hence (4.5) follows and we show next (4.7). Similarly, for x_0, y_0 as above, one has that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{(0,\infty)^2} n[1-G_n(x_1,x_2)]\,dF_n^n(x_1,x_2)\\
&= \limsup_{n\to\infty}\Bigg[\int_{([x_0,\infty)\times[y_0,\infty))^c} n[1-G_n(x_1,x_2)]\,dF_n^n(x_1,x_2)\\
&\quad + \int_{[x_0,\infty)\times[y_0,\infty)} n[1-G_n(x_1,x_2)]\,dF_n^n(x_1,x_2)\Bigg]\\
&\le C_2 \limsup_{n\to\infty} \int_{([x_0,\infty)\times[y_0,\infty))^c} n[\bar F_{n1}(x_1)+\bar F_{n2}(x_2)]\,dF_n^n(x_1,x_2)\\
&\quad + \limsup_{n\to\infty} \int_{[x_0,\infty)\times[y_0,\infty)} n[1-G_n(x_1,x_2)]\,dF^n(x_1,x_2)\\
&\le C_2 \limsup_{n\to\infty}(F_{n1}^n(x_0)+F_{n2}^n(y_0))\left[n\bar F_{n1}(x_0)+n\bar F_{n2}(y_0)\right]\\
&\quad - \int_{[x_0,\infty)\times[y_0,\infty)} \ln Q(x_1,x_2)\,dH(x_1,x_2)\\
&= C_2\left[e^{-1/x_0}+e^{-1/y_0}\right]\left[\frac{1}{x_0}+\frac{1}{y_0}\right] - \int_{[x_0,\infty)\times[y_0,\infty)} \ln Q(x_1,x_2)\,dH(x_1,x_2)\\
&\to -\int_{(0,\infty)^2} \ln Q(x_1,x_2)\,dH(x_1,x_2), \quad x_0\downarrow 0, y_0 \downarrow 0,
\end{aligned}$$

hence the proof follows. □

Remark 4.3. The validity of (4.4) has been shown under the assumption that G_n is a continuous df. From the proof above it is easy to see that (4.4) still holds if we assume instead that G_n is continuous and positive such that G_n^n is a df. Similarly, for the validity of (4.7) it is enough to assume that F_n^n is a continuous df.

Corollary 4.4. *If H is a bivariate max-stable df with unit Fréchet marginal dfs H_1 and H_2, then for u,t positive*

$$\int_{(0,\infty)^2} \min\left(H_1^{1/u}(x_1),\, H_2^{1/t}(x_2)\right) d\upsilon(x_1,x_2) \quad = \quad u+t+\ln H(1/u,1/t). \tag{4.9}$$

Proof of Corollary 4.4 The proof follows using Fubini Theorem and the homogeneity property of the exponent measure inherited by (1.1). We give below an alternative proof. Let (V_1,V_2) have df H and set $U_i = H_i(V_i), i=1,2$. By the assumptions since the df H is continuous, applying Theorem 2.3 and (4.1) with $u,t>0$ we obtain

$$\begin{aligned}
&\int_{(0,\infty)^2} \min\left(H_1^{1/u}(x_1), H_2^{1/t}(x_2)\right) d\upsilon(x_1,x_2)\\
&= \lim_{n\to\infty} n \int_{(0,\infty)^2} \min\left(H_1^{n/u}(x_1), H_2^{n/t}(x_2)\right) dH(x_1,x_2)\\
&= \lim_{n\to\infty} n\mathbb{P}\left\{\min\left(H_1^{1/u}(V_1), H_2^{1/t}(V_2)\right) > 1-\frac{1}{n}\right\}
\end{aligned}$$

$$= \lim_{n\to\infty} n\mathbb{P}\Big\{U_1 > 1 - \frac{u}{n}, U_2 > 1 - \frac{t}{n}\Big\} = u + t + \ln H(1/u, 1/t),$$

establishing the proof. □

5 Proofs

PROOF OF THEOREM 2.1 For $n > 0$ set $A_n = Q^{1/n}$ and $B_n = H^{1/n}$. Since H is a max-id df, then B_n is a df for any $n > 0$. Furthermore, since $H_i = Q_i, i \le d$ (recall H_i, Q_i are the marginal dfs of H and Q, respectively), it can be easily checked that we can apply Proposition 4.2, which together with Remark 4.3 implies

$$\begin{aligned}
&\lim_{n\to\infty} n \int_{\mathbb{R}^d} [1 - H^{1/n}(x_1, \dots, x_d)]\, dQ(x_1, \dots, x_d) \\
&= \lim_{n\to\infty} n \int_{\mathbb{R}^d} [1 - B_n(x_1, \dots, x_d)]\, dA_n^n(x_1, \dots, x_d) \\
&= -\int_{\mathbb{R}^d} \ln H(x_1, \dots, x_d)]\, dQ(x_1, \dots, x_d) = \mu(H, Q).
\end{aligned} \tag{5.1}$$

The second claim in (2.1) follows with similar arguments and therefore we omit its proof.
Next, for any non-empty subset K of $\{1, \dots, d\}$ with $m = |K|$ elements by (2.1) one has

$$\mu(H_K, Q_K) = \lim_{n\to\infty} n \int_{\mathbb{R}^m} [1 - F_{n,K}(x_1, \dots, x_d)]\, dQ_K(x_1, \dots, x_d)$$

and

$$\lambda(Q_K, H_K) = \lim_{n\to\infty} n \int_{\mathbb{R}^m} Q_K(x_1, \dots, x_d)\, dF_{nK}(x_1, \dots, x_d),$$

where F_{nK}, Q_K are the marginals of F_n and Q with respect to K. Note that for notational simplicity we write the marginal dfs with respect to K as functions of $x_1, \dots, x_d$ and not as functions of $x_{j1}, \dots, x_{jm}$ where $K = \{j_1, \dots, j_m\}$ has $m = |K|$ elements. By Fubini Theorem

$$\int_{\mathbb{R}^m} Q_K(x_1, \dots, x_d)\, dF_{nK}(x_1, \dots, x_d) = \int_{\mathbb{R}^m} \overline{F}_{nK}(x_1, \dots, x_d)\, dQ_K(x_1, \dots, x_d),$$

where $\overline{F}_{nK}$ stands for the joint survival function of F_{nK}. In the light of the inclusion-exclusion formula

$$1 - F_n(x_1, \dots, x_d) = \sum_{1\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\}, |K|=i} \overline{F}_{nK}(x_1, \dots, x_d), \quad (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Using further the fact that H and Q have the same marginal dfs, for any index set K with only one element we have

$$\lim_{n\to\infty} n \int_{\mathbb{R}^d} \overline{H}_{nK}(x_1, \dots, x_d)\, dQ(x_1, \dots, x_d) = \lim_{n\to\infty} n \int_0^1 (1 - t^{1/n}) dt = 1,$$

hence

$$\begin{aligned}
\mu(H, Q) &= \lim_{n\to\infty} n \int_{\mathbb{R}^d} [1 - F_n(x_1, \dots, x_d)]\, dQ(x_1, \dots, x_d) \\
&= d + \lim_{n\to\infty} n \int_{\mathbb{R}^d} \sum_{2\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\}, |K|=i} \overline{F}_{nK}(x_1, \dots, x_d)\, dQ(x_1, \dots, x_d) \\
&= d + \sum_{2\le i\le d} (-1)^{i+1} \lim_{n\to\infty} n \int_{\mathbb{R}^d} \sum_{K\subset\{1,\dots,d\}, |K|=i} \overline{F}_{nK}(x_1, \dots, x_d)\, dQ_K(x_1, \dots, x_d)
\end{aligned}$$

$$= \quad d + \sum_{2\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\},|K|=i} \lambda(Q_K, H_K)$$

and thus (2.2) follows. Since by the inclusion-exclusion formula we have further

$$\overline{F}_n(x_1, \dots, x_d) = \sum_{1\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\},|K|=i} [1 - F_{nK}(x_1, \dots, x_d)], \quad (x_1, \dots, x_d) \in \mathbb{R}^d$$

the claim in (2.3) follows with similar arguments as above. □

PROOF OF THEOREM 2.5 The claim in (2.8) follows by the de Haan and inf-argmax representation of H. Since by the independence of Y_i's and Z_i's and the fact that $\mathbb{E}\{Y_i\} = \mathbb{E}\{1/Z_i\} = 1$ we have that

$$\mathbb{E}\{Y_i/Z_i\} = \mathbb{E}\{Y_i\}\mathbb{E}\{1/Z_i\} = 1 \tag{5.2}$$

is valid for any $i \le d$. Consequently, by (2.3), (2.8) and the fact that for given constants $c_1, \dots, c_d$

$$\min_{1\le i\le d} c_i = \sum_{i=1}^{d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\}:|K|=i} \max_{j\in K} c_j,$$

then we have

$$\begin{aligned}\lambda(Q, H) &= \sum_{i=1}^{d} \mathbb{E}\left\{\frac{Y_i}{Z_i}\right\} + \sum_{2\le i\le d} (-1)^{i+1} \sum_{K\subset\{1,\dots,d\},|K|=i} \mathbb{E}\left\{\max_{j\in K} \frac{Y_j}{Z_j}\right\}\\ &= \mathbb{E}\left\{\min_{1\le j\le d} \frac{Y_j}{Z_j}\right\},\end{aligned}$$

establishing (2.9).

Further, since (5.2) holds, then by de Haan representation of max-stable dfs we have that the dfs $H^\star$, $\tilde{H}$ defined in (2.6) and (2.7), respectively are max-stable with unit Fréchet marginal dfs. Hence (2.8) implies that $\mu(H, Q) = \theta(H^\star)$. Note in passing that for $Q = H$ this follows also from [8, Pr. 2.2].

Using again that Y_i's are independent of Z_i's and $\mathbb{E}\{Y_i\} = 1, i \le d$ we obtain (recall Y_i's and Z_i's are non-negative random variables)

$$\begin{aligned}\mu(H, Q) &= \mathbb{E}\left\{\mathbb{E}\left\{\max_{1\le i\le d} \frac{Y_i}{Z_i}\middle|(Z_1, \dots, Z_d)\right\}\right\}\\ &\ge \mathbb{E}\left\{\max_{1\le i\le d} \frac{\mathbb{E}\{Y_i\}}{Z_i}\right\}\\ &\ge \mathbb{E}\left\{\max_{1\le i\le d} \frac{1}{Z_i}\right\} = \theta(\tilde{H})\\ &\ge \max_{1\le i\le d} \mathbb{E}\left\{\frac{1}{Z_i}\right\} = 1.\end{aligned}$$

With the same arguments using now that $\mathbb{E}\{1/Z_i\} = 1, i \le d$ we have

$$\begin{aligned}\mu(H, Q) &= \mathbb{E}\left\{\mathbb{E}\left\{\max_{1\le i\le d} \frac{Y_i}{Z_i}\middle|(Y_1, \dots, Y_d)\right\}\right\}\\ &\ge \mathbb{E}\left\{\max_{1\le i\le d} Y_i\right\} = -\ln H(1, \dots, 1) = \theta(H).\end{aligned}$$

The lower bound in (2.11) follows with similar arguments, hence the proof is complete. □

PROOF OF THEOREM 2.3 Suppose without loss of generality that F satisfies (1.5). If $F_i = G_i, i = 1, 2$, then the claim follows from Lemma 4.1 and Proposition 4.2. We consider next the general case that F_i's are tail equivalent to G_i's and suppose for simplicity that $d = 2$. In view of [16, Lemma 2.4] we have

$$\lim_{n\to\infty} n \int_{[0,\infty)} G_i^n(x) dF_i(x) = c_i \in [0, \infty), \quad i = 1, 2$$

if and only if $\lim_{n\to\infty} n\mathbb{P}\{G_i(X_i) > 1 - 1/n\} = c_i$ or equivalently

$$\lim_{x\to\infty} \frac{\bar{F}_i(x)}{\bar{G}_i(x)} = c_i.$$

By the assumption $c_i \in (0, \infty)$ for $i = 1, 2$. Consequently, for all $x > 0$ there exist a_1, a_2 positive such that

$$a_1\bar{F}_i(x) \le \bar{G}_i(x) \le a_2\bar{F}_i(x).$$

Assume for simplicity that $c_i = 1, i = 1, 2$. By the assumptions

$$n\bar{F}_i(nx) \to 1/x, \quad n\bar{G}_i(nx) \to 1/x, \quad n \to \infty$$

uniformly for x in $[t, \infty)$, $t > 0$. Further, for $i = 1, 2$ we have

$$\lim_{t\downarrow 0}\lim_{n\to\infty} n \int\limits_{[0,t]} G_i^n(nx)\, dF_i(nx) = \lim_{t\downarrow 0}\lim_{n\to\infty} n \int\limits_{[0,t]} \bar{G}_i(nx)\, dF_i^n(nx) = 0,$$

which implies

$$\lim_{t\downarrow 0}\lim_{n\to\infty} n \int\limits_{[0,t]^2} G^n(nx, ny)\, dF(nx, ny) = \lim_{t\downarrow 0}\lim_{n\to\infty} n \int\limits_{[0,t]^2} [1 - G(nx, ny)]\, dF^n(nx, ny) = 0.$$

As in the proof of Proposition 4.2, using that F and G are in the MDA of H and Q, respectively, it follows that for any integer k

$$\lim_{n\to\infty} n \int\limits_{[0,\infty)^2} G^{n-k}(x_1, x_2)\, dF(x_1, x_2) = \int\limits_{(0,\infty)^2} Q(x_1, x_2)\, d\upsilon(x_1, x_2) = \lambda(Q, H),$$

and further

$$\lim_{n\to\infty} n \int\limits_{[0,\infty)^2} [1 - F(x_1, x_2)]\, dG^{n-k}(x_1, x_2) = -\int\limits_{(0,\infty)^2} \ln H(x_1, x_2)\, dQ(x_1, x_2) = \mu(H, Q),$$

establishing the proof. □

Proof of Proposition 3.1 In view of Theorem 2.1, since H being a max-id df implies that $H^{1/n}$ is a df for any $n \ge 1$ we have, with $F_n = Q^{1/n}$, that

$$\begin{aligned}
\int\limits_{(0,\infty)^2} Q(x_1, x_2) d\upsilon(x_1, x_2) &= \lim_{n\to\infty} n \int\limits_{(0,\infty)^2} Q(x_1, x_2)\, dH^{1/n}(x_1, x_2) \\
&= 2 - \lim_{n\to\infty} n \int\limits_{[0,\infty)^d} [1 - H^{1/n}(x_1, x_2)]\, dF_n^n(x_1, x_2) \\
&= \int\limits_{(0,\infty)^2} [2 + \ln H(x_1, x_2)]\, dQ(x_1, x_2).
\end{aligned}$$

Since further by [1, Th. 7] the restriction of υ on $(0, \infty)^2$ denoted by υ_0 satisfies

$$\frac{d\upsilon_0}{dH} = \frac{1 - \xi_H}{H}$$

and $\xi_H(x_1, x_2) \in [0, 1]$ almost everywhere dH, then the first claim follows.

The equivalence of i) and ii) is known as Takahashi Theorem, see [9, Th. 4.3.2]. Since $\xi_H \in [0, 1]$ almost everywhere dH, then the equivalence of ii) and iii) is a direct consequence of (3.3) and the fact that $\lambda(Q, H) = 2 - \mu(H, Q)$, see (2.3). Clearly, by (3.3) we have thus $\xi_H = 1$ almost everywhere dH is equivalent to $H = H_0$, whereas iv) is equivalent to v) is consequence of [1, Th. 7]. □

Proof of Proposition 3.3 If F (1.6) holds, then clearly ii) is satisfied and thus i) implies ii). If ii) holds, then

$$\begin{aligned}\limsup_{n\to\infty} F^n(nx_1, nx_2) &= \exp\Big(\limsup_{n\to\infty} n\ln(1-[1-F(nx_1,nx_2)])\Big)\\ &= \exp\Big(-\limsup_{n\to\infty} n(1-F(nx_1,nx_2))\Big)\\ &\le \exp\Big(-\limsup_{n\to\infty}[n\bar F_1(nx_1)+n\bar F_2(nx_2)-n\mathbb{P}\{X_1>n\min(x_1,x_2), X_2>n\min(x_1,x_2)\}]\Big)\\ &= \exp\big(-1/x_1-1/x_2\big), \quad x_1,x_2>0.\end{aligned}$$

As for the derivation of (1.2) we obtain further

$$\liminf_{n\to\infty} F^n(nx_1,nx_2) \ge \exp\big(-1/x_1-1/x_2\big), \quad x_1,x_2>0 \tag{5.3}$$

implying that $F \in MDA(H_0)$, hence i) follows.
Assuming iii) and since the marginal dfs of G are in the MDA of Φ, with the same calculations as in (5.3) for the df G we obtain

$$\begin{aligned}0 = \lim_{n\to\infty} n \int_{[0,\infty)^2} G^n(x_1,x_2)\,dF(x_1,x_2) &\ge \lim_{n\to\infty} n\mathbb{P}\{X_1>n, X_2>n\}G^n(n,n)\\ &\ge c\lim_{n\to\infty} n\mathbb{P}\{X_1>n, X_2>n\},\end{aligned}$$

for some $c \in (0, e^{-2})$, hence ii) follows.
Next, assume that ii) holds. We have that $G(x_1,x_2) \le G_1(x_1)G_2(x_2) =: K(x_1,x_2)$ and by the assumption that G_i's are in the MDA of Φ it follows that K is in the MDA of H_∞. Further ii) implies that $F \in MDA(H_0)$ and $-\ln H(1,1) = 2$. Consequently, Theorem 2.3 yields

$$\lim_{n\to\infty} \lambda_n(K^n, F) = \lambda(H_\infty, H_0).$$

But from Corollary 4.4 we have that $\lambda(H_\infty, H_0) = 0$, hence ii) implies iii).
Let $\overline{G}$ be the joint survival function of the bivariate df G. For any positive integer n, we have that F^n is a bivariate df. Hence by Fubini theorem and the fact that $F_i = G_i, i = 1, 2$ are continuous dfs, for any positive integer n we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} F^n(x_1,x_d)dG(x_1,x_2) &= \int_{\mathbb{R}^2} \overline{G}(x_1,x_2)dF^n(x_1,x_2)\\ &= 2n/(n+1) - \int_{\mathbb{R}^2}[1-G(x_1,x_2)]dF^n(x_1,x_2)\end{aligned}$$

and thus the equivalence of iii) and iv) follows. The equivalence iv) and v) follows from Lemma 4.1 and thus the proof is complete. □

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