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Copula-based dependence measures for piecewise monotonicity

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Abstract: The aim of the present paper is to develop and examine association coefficients which can be helpfully applied in the framework of regression analysis. The construction of the coefficients is connected with the well-known Spearman coefficient and extensions of it (see Liebscher [5]). The proposed coefficient measures the discrepancy between the data points and a function which is strictly increasing on one interval and strictly decreasing in the remaining domain. We prove statements about the asymptotic behaviour of the estimated coefficient (convergence rate, asymptotic normality).

Keywords: dependence measures, Spearman's ρ, Spearman's footrule, estimators for dependence measures, piecewise monotonicity

MSC: 62H20

1 Introduction

The aim of this paper is to investigate measures for dependence which can be useful in performing regression analysis. The measures to be introduced are based on copulas since one intention is to eliminate the influence of the marginal distribution. These measures are then robust against outliers. Moreover, they are invariant under monotone transformations. In the modelling step of regression analysis, the task often arises: How well can data points be approximated by a function which is increasing on one interval and decreasing on the remaining region? For this reason, we develop special dependence coefficients for monotonicity in certain subdomains which can be helpfully applied in the framework of regression analysis. Especially, the coefficients can support checks on whether the regression function is convex because nonconvex functions can be ruled out. The result of the dependence analysis should give us an idea what kind of functions can be used to describe a regression relationship or to improve an established regression function.

We consider a coefficient on the basis of the dependence measure introduced in Liebscher [5]. This measure represents a generalization of Spearman's rho. In [6], Chapter 4, the reader finds a review of classical copula-based dependence measures. Schmid et al. [10] give a good survey on multivariate dependence measures. We refer also to the references and the discussion in [5]. In the paper Grothe et al. [3], the authors deal with dependence measures of two random vectors. Though the aim is different than ours, these association measures can also be applied in the framework of regression analysis, especially for model-building.

Several authors examined methods for checking the regression function to be convex. Juditsky and Nemirovski [4] considered tests for convexity in a general framework by using kernel estimators for the regression function. Incorporating spline estimates, Wang and Meyer developed tests for monotonicity and convexity in [14]. The advantage of our approach is that we avoid using nonparametric estimators for the regression function and so obtain estimators with a faster convergence rate. Moreover, we do not have the difficulty of choosing knots in splines or bandwidths for kernel estimators.

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The paper is organized as follows: In Section 2 we discuss known dependence coefficients. In Section 3 we introduce a coefficient of monotonicity of order 2, and discuss properties of it. Section 4 is devoted to the estimation of the measure introduced in Section 3. We provide results on the asymptotic behaviour of the estimated coefficient including the convergence rate and the asymptotic normality of the estimator. The reader finds the proofs of the theorems of Section 4 in Section 5.

2 Dependence coefficients of two variables

In this section we consider dependence measures of real random variables X and Y. Let F and G be the distribution functions of X, Y, respectively. It is assumed that F and G are continuous. H denotes the joint distribution function of X and Y. In view of Sklar's Theorem in [12], we have

$$H(x,y) = C(F(x), G(y)) \quad \text{for } x, y \in \mathbb{R}.$$

Hereby $C \in \mathcal{C}$ is the uniquely determined copula of X, Y. $\mathcal{C}$ denotes the set of all bivariate copulas. Spearman's ρ can be computed by the following formulas (cf. [6], p. 167):

$$\begin{aligned} \rho_S(C) &= 1 - 6 \cdot \int_{[0,1]^2} (u-v)^2 \,\mathrm{d}C(u,v) = 12 \int_{[0,1]^2} uv \,\mathrm{d}C(u,v) - 3 \\ &= 12 \int_0^1 \int_0^1 C(u,v) \,\mathrm{d}u\mathrm{d}v - 3 \qquad (1) \\ &= 1 - 6\mathbb{E}(F(X) - G(Y))^2. \end{aligned}$$

Note that in theses formulas, ρ_S is a map $\mathcal{C} \to [-1, 1]$. This notation emphasizes that the coefficient only depends on the copula and not on the marginal distributions. This property holds for all coefficients in this paper. Next we consider a dependence coefficient as a generalization of Spearman's rho. Let $\psi : [-1, 1] \to \mathbb{R}$ be a function satisfying the following assumption:

Assumption $\mathcal{A}_1$: $\psi(0) = 0$, $\psi(x) \geq 0$ and $\psi(-x) = \psi(x)$ holds for $x \in [0, 1]$. Assume that ψ is strictly increasing on $[0, 1]$ and Lipschitz-continuous.

According to [5], the generalized coefficient of monotonically increasing dependence of X and Y is defined by:

$$\zeta(C) = 1 - \int_{[0,1]^2} \psi(u-v) \,\mathrm{d}C(u,v) \cdot \bar{\psi}^{-1}, \qquad (2)$$

where ζ is a map $\mathcal{C} \to [\zeta_{\min}, 1]$, and

$$\bar{\psi} = 2 \int_0^1 (1-u)\psi(u) \,\mathrm{d}u, \quad \zeta_{\min} = 1 - \int_0^1 \psi(u) \,\mathrm{d}u \cdot \bar{\psi}^{-1}. \qquad (3)$$

This implies

$$\zeta(C) = 1 - \mathbb{E}\psi(F(X) - G(Y)) \cdot \bar{\psi}^{-1}.$$

The coefficients $\rho_S(C)$ and $\zeta(C)$ measure the discrepancy between the random data points (X, Y) and a (suitably chosen) monotonically increasing function, and simultaneously, the discrepancy between the points $(F(X), G(Y))$ having distribution function C and the line from $(0, 0)$ to $(1, 1)$. Two copulas are of special interest here:

$$M(u,v) = \min(u,v), \quad \Pi(u,v) = uv.$$

Coefficient ζ achieves the maximum value 1 only in the case of $C = M$ (comonotonicity); i.e. in the case where Y is a strictly increasing function of X almost surely. Coefficient ζ gives the value 0 in the case $C = \Pi$ where X and Y are independent. Thus the coefficient ζ is a linear transformation of the integral in (2) such that $\zeta(\Pi) = 0$ and $\zeta(M) = 1$. Moreover, $\zeta_{\min} = \zeta(W)$ where $W(u, v) = \max(0, u + v - 1)$ (antimonotonicity).

The following table shows several potential functions ψ.

coefficient	$\psi(t)$	$\bar{\psi}$
Spearman coefficient	t^2	$\frac{1}{6}$
Spearman footrule	$\lvert t\rvert$	$\frac{1}{3}$
power with $p \geq 1$	$\lvert t\rvert^p$	$\frac{2}{p^2+3p+2}$
Huber function with $\kappa = 0.5$	$\frac{1}{2}t^2$ for $\lvert t\rvert \leq \frac{1}{2}$, $\frac{1}{2}\lvert t\rvert - \frac{1}{8}$ otherwise	$\frac{5}{64}$

A detailed discussion about the dependence measure $\zeta(C)$ according to (2) can be found in [5]. In the following, we adopt this approach to develop association measures in the situation of piecewise monotonic functions.

3 Measures of piecewise monotonicity

We call a function $h : \mathbb{R} \to \mathbb{R}$ *piecewise monotonic of order 2* iff h is strictly increasing on D and strictly decreasing on $\mathbb{R}\backslash(D \cup \{x_0\})$ where $D = (-\infty, x_0)$ or $(x_0, +\infty)$. Non-monotonic convex functions and non-monotonic concave ones represent prominent classes of functions which are piecewise monotonic of order 2. Furthermore there are non-convex functions being piecewise monotonic of order 2. One example is $h(x) = \frac{x^2}{x^2+1}$. Now we are searching for a statistical coefficient describing how well the response variable Y can be approximated by a functional value $h(X)$ of the regressor X where h is a suitable function.

Let all settings of the previous section (e.g. $X, Y, H, F, G, \psi\ldots$) be also valid in Section 3. Suppose that F, G are continuous and H is strictly increasing. Let I be a closed subinterval of $(0, 1)$, and $a \in I$. The conditional distribution functions of X and Y given $F(X) \leq a$ are denoted by $F_{\leq a}$ and $G_{\leq a}$, respectively. $F_{>a}$ and $G_{>a}$ are these conditional distribution function given $F(X) > a$. Hence

$$\begin{aligned}
F_{\leq a}(x) &= \mathbb{P}\left\{X \leq x \mid F(X) \leq a\right\} = \min\{\tfrac{1}{a}F(x), 1\}, \\
F_{>a}(x) &= \mathbb{P}\left\{X \leq x \mid F(X) > a\right\} = \max\{\tfrac{1}{1-a}\left(F(x) - a\right), 0\}, \\
G_{\leq a}(y) &= \mathbb{P}\left\{Y \leq y \mid F(X) \leq a\right\} = \frac{H(F^{-1}(a), y)}{a}, \text{ and} \\
G_{>a}(y) &= \mathbb{P}\left\{Y \leq y \mid F(X) > a\right\} = \frac{G(y) - H(F^{-1}(a), y)}{1 - a}
\end{aligned}$$

for $x, y \in \mathbb{R}$. In the following, we establish a coefficient for piecewise monotonicity in the case $D = (-\infty, x_0)$. Here two main ideas are involved: First, we split the copula domain into two parts $[0, a] \times [0, 1]$ and $[a, 1] \times [0, 1]$. Secondly, the copula distribution in the second region $[a, 1] \times [0, 1]$ is reflected over the line $v = \frac{1}{2}$. Here $F^{-1}(y) = \inf\{x : F(x) \geq y\}$ is the generalised inverse function of F.

Define $U = F(X)$, $V = G(Y)$. Let $\tilde{C}_a^1$ be the copula of X and Y given $F(X) \leq a$, which is the distribution function of $F_{\leq a}(X)$ and $G_{\leq a}(Y)$ given $F(X) \leq a$. Observe that given $F(X) \leq a$, $G_{\leq a}(Y) = a^{-1}C(a, V)$ $a.s.$ holds and

$$\begin{aligned}
\tilde{C}_a^1(u, v) &= \mathbb{P}\left\{F_{\leq a}(X) \leq u, G_{\leq a}(Y) \leq v \mid F(X) \leq a\right\} \\
&= \frac{1}{a}\mathbb{P}\left\{U \leq au, a^{-1}C(a, V) \leq v\right\} \\
&= \frac{1}{a}\int_{[0,au]\times[0,1]} \mathbf{1}\left(a^{-1}C(a, t) \leq v\right)\, \mathrm{d}C(s, t) \qquad (4)
\end{aligned}$$

($u, v \in [0,1]$). Here $\mathbf{1}(.)$ is the indicator of an inequality being 1 exactly in the case where the inequality between the parentheses is fulfilled. According to (4), $\tilde{C}_a^1$ is completely based on the copula C and it is independent of F and G. Now we consider the copula in the second region $[a, 1] \times [0, 1]$. Let $\bar{G}_a$ be the distribution function of $-Y$ given $F(X) > a$. This distribution function is related to $G_{>a}$ as follows:

$$\bar{G}_a(y) := \mathbb{P}\left\{-Y \le y \mid F(X) > a\right\} = 1 - G_{>a}(-y)$$

for $y \in \mathbb{R}$. Therefore

$$\bar{G}_a(-Y) = 1 - G_{>a}(Y) = 1 - \frac{V - C(a, V)}{1-a} \quad a.s.$$

Further the copula $\tilde{C}_a^2$ of X and $-Y$ given $F(X) > a$ can be computed as the distribution function of $F_{>a}(X)$ and $\bar{G}_a(-Y)$ given $F(X) > a$:

$$\begin{aligned} \tilde{C}_a^2(u, v) &= \mathbb{P}\left\{F_{>a}(X) \le u, \bar{G}_a(-Y) \le v \mid F(X) > a\right\} \\ &= \tfrac{1}{1-a}\mathbb{P}\left\{\tfrac{1}{1-a}(U - a) \le u, 1 - \tfrac{1}{1-a}(V - C(a, V)) \le v, U > a\right\} \\ &= \tfrac{1}{1-a} \int\limits_{[a, a+u-ua]\times[0,1]} \mathbf{1}\left(1 - \tfrac{1}{1-a}\left(t - C(a, t)\right) \le v\right) \mathrm{d}C(s, t) \end{aligned} \tag{5}$$

($u, v \in [0, 1]$). Geometrically, copula $\tilde{C}_a^2$ of X and $-Y$ given $F(X) > a$ is obtained from the copula of X and Y given $F(X) > a$ by reflecting the distribution over the line $v = \frac{1}{2}$:

$$\tilde{C}_a^2(u, v) = u - \mathbb{P}\left\{F_{>a}(X) \le u, G_{>a}(Y) \le 1 - v \mid F(X) > a\right\}.$$

The reason for considering here the copula of X and $-Y$ (i.e. the distribution function of U and $1 - V$) and not the copula of X and Y is that we measure the deviation of Y from a decreasing function. As in the case of $\tilde{C}_a^1$, we see that $\tilde{C}_a^2$ is completely based on the copula C and it is independent of F and G according to (5).

Let Assumption $\mathcal{A}_1$ be satisfied. For fixed a, we define the *first coefficient of piecewise monotonicity of order 2* of X, Y as the combination of the coefficients for increasing behaviour on the two subintervals:

$$\zeta_a^{+-}(C) := \zeta(\tilde{C}_a^1)a + \zeta(\tilde{C}_a^2)(1-a) \tag{6}$$

where $\zeta(.)$ is the dependence coefficient defined in (2). Define $\mathbf{1}\{A\}$ of an event A to be 1 if A occurs, and to be 0 if A not occurs. Hence

$$\begin{aligned} \zeta_a^{+-}(C) &= \left(1 - \mathbb{E}\left(\psi\left(F_{\le a}(X) - G_{\le a}(Y)\right) \mid F(X) \le a\right) \cdot \bar{\psi}^{-1}\right) a \\ &\quad + \left(1 - \mathbb{E}\left(\psi\left(F_{>a}(X) - \bar{G}_a(-Y)\right) \mid F(X) > a\right) \cdot \bar{\psi}^{-1}\right)(1-a) \\ &= 1 - \mathbb{E}\left(\psi(F_{\le a}(X) - G_{\le a}(Y))\mathbf{1}\left\{F(X) \le a\right\} + \psi\left(F_{>a}(X) - \bar{G}_a(-Y)\right)\mathbf{1}\left\{F(X) > a\right\}\right) \cdot \bar{\psi}^{-1}. \\ &= 1 - \int\limits_{[0,1]^2} \left(\psi(\tfrac{1}{a}u - \tfrac{1}{a}C(a, v))\mathbf{1}(u \le a) + \psi\left(\tfrac{1}{1-a}(u - 1 + v - C(a, v))\right)\mathbf{1}(u > a)\right) \mathrm{d}C(u, v) \cdot \bar{\psi}^{-1}. \end{aligned} \tag{7}$$

We introduce the *first total coefficient of piecewise monotonicity of order 2* of X, Y by

$$\zeta^{+-}(C) := \max_{a \in I} \zeta_a^{+-}(C). \tag{8}$$

The maximizer in (8) exists (not necessarily unique) since $a \rightsquigarrow \zeta_a^{+-}(C)$ is a continuous function in view of Lemma 5. The coefficient $\zeta_a^{+-}(C)$ describes the discrepancy between the random data points (X, Y) and a function which is strictly increasing on $(-\infty, F^{-1}(a))$ and strictly decreasing on $(F^{-1}(a), +\infty)$, see also the detailed explanation about ζ in [5]. Moreover, $\zeta^{+-}(C)$ represents a measure on how Y approaches a piecewise monotonic function of X which is initially increasing.

Important properties of the ζ-coefficients are summarized in the next theorem. These properties are similar to that in the definition of a "measure of concordance" by Scarsini; see [9]. Since some assertions uses several copulas depending on the corresponding random variables, we denote the copula of random variables X, Y by $C_{X,Y}$.

Theorem 1. *Suppose that Assumption $\mathcal{A}_1$ is satisfied. Then for $C \in \mathcal{C}$, $a \in (0,1)$,*

a) $\zeta_{\min} \le \zeta^{+-}(C) \le 1$ and $\zeta_{\min} \le \zeta_a^{+-}(C) \le 1$, $\zeta_{\min}$ as in (3),

b) $\zeta_a^{+-}(\Pi) = 0$ and $\zeta^{+-}(\Pi) = 0$,

c) $\zeta_a^{+-}(M) = a + \zeta_{\min}(1-a)$ and $\zeta_a^{+-}(W) = a\zeta_{\min} + 1 - a$.

For random variables X, Y with copula $C_{X,Y} \in \mathcal{C}$, we have:

d) For every $a \in (0,1)$, the identity $\zeta_a^{+-}(C_{X,Y}) = 1$ holds iff $Y = h_1(X)$ a.s. for $\omega : F(X) \le a$ and $Y = h_2(X)$ a.s. for $\omega : F(X) > a$ with a suitable strictly increasing function h_1 and a suitable strictly decreasing function h_2.

e) The identity $\zeta^{+-}(C_{X,Y}) = 1$ holds iff $Y = h(X)$ a.s. with a suitable strictly piecewise monotonic function h of order 2 which is initially increasing.

f) $\zeta_a^{+-}(C_{-X,Y}) = \zeta_{1-a}^{+-}(C_{X,Y})$ for $a \in (0,1)$ and $\zeta^{+-}(C_{-X,Y}) = \zeta^{+-}(C_{X,Y})$ provided that $I = [\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$.

g) $\zeta_a^{+-}(C_{g(X),h(Y)}) = \zeta_a^{+-}(C_{X,Y})$ and $\zeta^{+-}(C_{g(X),h(Y)}) = \zeta^{+-}(C_{X,Y})$ hold for all strictly increasing functions g, h and $a \in (0,1)$.

Moreover, for copulas $C, C^\star \in \mathcal{C}$, the following properties hold:

h) Let $\tilde{C}_a^1$ and $\tilde{C}_a^2$ be the copulas defined by (4) and (5). If C is replaced by $C^\star$ in (4) and (5), we obtain copulas $\check{C}_a^1$ and $\check{C}_a^2$. Then, for every $a \in I$, $\tilde{C}_a^j \prec \check{C}_a^j$ for $j = 1, 2$ implies that $\zeta_a^{+-}(C) \le \zeta_a^{+-}(C^\star)$. Hereby $C_1 \prec C_2$ means $C_1(u,v) \le C_2(u,v)$ for all $u, v \in [0,1]$.

i) Let $\{C_n\}$ be any sequence of copulas tending pointwise to C. Then $\zeta_a^{+-}(C_n) \to \zeta_a^{+-}(C)$ for every $a \in (0,1)$ and $\zeta^{+-}(C_n) \to \zeta^{+-}(C)$.

Proof. a) Obvious in view of (6).
b) In the case $C = \Pi$, $\tilde{C}_a^1 = \tilde{C}_a^2 = \Pi$ holds for all a and

$$\zeta(\tilde{C}_a^1) = \zeta(\tilde{C}_a^2) = 0.$$

c) Observe that $\tilde{C}_a^1 = M$ and $\tilde{C}_a^2 = W$ if $C = M$, and $\tilde{C}_a^1 = W$ and $\tilde{C}_a^2 = M$ if $C = W$. This leads directly to assertion c).
d) Equation $\zeta_a^{+-}(C_{X,Y}) = 1$ holds exactly in the case where $\zeta(\tilde{C}_a^1) = 1$ and $\zeta(\tilde{C}_a^2) = 1$. This is, in turn, equivalent to $\tilde{C}_a^j(u,v) = \min(u,v)$ for $j = 1, 2$, $u, v \in [0,1]$, and equivalent to $F_{\le a}(X(\omega)) = G_{\le a}(Y(\omega))$ $a.s.$ for $\omega : F(X(\omega)) \le a$ and $F_{>a}(X(\omega)) = \bar{G}_a(-Y(\omega))$ $a.s.$ for $\omega : F(X(\omega)) > a$ in view of (4) and (5). Now the claim c) follows since F, $G_{\le a}^{-1}$ and $\bar{G}_a^{-1}$ are strictly increasing functions.
e) Equation $\zeta^{+-}(C_{X,Y}) = 1$ is fulfilled iff $\zeta_{a_0}^{+-}(C_{X,Y}) = 1$ for some a_0. Hence part d) yields claim e).
Claim f) follows immediately from (7) by a change of variables.
Claim g) is trivial since $C_{g(X),h(Y)} = C_{X,Y}$.
h) From Theorem 2.1b) in [5], it follows that

$$\zeta(\tilde{C}_a^j) \le \zeta(\check{C}_a^j) \text{ for } j = 1, 2$$

which implies claim h).
Claim i) follows from Lemma 7. □

Property b) of Theorem 1 means that for independent random variables X and Y, the coefficient is equal to 0. Property e) shows that the coefficient equals 1 exactly in the case where the data points (X, Y) lie almost surely on a piecewise monotonic function which is initially increasing. Claim h) states that the concordance inequality carries over from ζ to ζ_a^{+-} in a certain way. Part i) represents the continuity property of the coefficients w.r.t. the copula.

Observe that

$$C_{X,-Y}(u,v) := u - C_{X,Y}(u, 1-v) \text{ for } u, v \in [0,1]$$

is the copula of $(X, -Y)$. Further we define the *second (total) coefficient of piecewise monotonicity of order 2* of X, Y by

$$\zeta_a^{-+}(C_{X,Y}) := \zeta_a^{+-}(C_{X,-Y}) \text{ and } \zeta^{-+}(C_{X,Y}) := \max_{a \in I} \zeta_a^{-+}(C_{X,Y}).$$

Hence

$$\zeta^{-+}(C_{X,Y}) = \zeta^{+-}(C_{X,-Y}).$$

Similarly to the above identity (6), we have

$$\zeta_a^{+-}(C) = \zeta(\tilde{C}_a^3)a + \zeta(\tilde{C}_a^4)(1-a),$$

where $\tilde{C}_a^3$ is the the copula of X and $-Y$ given $F(X) \le a$, and $\tilde{C}_a^4$ is the copula of X and Y given $F(X) > a$. The coefficient $\zeta^{-+}(C_{X,Y})$ describes the discrepancy between the data points (X, Y) and a function which is strictly decreasing on an interval $(-\infty, x_0)$ and strictly increasing on $(x_0, +\infty)$ for some x_0. From these identities, one can see that there is a close relationship between ζ^{-+} and ζ^{+-}, and properties of ζ^{-+} can be derived immediately from Theorem 1.

Theorem 2. *Let Assumption $\mathcal{A}_1$ be satisfied. Then, for $C \in \mathcal{C}$, $a \in (0,1)$,*

a) $\zeta_{\min} \le \zeta^{-+}(C) \le 1$ and $\zeta_{\min} \le \zeta_a^{-+}(C) \le 1$, $\zeta_{\min}$ as in (3),

b) $\zeta_a^{-+}(\Pi) = 0$ and $\zeta^{-+}(\Pi) = 0$,

c) $\zeta_a^{-+}(M) = a\zeta_{\min} + 1 - a$ and $\zeta_a^{-+}(W) = a + \zeta_{\min}(1-a)$.

For random variables X, Y with copula $C_{X,Y} \in \mathcal{C}$, we have:

d) For every $a \in (0,1)$, the identity $\zeta_a^{-+}(C_{X,Y}) = 1$ holds iff $Y = h_1(X)$ $a.s.$ for $\omega : F(X) \le a$ and $Y = h_2(X)$ $a.s.$ for $\omega : F(X) > a$ with a suitable strictly decreasing function h_1 and a suitable strictly increasing function h_2.

e) The identity $\zeta^{-+}(C_{X,Y}) = 1$ holds iff $Y = h(X)$ $a.s.$ with a suitable strictly piecewise monotonic function h of order 2 which is initially decreasing.

f) $\zeta_a^{-+}(C_{-X,Y}) = \zeta_{1-a}^{-+}(C_{X,Y})$ for $a \in (0,1)$ and $\zeta^{-+}(C_{-X,Y}) = \zeta^{-+}(C_{X,Y})$ provided that $I = [\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$.

g) $\zeta_a^{-+}(C_{g(X),h(Y)}) = \zeta_a^{-+}(C_{X,Y})$ and $\zeta^{-+}(C_{g(X),h(Y)}) = \zeta^{-+}(C_{X,Y})$ hold for all strictly increasing functions g, h and $a \in (0,1)$.

Moreover, for copulas $C, C^\star \in \mathcal{C}$, the following properties hold:

h) Let $\{C_n\}$ be a sequence of copulas tending pointwise to C. Then $\zeta_a^{-+}(C_n) \to \zeta_a^{-+}(C)$ for every $a \in (0,1)$ and $\zeta^{-+}(C_n) \to \zeta^{-+}(C)$.

The remarks after Theorem 1 apply to this theorem analogously.

The concept presented in this section can be carried over to monotonicity of higher order. For this purpose the copula domain $[0,1]^2$ is split into more than two subdomains. The corresponding coefficient of piecewise monotonicity of higher order can then be established as a weighted sum of the appropriately defined coefficients for the subdomains.

4 Estimating the coefficient of piecewise monotonicity

4.1 Estimator

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sample of independent random vectors with distribution function H and copula C. F and G are the marginal distribution functions of X_i and Y_i, respectively. Suppose that F and G are continuous with densities f and g, respectively. In Section 4, we deal with properties of estimators for association measures $\zeta_a^{+-}(C)$ and $\zeta^{+-}(C)$. Similarly, one can treat estimators for the other coefficients. First, we introduce estimators for the distribution functions F, $G_{\le a}$ and $\bar{G}_a$:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{X_i \le x\right\},$$
$$G_{n,a}(y) = \frac{1}{an}\sum_{i=1}^{n} \mathbf{1}\left\{Y_i \le y, F_n(X_i) \le a\right\},$$
$$\bar{G}_{n,a}(y) = \frac{1}{n-an}\sum_{i=1}^{n} \mathbf{1}\left\{-Y_i \le y, F_n(X_i) > a\right\}.$$

The coefficient $\zeta_a^{+-}(C)$ can be estimated by

$$\hat{\zeta}_n^{+-}(a) = 1 - \frac{1}{n\bar{\psi}}\sum_{i=1}^{n}\Big(\psi(\tfrac{1}{a}F_n(X_i) - G_{n,a}(Y_i))\mathbf{1}\left\{F_n(X_i) \le a\right\}$$
$$+\psi(\tfrac{1}{1-a}\left(F_n(X_i) - a\right) - \bar{G}_{n,a}(-Y_i))\mathbf{1}\left\{F_n(X_i) > a\right\}\Big).$$

Let $I \subset (0,1)$ be a given closed interval. For defining an estimator of $\zeta^{+-}(C)$, it is reasonable to consider a maximization on a grid $I_n = \{\frac{j}{n} \in I, j \in \{0,\dots,n\}\}$:

$$\hat{\zeta}_n^{+-} = \max_{a\in I_n} \hat{\zeta}_n^{+-}(a) = \hat{\zeta}_n(a_n)$$

with a suitable a_n. Applying this approach to the modified data $(X_1,-Y_1),\dots,(X_n,-Y_n)$, we arrive at the coefficients $\hat{\zeta}_n^{-+}(a)$ and $\hat{\zeta}_n^{-+}$. These coefficients can support the detection of non-monotonic convex or concave functions in data points. If $\hat{\zeta}_n^{+-}$, respectively $\hat{\zeta}_n^{-+}$, is rather small, then we cannot expect that the Y values can be approximated well by a non-monotonic concave function of the X values, respectively a non-monotonic convex function of the X values.

The following theorem gives a convergence rate of the estimator ζ_n^{+-} to the true underlying coefficient $\zeta^{+-}(C)$.

Theorem 3. *Assume that Assumption $\mathcal{A}_1$ is satisfied.*

a) Then

$$\hat{\zeta}_n^{+-} = \zeta^{+-}(C) + O\left(\sqrt{\frac{\ln n}{n}}\right) \quad a.s.$$

b) If $a \rightsquigarrow \zeta_a^{+-}(C)$ has a unique maximizer a_0 on I, then $a_n \to a_0$ a.s.

The a.s. convergence rate $O\left(n^{-1/2}\sqrt{\ln n}\right)$ of the estimator $\hat{\zeta}_n^{+-}$ is only slightly worse than the usual a.s. rate $O\left(n^{-1/2}\sqrt{\ln\ln n}\right)$ of estimators for dependence coefficients. The reason for this difference can be seen in the maximization w.r.t. a.

Define $\bar{a} := F^{-1}(a)$. Let us now introduce the conditional distribution function $\gamma(y,x) = \mathbb{P}\{Y_1 \le y \mid X_1 = x\}$, $\tilde{\psi}(x) = \mathbb{E}\left(\psi(1 - G_{\le a}(Y_1)) - \psi(\bar{G}_a(-Y_1)) \mid X_1 = x\right)$, and assumptions which will be needed in the next Theorem 4:

Assumption $\mathcal{A}_2$: Suppose that for any sequence $\eta_n \to 0$,

$$\lim_{n\to\infty}\ \sup_{x:\bar{a}-\eta_n \le x \le \bar{a}+\eta_n}\ \sup_{y\in\mathbb{R}} \left|\gamma(y,x) - \gamma(y,\bar{a})\right| = 0. \quad \square$$

Assumption $\mathcal{A}_3$: Assume that

$$\sup_{x_1,x_2\in[0,1]} \frac{\left|\psi'(x_1) - \psi'(x_2)\right|}{\left|x_1 - x_2\right|^{\alpha}} < +\infty \text{ for some } \alpha \in (0,1]. \quad \square$$

The following theorem provides the asymptotic normality result for the coefficient $\zeta_a^{+-}(C)$ with fixed parameter a.

Theorem 4. *Let $a \in I$ be fixed. Suppose that Assumptions $\mathcal{A}_1$ to $\mathcal{A}_3$ are satisfied. Assume that f and $\tilde{\psi}$ are continuous at $\bar{a}$, and $f(\bar{a}) > 0$. Then*

$$\sqrt{n}\left(\hat{\zeta}_n^{+-}(a) - \zeta_a^{+-}(C)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

where $\xrightarrow{d}$ *denotes convergence in distribution,* $\sigma^2 = 4 \operatorname{Var} \Phi(X_1, Y_1)\bar{\psi}^{-2}$,

$$\begin{aligned}
&\Phi(\xi, \chi) \\
= &\int_{(-\infty,\bar{a}]\times\mathbb{R}} \psi'(F_{\le a}(x) - G_{\le a}(y)) \left(\tfrac{1}{a}\left(\mathbf{1}(\xi \le x) - F(x)\right)\right. \\
&\left. -\tfrac{1}{a}\mathbf{1}(\chi \le y, \xi \le \bar{a}) + G_{\le a}(y) + \tfrac{1}{a}\left(\mathbf{1}(\xi \le \bar{a}) - a\right)\gamma(y, \bar{a})\right) dH(x, y) \\
&+ \int_{[\bar{a},\infty)\times\mathbb{R}} \psi'(F_{>a}(x) - \bar{G}_a(-y)) \left(\tfrac{1}{1-a}\left(\mathbf{1}(\xi \le x) - F(x)\right) + \bar{G}_a(-y)\right. \\
&\left. -\tfrac{1}{1-a}\mathbf{1}(y \le \chi, \xi > \bar{a}) - \tfrac{1}{1-a}\left(\mathbf{1}(\xi \le \bar{a}) - a\right)\left(1 - \gamma(y, \bar{a})\right)\right) dH(x, y) \\
&-\tilde{\psi}(\bar{a})\left(\mathbf{1}(\xi \le \bar{a}) - a\right) + \psi(F_{\le a}(\xi) - G_{\le a}(\chi))\mathbf{1}(\xi \le \bar{a}) \\
&+\psi(F_{>a}(\xi) - \bar{G}_a(-\chi))\mathbf{1}(\xi > \bar{a}) - \check{\zeta}(a).
\end{aligned}$$

The structure of the variance is rather complicated, and it is not easy to establish a reasonable estimator of it. The conditional expectations are problematic in this respect.

4.2 Applications

In connection with this paper, an R script was prepared and used for the computations. In this section we report on the results for one example dataset and the results of simulations.

Example: We consider the dataset "engine" of [1]. Variable X contains the values of the fuel/air ratio whereas variable Y gives the nitrogen oxides output of the engine. The dataset comprises $n = 80$ data points.

The unit two-dimensional interval $[0, 1]^2$ is split in two parts at $a^* = 0.52$, the maximizer of $\hat{\zeta}_n^{+-}(a)$. In the left subarea and in the right subarea, we measure the distance of the data points to the dashed line separately. The estimated coefficient gives information as to how far the data points are from the ideal situation in which they would lie on the dashed line.

The following table provides the values of the coefficients.

	$\hat{\zeta}_n^{+-}$
Spearman (S)	0.9497953
Spearman's footrule (F)	0.7975014
Huber $\kappa = 0.5$ (H)	0.9464483
power $p = 1.5$ (P)	0.9011831

For all coefficients, the maximum is achieved at the same point $\frac{64}{n}$ on the u-axis. We can see that the data points are rather close to a piecewise monotonic function of order 2 which is initially increasing. In the context of regression analysis, it makes sense to look for a suitable piecewise monotonic regression function describing the trend of the data.

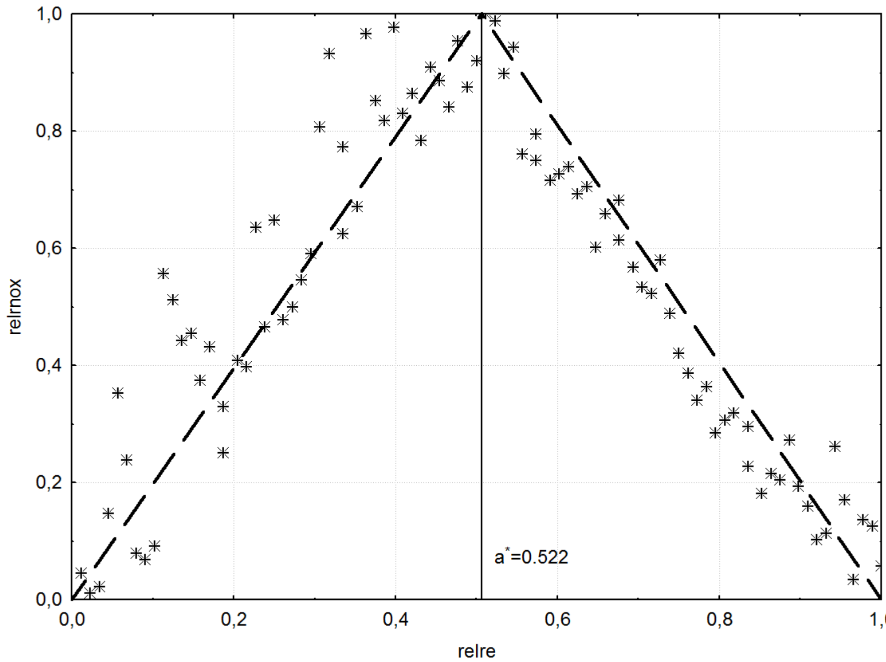

Figure 1: empirical copula of the dataset "engine". The dashed line shows the ideal line for perfect piecewise monotonicity.

Simulations: Let us consider the copula

$$C(u,v) = \begin{cases} a\check{C}_1(\frac{u}{a}, v) & \text{for } u \le a \\ av + (u-a) - (1-a)\check{C}_2(\frac{u-a}{1-a}, 1-v) & \text{for } u > a \end{cases}$$

where $\check{C}_1$ and $\check{C}_2$ are Clayton copulas with parameter θ. We put $a = 0.6$. This copula C can simulated by the following algorithm

1) Generate random value $W \sim$ U[0,1] from uniform distribution, and random vectors $(U_1, V_1) \sim \check{C}_1, (U_2, V_2) \sim \check{C}_2$.

2) If $W \le a$, then $V = V_1$, $U = U_1 \cdot a$.

3) If $W > a$, then $V = 1 - V_2$, $U = a + U_1(1-a)$.

We generated 50000 samples of different sizes ($n = 100, 200$) for (U, V). For each sample, the estimator $\hat{\zeta}_n^{+-}$ is computed (S=Spearman, F=Spearman's footrule, H=Huber function, P=power function with $p = 1.5$). On the other hand, the exact theoretical value of the coefficients is calculated for comparisons. Some simulation results are summarized in the following table:

θ	n	coeff.	average	bias	standard deviation
0.5	100	F	0.19685	0.01004	0.06211
	200	F	0.19236	0.00556	0.04453
	100	S	0.31053	0.01559	0.09054
	200	S	0.30461	0.00966	0.06492
	100	P	0.25816	0.01161	0.07826
	200	P	0.25340	0.00685	0.05612
	100	H	0.29350	0.01418	0.08818
	200	H	0.28806	0.00874	0.06321
2	100	F	0.48146	-0.00382	0.05322
	200	F	0.48330	-0.00198	0.03776
	100	S	0.68268	0.00045	0.05991
	200	S	0.68398	0.00175	0.04240
	100	P	0.59908	-0.00230	0.05854
	200	P	0.60106	-0.00033	0.04145
	100	H	0.66540	0.00014	0.06162
	200	H	0.66687	0.00160	0.04359
20	100	F	0.89297	-0.0126	0.01664
	200	F	0.89843	-0.00717	0.01034
	100	S	0.98251	-0.00455	0.00592
	200	S	0.98498	-0.00209	0.00333
	100	P	0.95776	-0.00843	0.00995
	200	P	0.96205	-0.00414	0.00596
	100	H	0.98135	-0.00486	0.00631
	200	H	0.98398	-0.00223	0.00355

From this table, we see that the bias is negligibly small in comparison to the standard deviation. In few cases, the bias increases from $n = 100$ to $n = 200$, but it keeps significantly smaller than the standard deviation. Furthermore, the standard deviation decreases as n increases. Consequently, the mean square error decreases as n increases.

5 Proofs

5.1 Auxiliary statements

In this section, we prove continuity properties of the ζ-coefficient and later, a convergence property of the maximum of the coefficient.

Lemma 5. *Under the Assumption $\mathcal{A}_1$, the function $a \rightsquigarrow \zeta_a^{+-}(C)$ is Lipschitz continuous.*

Proof. Let $I = [\underline{a}, \bar{a}]$. We obtain

$$\sup_{a,a' \in I: a \le a' \le a+\Delta} \left|\zeta_{a'}^{+-}(C) - \zeta_a^{+-}(C)\right| \le \bar{\psi}^{-1} \left(A_n^\star(\Delta) + B_n^\star(\Delta) + D_n^\star(\Delta)\right),$$

where

$$A_n^\star(\Delta) = \sup_{a,a'\in I: a\le a'\le a+\Delta} \int_{[0,a]\times[0,1]} \left|\psi(\tfrac{1}{a'}u - \tfrac{1}{a'}C(a',v)) - \psi(\tfrac{1}{a}u - \tfrac{1}{a}C(a,v))\right| \mathrm{d}C(u,v),$$

$$B_n^\star(\Delta) = \sup_{a,a'\in I: a\le a'\le a+\Delta} \int_{(a',1]\times[0,1]} \Big|\psi\left(\tfrac{1}{1-a'}(u-1+v-C(a',v))\right) - \psi\left(\tfrac{1}{1-a}(u-1+v-C(a,v))\right)\Big| \mathrm{d}C(u,v),$$

$$D_n^\star(\Delta) = \sup_{a,a'\in I: a\le a'\le a+\Delta} \int_{(a,a']\times[0,1]} \left(\psi(\tfrac{1}{a'}u - \tfrac{1}{a'}C(a',v)) + \psi\left(\tfrac{1}{1-a}(u-1+v-C(a,v))\right)\right) \mathrm{d}C(u,v)$$

Observe that C is Lipschitz continuous on $[0,1]^2$ with Lipschitz constant 1. Hence, by the Lipschitz continuity of ψ,

$$\begin{aligned} A_n^\star(\Delta) &\le \kappa_0 \sup_{a,a'\in I: a\le a'\le a+\Delta} \int_{[0,a]\times[0,1]} \left(a^{-2}\Delta u + \left|\tfrac{1}{a'}C(a',v) - \tfrac{1}{a}C(a,v)\right|\right) \mathrm{d}C(u,v) \\ &\le \kappa_0 \left(2\underline{a}^{-2}\Delta + \underline{a}^{-1}\Delta\right), \\ B_n^\star(\Delta) &\le \kappa_0 \sup_{a,a'\in I: a\le a'\le a+\Delta} \int_{(a',1]\times[0,1]} \left((1-a')^{-2}\Delta\,|u-1+v| + \left|\tfrac{1}{1-a'}C(a',v)) - \tfrac{1}{1-a}C(a,v)\right|\right) \mathrm{d}C(u,v) \\ &\le \kappa_0 \left(2(1-\bar{a})^{-2}\Delta + (1-\bar{a})^{-1}\Delta\right), \text{ and} \\ D_n^\star(\Delta) &\le \kappa_0 \sup_{a,a'\in I: a\le a'\le a+\Delta} \left(C(a',1) - C(a,1)\right) \le \kappa_0\Delta \end{aligned}$$

for any $\Delta > 0$ with a suitable constant $\kappa_0 > 0$. This proves the lemma. □

From Proposition 2.3 of [2] or from Theorem 7.33 of [8], one obtains immediately the following statement:

Proposition 6. *Let $\{\varphi_n\}$ be a sequence of continuous functions on a compact set $D \subset \mathbb{R}$. Assume that for every sequence $\{x_n\}$ with $x_n \to \bar{x} \in D$,*

$$\lim_{n\to\infty} \varphi_n(x_n) = \varphi(\bar{x})$$

holds with a continuous function φ. Then

$$\lim_{n\to\infty} \max_{x\in D} \varphi_n(x) = \max_{x\in D} \varphi(x).$$

Lemma 7. *Suppose that Assumption $\mathcal{A}_1$ is satisfied and $\{C_n\}$ is a sequence of copulas tending pointwise to C. Let $\varphi_n(a) = \zeta_a^{+-}(C_n)$, and $\varphi(a) = \zeta_a^{+-}(C)$ for $a \in I$. Then*

$$\begin{aligned} a)\ \lim_{n\to\infty} \varphi_n(a) &= \varphi(a) \text{ for any } a \in I \text{ and} \\ b)\ \lim_{n\to\infty} \max_{a\in I} \varphi_n(a) &= \max_{a\in I} \varphi(a). \end{aligned}$$

Proof. Here we show that the assumptions of Proposition 6 are satisfied such that the claim of Lemma 7 follows from this proposition. In view of Lemma 5, φ_n and φ are continuous. Let a be any real number belonging to I, and $\{a_n\}$ be any sequence of real numbers with $a_n \to a$. We have

$$\lim_{n\to\infty} \left(\zeta_{a_n}^{+-}(C_n) - \zeta_a^{+-}(C)\right) = \bar{\psi}^{-1} \lim_{n\to\infty} \left(b_n + d_n - e_n - f_n - g_n\right), \tag{9}$$

where

$$
\begin{aligned}
b_n &= \int\limits_{[0,a]\times[0,1]} \left(\psi(\tfrac{1}{a}u - \tfrac{1}{a}C(a,v)) - \psi(\tfrac{1}{a_n}u - \tfrac{1}{a_n}C_n(a_n,v))\right) \mathrm{d}C_n(u,v),\\
d_n &= \int\limits_{(a,1]\times[0,1]} \left(\psi\left(\tfrac{1}{1-a}(u-1+v-C(a,v))\right) - \psi\left(\tfrac{1}{1-a_n}(u-1+v-C_n(a_n,v))\right)\right)\\
&\qquad \mathrm{d}C_n(u,v),\\
e_n &= \int\limits_{[0,a]\times[0,1]} \psi(\tfrac{1}{a}u - \tfrac{1}{a}C(a,v))\mathrm{d}\left(C_n(u,v) - C(u,v)\right)\\
f_n &= \int\limits_{(a,1]\times[0,1]} \psi\left(\tfrac{1}{1-a}(u-1+v-C(a,v))\right)\mathrm{d}\left(C_n(u,v) - C(u,v)\right)\\
g_n &= \int\limits_{[0,1]^2} \Big(\psi(\tfrac{1}{a_n}u - \tfrac{1}{a_n}C_n(a_n,v))\left(\mathbf{1}(u \le a_n) - \mathbf{1}(u \le a)\right)\\
&\qquad + \psi\left(\tfrac{1}{1-a_n}(u-1+v-C_n(a_n,v))\right)\left(\mathbf{1}(u > a_n) - \mathbf{1}(u > a)\right)\Big)\mathrm{d}C_n(u,v)
\end{aligned}
$$

Since C is continuous and C_n is increasing for every n, the convergence of C_n to C is uniform. Using this uniform convergence and Lipschitz continuity of ψ, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} |b_n| &\le \text{const}\cdot \lim_{n\to\infty}\left(\tfrac{2}{a} - \tfrac{2}{a_n} + \tfrac{1}{a_n}\int\limits_{[0,a]\times[0,1]} \left|C(a,v) - C_n(a_n,v)\right| \mathrm{d}C_n(u,v)\right)\\
&\le \text{const}\cdot \lim_{n\to\infty}\left(\tfrac{2}{a} - \tfrac{2}{a_n} + \tfrac{1}{a_n}|a - a_n| + \sup_{v\in[0,1]}\left|C(a,v) - C_n(a,v)\right|\right)\\
&= 0.
\end{aligned}
$$

Analogously, one shows that

$$\lim_{n\to\infty} d_n = 0.$$

By the Portmanteau Theorem (cf. [13], p. 6),

$$\lim_{n\to\infty} e_n = 0 \text{ and } \lim_{n\to\infty} f_n = 0.$$

Moreover, we can derive

$$|g_n| \le 2 \sup_{t\in[-1,1]} \psi(t) \left|C_n(a_n,1) - C_n(a,1)\right| = 2 \sup_{t\in[-1,1]} \psi(t)\, |a_n - a|\,,$$

which implies $\lim_{n\to\infty} g_n = 0$. Consequently, the limit in (9) is equal to zero and the assumptions of Proposition 6 are proved. □

The next lemma provides convergence rates of the joint empirical distribution function H_n, of the marginal empirical distribution function F_n of X and conditional empirical distribution functions $G_{n,a}$ and $\bar{G}_{n,a}$ of Y which are defined in Section 4.1.

Lemma 8. *We have*

a)

$$\sup_{x,y\in\mathbb{R}} \left|H_n(x,y) - H(x,y)\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \; a.s.,$$

b)

$$\sup_{x\in\mathbb{R}} \left|F_n(x) - F(x)\right| \le \kappa_1 \sqrt{\frac{\ln\ln n}{n}} \; a.s.$$

for $n \geq n_0(\omega)$ *with a constant* $\kappa_1 > \frac{1}{2}\sqrt{2}$,

c)

$$\frac{1}{n}\sup_{a\in I}\sum_{i=1}^{n}\left|\mathbf{1}\left\{F_n(X_i)\leq a\right\} - \mathbf{1}\{F(X_i)\leq a\}\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right)\ a.s.,$$

d)

$$\sup_{y\in\mathbb{R},a\in I}\left|G_{n,a}(y) - G_{\leq a}(y)\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right)\ a.s.\ and$$

$$\sup_{y\in\mathbb{R},a\in I}\left|\bar{G}_{n,a}(y) - \bar{G}_a(y)\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right)\ a.s.$$

Proof. Claim a) follows immediately from the law of iterated logarithm for the empirical process (cf. [13], p. 268, for example). Furthermore, the law of iterated logarithm implies

$$\limsup_{n\to\infty}\sqrt{\frac{n}{\ln\ln n}}\sup_{x\in\mathbb{R}}\left|F_n(x) - F(x)\right| \leq \frac{1}{2}\sqrt{2}.$$

Claim b) is a consequence of this inequality. Let $\lfloor x\rfloor$ be the largest integer less than or equal to x. Further

$$\begin{aligned}
&\frac{1}{n}\sup_{a\in I}\sum_{i=1}^{n}\left|\mathbf{1}\left\{F_n(X_i)\leq a\right\} - \mathbf{1}\{F(X_i)\leq a\}\right| \\
= &\sup_{a\in I}\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{1}\left\{F_n(X_i)\leq a, F(X_i) > a\right\} + \mathbf{1}\{F_n(X_i) > a, F(X_i)\leq a\}\right) \\
= &\sup_{a\in I}\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{1}\left\{F^{-1}(a) < X_i \leq F_n^{-1}(\lfloor an\rfloor/n)\right\} + \mathbf{1}\{F_n^{-1}(\lfloor an\rfloor/n) < X_i < F^{-1}(a)\}\right) \\
\leq &\sup_{a\in I}\left|\lfloor an\rfloor/n - F_n(F^{-1}(a))\right| \leq \sup_{a\in I}\left|F_n(F^{-1}(a)) - F(F^{-1}(a))\right| + O(n^{-1}) \\
= &O\left(\sqrt{\frac{\ln\ln n}{n}}\right)\ a.s.
\end{aligned}$$

which is assertion c). Next we prove part d). In view of a) and c), we can derive

$$\begin{aligned}
\sup_{y\in\mathbb{R},a\in I}\left|G_{n,a}(y) - G_{\leq a}(y)\right| &\leq \sup_{y\in\mathbb{R},a\in I}\frac{1}{an}\left|\sum_{i=1}^{n}\mathbf{1}\left\{Y_i\leq y, X_i\leq F_n^{-1}(\lfloor an\rfloor/n)\right\} - nH(F_n^{-1}(\lfloor an\rfloor/n), y)\right| \\
&\quad + \sup_{y\in\mathbb{R},a\in I}a^{-1}\left|\left(H(F_n^{-1}(\lfloor an\rfloor/n), y) - H(F^{-1}(a), y)\right)\right| \\
&\leq O(1)\cdot\sup_{x,y\in\mathbb{R}}\left|H_n(x,y) - H(x,y)\right| + \sup_{a\in I}a^{-1}\left|F(F_n^{-1}(\lfloor an\rfloor/n)) - a\right| \\
&\leq O\left(\sqrt{\frac{\ln\ln n}{n}}\right) + \sup_{a\in I}a^{-1}\left|F(F_n^{-1}(\lfloor an\rfloor/n)) - F_n(F_n^{-1}(\lfloor an\rfloor/n))\right| + O(n^{-1}) \\
&= O\left(\sqrt{\frac{\ln\ln n}{n}}\right)\ a.s.
\end{aligned}$$

The corresponding assertion about $\bar{G}_{n,a}$ is proved analogously. □

5.2 Proof of convergence rate of the estimated coefficient

Throughout the remainder of Section 5, we assume that Assumption $\mathcal{A}_1$ is fulfilled. We start with the proof of strong convergence rate of the coefficient of piecewise monotonicity. Now we prove a lemma which is used in the subsequent proof:

Lemma 9. *We have*

$$\sup_{a\in I}\left|\Lambda_n(a)-\mathbb{E}\Lambda_n(a)\right| = O\left(\sqrt{\frac{\ln n}{n}}\right) \text{ and}$$
$$\sup_{a\in I}\left|\bar{\Lambda}_n(a)-\mathbb{E}\bar{\Lambda}_n(a)\right| = O\left(\sqrt{\frac{\ln n}{n}}\right) \text{ a.s.,}$$

where

$$\Lambda_n(a) = \frac{1}{n}\sum_{i=1}^{n}\psi(\tfrac{1}{a}F(X_i)-G_{\le a}(Y_i))\mathbf{1}\left\{F(X_i)\le a\right\},$$
$$\bar{\Lambda}_n(a) = \frac{1}{n}\sum_{i=1}^{n}\psi(\tfrac{1}{1-a}\left(F(X_i)-a\right)-\bar{G}_a(-Y_i))\mathbf{1}\left\{F(X_i)> a\right\}$$

Proof. We divide I into n closed intervals $J_1,\dots,J_n\subset I$ of length $\frac{1}{n}$ such that $\bigcup_{i=1}^{n}J_n=I$, a_k is the centre of the interval J_k. Observe that

$$\begin{aligned}\sup_{a\in I}\left|\Lambda_n(a)-\mathbb{E}\Lambda_n(a)\right| &\le \max_{k=1,\dots,n}\sup_{a\in J_k}\left|\Lambda_n(a)-\mathbb{E}\Lambda_n(a)\right|\\ &\le \max_{k=1,\dots,n}\left|\Lambda_n(a_k)-\mathbb{E}\Lambda_n(a_k)\right| + \sup_{a,a'\in I:a\le a'\le a+n^{-1}}\left|\Lambda_n(a)-\Lambda_n(a')\right|\\ &\quad+\sup_{a,a'\in I:a\le a'\le a+n^{-1}}\left|\mathbb{E}\Lambda_n(a)-\mathbb{E}\Lambda_n(a')\right|. \end{aligned}\tag{10}$$

Let $Z_1,\dots,Z_n$ be independent random variables with $|Z_i|\le M$ *a.s.*, $M>0$ is a constant. Bernstein's inequality (see [7], p.193) says that

$$\mathbb{P}\left\{\left|\sum_{i=1}^{n}(Z_i-\mathbb{E}Z_i)\right|>\varepsilon\right\}\le 2\exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^{n}\operatorname{Var}Z_i+\frac{4}{3}M\varepsilon}\right)\tag{11}$$

for all $\varepsilon>0$. Let $Z_i=\frac{1}{n}\psi(\frac{1}{a}F(X_i)-G_{\le a}(Y_i))\mathbf{1}\{F(X_i)\le a\}$, $M_0=\sup_{t\in[-1,1]}|\psi(t)|$. Then $M=M_0n^{-1}$ and

$$\sum_{i=1}^{n}\operatorname{Var}Z_i\le n^{-2}\sum_{i=1}^{n}\mathbb{E}\psi(\tfrac{1}{a}F(X_i)-G_{\le a}(Y_i))^2\le n^{-1}M_0^2 \text{ for every } a\in I.$$

Let $\lambda_n=\sqrt{\ln n}\,n^{-1/2}$. By Bernstein's inequality (11), we obtain

$$\begin{aligned}\mathbb{P}\left\{\max_{k=1,\dots,n}\left|\Lambda_n(a_k)-\mathbb{E}\Lambda_n(a_k)\right|>\varepsilon\lambda_n\right\} &\le \sum_{k=1}^{n}\mathbb{P}\left\{\left|\Lambda_n(a_k)-\mathbb{E}\Lambda_n(a_k)\right|>\varepsilon\lambda_n\right\}\\ &\le 2n\exp\left(-\frac{\varepsilon^2\lambda_n^2}{n^{-1}M_0^2+\frac{4}{3}M_0n^{-1}\varepsilon\lambda_n}\right)\\ &\le 2n\exp\left(-\frac{\varepsilon^2\ln n}{M_0^2+\frac{4}{3}M_0\varepsilon\sqrt{\ln n}\,n^{-1/2}}\right)\\ &\le 2n\exp\left(-\frac{\kappa_2\varepsilon^2\ln n}{1+\varepsilon}\right)\end{aligned}$$

for all $\varepsilon>0$ and $n\ge n_0$, where $\kappa_2>0$ is an appropriate constant. Hence,

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\max_{k=1,\dots,n}\left|\Lambda_n(a_k)-\mathbb{E}\Lambda_n(a_k)\right|>\varepsilon\lambda_n\right\}<+\infty$$

for large ε. Then, by the Borel-Cantelli lemma,

$$\mathbb{P}\left\{\max_{k=1,\dots,n}\left|\Lambda_n(a_k)-\mathbb{E}\Lambda_n(a_k)\right|>\varepsilon\lambda_n \text{ for a finite number of } n\text{'s}\right\}=1.$$

Therefore

$$\max_{k=1,\dots,n} \left|\Lambda_n(a_k) - \mathbb{E}\Lambda_n(a_k)\right| = O\left(\sqrt{\frac{\ln n}{n}}\right) \; a.s. \tag{12}$$

Next we analyse the two remaining terms in (10) concerning the convergence rate. Note that

$$\begin{aligned}
\sup_{a,a'\in I: a\le a'\le a+n^{-1}} \sup_{y\in\mathbb{R}} \left|G_{\le a}(y) - G_{\le a'}(y))\right| &\le \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left(\left|\tfrac{1}{a} - \tfrac{1}{a'}\right| + \tfrac{1}{a'} \sup_{y\in\mathbb{R}} \left|H(F^{-1}(a'), y) - H(F^{-1}(a), y)\right|\right) \\
&\le O(n^{-1}) + \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left(\tfrac{1}{a'} \left|F(F^{-1}(a')) - F(F^{-1}(a))\right|\right) \\
&= O(n^{-1}).
\end{aligned} \tag{13}$$

By the Lipschitz property of ψ, Lemma 8b) and (13), we obtain

$$\begin{aligned}
sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left|\Lambda_n(a) - \Lambda_n(a')\right| &\le O\left(\frac{1}{n}\right)\left(\sup_{a,a'\in I: a\le a'\le a+n^{-1}} \sum_{i=1}^n \left(F(X_i)\left|\tfrac{1}{a} - \tfrac{1}{a'}\right| + \left|G_{\le a}(Y_i) - G_{\le a'}(Y_i))\right|\right)\right. \\
&\quad \left. + \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \sum_{i=1}^n \left|\mathbf{1}\left\{F(X_i)\le a\right\} - \mathbf{1}\left\{F(X_i)\le a'\right\}\right|\right) \\
&\le O(n^{-1}) + O(1)\cdot \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left(F_n(F^{-1}(a')) - F_n(F^{-1}(a))\right) \\
&\le O(n^{-1}) + O(1)\cdot\left(2\sup_{x\in\mathbb{R}}\left|F_n(x) - F(x)\right| + \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left|a - a'\right|\right) \\
&= O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \; a.s.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left|\mathbb{E}\Lambda_n(a) - \mathbb{E}\Lambda_n(a')\right| \\
\le\ & O(1)\cdot \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left(\left|\tfrac{1}{a} - \tfrac{1}{a'}\right| \mathbb{E}F(X_1) + \mathbb{E}\left|G_{\le a}(Y_1) - G_{\le a'}(Y_1)\right|\right. \\
& \qquad \left. + \mathbb{E}\left|\mathbf{1}\left\{F(X_1)\le a\right\} - \mathbf{1}\left\{F(X_1)\le a'\right\}\right|\right) \\
\le\ & O(n^{-1}) + O(1) \\
& \cdot \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left(\mathbb{E}\left|H(F^{-1}(a), Y_1) - H(F^{-1}(a'), Y_1)\right| + \mathbb{P}\left\{a \le F(X) \le a'\right\}\right) \\
\le\ & O(n^{-1}) + O(1) \sup_{a,a'\in I: a\le a'\le a+n^{-1}} \left|a - a'\right| = O(n^{-1}).
\end{aligned}$$

Consequently, the first assertion of the lemma is a consequence of (10) and (12). The second assertion can be proved analogously. □

Proof of Theorem 3. a) Observe that

$$\sup_{a\in I}\left|\hat{\zeta}_n^{+-}(a) - \zeta_a^{+-}(C)\right| \le \frac{1}{n\bar{\psi}}(S_n + T_n) + \frac{1}{\bar{\psi}} \sup_{a\in I}\left(\left|\Lambda_n(a) - \mathbb{E}\Lambda_n(a)\right| + \left|\bar{\Lambda}_n(a) - \mathbb{E}\bar{\Lambda}_n(a)\right|\right), \tag{14}$$

where Λ_n and $\bar{\Lambda}_n$ as in Lemma 9,

$$\begin{aligned}
S_n &= \sup_{a\in I}\left(\sum_{i=1}^n \left|\psi(\tfrac{1}{a}F_n(X_i) - G_{n,a}(Y_i)) - \psi(\tfrac{1}{a}F(X_i) - G_{\le a}(Y_i))\right|\right. \\
&\quad \left. + \sum_{i=1}^n \left|\psi(\tfrac{1}{1-a}\left(F_n(X_i) - a\right) - \bar{G}_{n,a}(-Y_i)) - \psi(\tfrac{1}{1-a}\left(F(X_i) - a\right) - \bar{G}_a(-Y_i))\right|\right), \\
T_n &= 2\sup_{t\in[-1,1]}|\psi(t)| \sup_{a\in I}\sum_{i=1}^n \left|\mathbf{1}\left\{F_n(X_i)\le a\right\} - \mathbf{1}\{F(X_i)\le a\}\right|.
\end{aligned}$$

Further, in view of Lemma 8, we can deduce

$$\begin{aligned} S_n &\le O(n)\cdot \sup_{a\in I}\left(\left(\tfrac{1}{a}+\tfrac{1}{1-a}\right)\sup_{x\in\mathbb{R}}\left|F_n(x)-F(x)\right| +\sup_{y\in\mathbb{R}}\left|G_{n,a}(y)-G_{\le a}(y)\right| + \sup_{y\in\mathbb{R}}\left|\bar{G}_{n,a}(y)-\bar{G}_a(y)\right|\right) \\ &= O\left(\sqrt{n\ln\ln n}\right)\ a.s. \end{aligned} \tag{15}$$

using the Lipschitz continuity of ψ. Lemma 8c) implies

$$T_n = O\left(\sqrt{n\ln\ln n}\right)\ a.s. \tag{16}$$

Combining (14) to (16) and using Lemma 9, we obtain

$$\sup_{a\in I}\left|\hat{\zeta}_n^{+-}(a)-\zeta_a^{+-}(C)\right| = O\left(\sqrt{\frac{\ln n}{n}}\right)\ a.s. \tag{17}$$

Let $\tilde{a}$ and a_n be maximizer of $a \rightsquigarrow \zeta_a^{+-}(C)$ on I, and of $\hat{\zeta}_n^{+-}(.)$ on I_n, respectively. Then we have

$$\zeta_{\tilde{a}}^{+-}(C) \ge \zeta_{a_n}^{+-}(C) \ge \hat{\zeta}_n^{+-}(a_n) - O\left(\sqrt{\frac{\ln n}{n}}\right)\ a.s.$$

There is a $\breve{a}_n \in I_n$ such that $|\breve{a}_n-\tilde{a}| \le \frac{1}{n}$. Since $a \rightsquigarrow \zeta_a^{+-}(C)$ is Lipschitz continuous in view of Lemma 5 and (17) holds, it follows that

$$\hat{\zeta}_n^{+-}(a_n) \ge \hat{\zeta}_n^{+-}(\breve{a}_n) \ge \zeta_{\breve{a}_n}^{+-}(C) - O\left(\sqrt{\frac{\ln n}{n}}\right) \ge \zeta_{\tilde{a}}^{+-}(C) - O\left(\sqrt{\frac{\ln n}{n}}\right)\ a.s.$$

Combining these inequalities, we obtain

$$\left|\zeta_{\tilde{a}}^{+-}(C) - \hat{\zeta}_n^{+-}(a_n)\right| = O\left(\sqrt{\frac{\ln n}{n}}\right)\ a.s.$$

This identity proves part a) of Theorem 3.
b) There exist subsequences $\{a_{m_n}\}$ and $\{a_{M_n}\}$ of $\{a_n\}$ almost surely such that

$$\bar{a}_0 := \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_{m_n} \le \tilde{a}_0 := \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_{M_n}\ a.s.$$

By part a),

$$\lim_{n\to\infty}\hat{\zeta}_{m_n}^{+-}(a_{m_n}) = \lim_{n\to\infty}\hat{\zeta}_{M_n}^{+-}(a_{M_n}) = \zeta_{a_0}^{+-}(C)\ a.s.$$

By (17),

$$\lim_{n\to\infty}\zeta_{a_{m_n}}^{+-}(C) = \lim_{n\to\infty}\zeta_{a_{M_n}}^{+-}(C) = \zeta_{a_0}^{+-}(C)\ a.s.$$

which implies $\zeta_{\bar{a}_0}^{+-}(C) = \zeta_{\tilde{a}_0}^{+-}(C) = \zeta_{a_0}^{+-}(C)$ by virtue of Lemma 5. Therefore $\bar{a}_0 = \tilde{a}_0 = a_0$ holds $a.s.$ since a_0 is the unique maximizer, and the the claim b) is proved. □

5.3 Proof of asymptotic normality of the estimated coefficient

Before giving the proof of Theorem 4, we introduce some definitions and provide three useful auxiliary statements:

$$\begin{aligned} \tilde{G}_{n,a}(y) &:= \frac{1}{an}\sum_{i=1}^{n}\mathbf{1}\left\{Y_i \le y, F(X_i)\le a\right\}, \\ \tilde{\bar{G}}_{n,a}(y) &:= \frac{1}{n-an}\sum_{i=1}^{n}\mathbf{1}\left\{-Y_i\le y, F(X_i)>a\right\}. \end{aligned}$$

Define $\bar{a} := F^{-1}(a)$, $\gamma(y,x) := \mathbb{P}\{Y_1\le y \mid X_1 = x\}$. From Corollary 21.5 in the book [13] by van der Vaart, we can derive the following Proposition:

Proposition 10. *Assume that $f(\bar{a}) > 0$. Then*

$$F_n^{-1}(a) - \bar{a} = -\frac{1}{f(\bar{a})}\left(F_n(\bar{a}) - a\right) + o_{\mathbb{P}}(n^{-1/2}).$$

Lemma 11. *Suppose that Assumption $\mathcal{A}_2$ is fulfilled. Assume that f is continuous at $\bar{a}$ and $f(\bar{a}) > 0$. Then*

$$a)\ \sup_{y\in\mathbb{R}}\left|G_{n,a}(y) - \tilde{G}_{n,a}(y) + \frac{1}{a}(F_n(\bar{a}) - a)\gamma(y,\bar{a})\right| = o_{\mathbb{P}}(n^{-1/2}),\ and$$

$$b)\ \sup_{y\in\mathbb{R}}\left|\bar{G}_{n,a}(y) - \tilde{\bar{G}}_{n,a}(y) - \frac{1}{1-a}(F_n(\bar{a}) - a)\left(1 - \gamma(-y,\bar{a})\right)\right| = o_{\mathbb{P}}(n^{-1/2}).$$

Proof. a) Denote the empirical distribution function of (X, Y) by H_n. Let $\bar{\delta} > 0$ such that f is bounded from below on $[F^{-1}(a-\bar{\delta}), F^{-1}(a+\bar{\delta})]$. Therefore, we can derive

$$\sup_{p:|p-a|<\bar{\delta}}\frac{\left|F^{-1}(p) - F^{-1}(a)\right|}{|p-a|} \le \sup_{q:|q-a|\le\bar{\delta}}\frac{1}{f(F^{-1}(q))} \le \kappa_3 \tag{18}$$

with a suitable constant $\kappa_3 > 0$. Obviously,

$$\left|F_n(F_n^{-1}(\lfloor an\rfloor/n)) - a\right| = \left|\lfloor an\rfloor/n - a\right| \le n^{-1}$$

for $n \ge n_0(\omega)$. Applying Lemma 8b), we obtain

$$\left|F(F_n^{-1}(\lfloor an\rfloor/n)) - a\right| \le \kappa_1\sqrt{\frac{\ln\ln n}{n}} + n^{-1}$$

for $n \ge n_0(\omega)$. By (18),

$$\left|F_n^{-1}(\lfloor an\rfloor/n) - \bar{a}\right| \le \kappa_3\left(\kappa_1\sqrt{\frac{\ln\ln n}{n}} + n^{-1}\right) =: \delta_n \tag{19}$$

for $n \ge n_0(\omega)$. We have

$$\begin{aligned} G_{n,a}(y) - \tilde{G}_{n,a}(y) &= \frac{1}{an}\sum_{i=1}^{n}\left(\mathbf{1}\left\{Y_i \le y, X_i \le F_n^{-1}(\lfloor an\rfloor/n)\right\} - \mathbf{1}\left\{Y_i \le y, X_i \le \bar{a}\right\}\right) \\ &= \frac{1}{a}\left(H_n(F_n^{-1}(\lfloor an\rfloor/n), y) - H_n(\bar{a}, y)\right) \\ &= \frac{1}{a}\left(H(F_n^{-1}(\lfloor an\rfloor/n), y) - H(\bar{a}, y) + \Delta_n(y, F_n^{-1}(\lfloor an\rfloor/n))\right), \end{aligned} \tag{20}$$

where $\Delta_n(y, x) := H_n(x, y) - H_n(\bar{a}, y) - H(x, y) + H(\bar{a}, y)$. Since the empirical process $\sqrt{n}\left(H_n(.) - H(.)\right)$ converges weakly to a Gaussian process, this process is asymptotically equicontinuous which in turn leads to

$$\sup_{y\in\mathbb{R}}\left|\Delta_n(y, F_n^{-1}(\lfloor an\rfloor/n))\right| = o_{\mathbb{P}}(n^{-1/2}) \tag{21}$$

by virtue of (19). Note that $F_n^{-1}(\lfloor an\rfloor/n) - F_n^{-1}(a) = o_{\mathbb{P}}(n^{-1/2})$ in view of Lemma 21.7 in [13]. Consequently, by Proposition 10 and Assumption $\mathcal{A}_2$, we have

$$\begin{aligned} H(F_n^{-1}(\lfloor an\rfloor/n), y) - H(\bar{a}, y) &= \int_{\bar{a}}^{F_n^{-1}(\lfloor an\rfloor/n)} \mathbb{P}\left\{Y_1 \le y \mid X_1 = x\right\} f(x)\,\mathrm{d}x \\ &= (F_n^{-1}(\lfloor an\rfloor/n) - \bar{a})\left(\gamma(y,\bar{a})f(\bar{a}) + o_{\mathbb{P}}(1)\right) \\ &= -(F_n(\bar{a}) - a)\left(\gamma(y,\bar{a}) + o_{\mathbb{P}}(1)\right) + o_{\mathbb{P}}(n^{-1/2}) \\ &= -(F_n(\bar{a}) - a)\gamma(y,\bar{a}) + o_{\mathbb{P}}(n^{-1/2}) \end{aligned}$$

uniformly in $y \in \mathbb{R}$. In view of (20) and (21), we obtain the first assertion of the lemma.
b) Further

$$\begin{aligned}\bar{G}_{n,a}(y) - \tilde{\tilde{G}}_{n,a}(y) &= \frac{1}{n-an}\sum_{i=1}^{n}\left(\mathbf{1}\left\{-Y_i \le y, X_i > F_n^{-1}(\lfloor an \rfloor/n)\right\} - \mathbf{1}\{-Y_i \le y, X_i > \bar{a}\}\right)\\ &= \frac{1}{1-a}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{1}\left\{X_i > F_n^{-1}(\lfloor an \rfloor/n)\right\} - \mathbf{1}\{X_i > \bar{a}\}\right)\right.\\ &\quad \left. -\bar{H}_n(F_n^{-1}(\lfloor an \rfloor/n), -y) + \bar{H}_n(\bar{a}, -y)\right)\\ &= \frac{1}{1-a}\left(\frac{n-\lfloor an \rfloor}{n} - \left(1 - F_n(\bar{a})\right)\right.\\ &\quad \left. -\bar{H}(F_n^{-1}(\lfloor an \rfloor/n), -y) + \bar{H}(\bar{a}, -y) + \bar{\Delta}_n(-y, F_n^{-1}(\lfloor an \rfloor/n))\right), \end{aligned} \tag{22}$$

where $\bar{H}(x, y) = \mathbb{P}\{Y_i < y, X_i > x\}$, $\bar{H}_n$ is the corresponding empirical counter-part, and $\bar{\Delta}_n(y, x) := \bar{H}_n(\bar{a}, y) - \bar{H}(\bar{a}, y) - \bar{H}_n(x, y) + \bar{H}(x, -y)$. Similarly to the first part, one shows that

$$\sup_{y\in\mathbb{R}}\left|\Delta_n(-y, F_n^{-1}(\lfloor an \rfloor/n))\right| = o_{\mathbb{P}}(n^{-1/2}) \tag{23}$$

Taking into account Assumption $\mathcal{A}_2$, we can deduce

$$\begin{aligned}\bar{H}(F_n^{-1}(\lfloor an \rfloor/n), -y) - \bar{H}(\bar{a}, -y) &= \int_{F_n^{-1}(\lfloor an \rfloor/n)}^{\bar{a}} \mathbb{P}\{Y_1 \le -y \mid X_1 = x\} f(x)\,\mathrm{d}x\\ &= (\bar{a} - F_n^{-1}(\lfloor an \rfloor/n))\left(\gamma(-y, \bar{a})f(\bar{a}) + o_{\mathbb{P}}(1)\right)\\ &= (F_n(\bar{a}) - a)\gamma(-y, \bar{a}) + o_{\mathbb{P}}(n^{-1/2})\end{aligned}$$

uniformly in $y \in \mathbb{R}$. Combing this identity with (22), (23) and Proposition 10, it follows that

$$\bar{G}_{n,a}(y) - \tilde{\tilde{G}}_{n,a}(y) = \frac{1}{1-a}(F_n(\bar{a}) - a)\left(1 - \gamma(-y, \bar{a})\right) + o_{\mathbb{P}}(n^{-1/2})$$

uniformly in $y \in \mathbb{R}$. This identity proves the second claim.

□

Lemma 12. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a bounded function. Define $\bar{\gamma}(x) := \mathbb{E}(\Psi(Y_1) \mid X_1 = x)$, and*

$$W_n := \frac{1}{n}\sum_{i=1}^{n}\Psi(Y_i)\left(\mathbf{1}\{F_n(X_i) \le a\} - \mathbf{1}\{F(X_i) \le a\}\right).$$

If $\bar{\gamma}$ and f are continuous at $\bar{a}$ with $f(\bar{a}) > 0$, then

$$W_n = -(F_n(\bar{a}) - a)\bar{\gamma}(\bar{a}) + o_{\mathbb{P}}(n^{-1/2}).$$

Proof. Let $\eta_n(x) = \mathbb{E}\left(\Psi(Y_1)\mathbf{1}\{X_1 \le x\}\right)$ and $\tilde{\Delta}_n = W_n - \eta_n(F_n^{-1}(\lfloor an \rfloor/n)) + \eta_n(\bar{a})$. Note that

$$W_n = \frac{1}{n}\sum_{i=1}^{n}\Psi(Y_i)\left(\mathbf{1}\left\{\bar{a} < X_i \le F_n^{-1}(\lfloor an \rfloor/n)\right\} - \mathbf{1}\left\{F_n^{-1}(\lfloor an \rfloor/n) < X_i \le \bar{a}\right\}\right).$$

Using (19), we deduce

$$\sqrt{n}\left|\tilde{\Delta}_n\right| \le \sup_{b\in[-\delta_n,\delta_n]} R_n(b),$$

where

$$R_n(b) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_{ni}(b),$$

$$Z_{ni}(b) = \left(\Psi(Y_i)\left(\mathbf{1}\{\bar a < X_i \le \bar a + b\} - \mathbf{1}\{\bar a + b < X_i \le \bar a\}\right) - \mathbb{E}\Psi(Y_i)\mathbf{1}\{\bar a < X_i \le \bar a + b\} + \mathbb{E}\Psi(Y_i)\mathbf{1}\{\bar a + b < X_i \le \bar a\}\right).$$

First we decompose the interval $[-\delta_n, \delta_n]$ into $N = 4n$ closed intervals $I_{n1}, \dots, I_{nN}$: $\bigcup_{l=1}^{2n} I_{nl} = [-\delta_n, 0]$, $\bigcup_{l=2n+1}^{4n} I_{nl} = [0, \delta_n]$ with $b_l, \bar b_l$ as the smallest/largest element of I_{nl} and $\bar b_l - b_l \le \frac{1}{n}\delta_n$ for $l = 1, \dots, N$. Then

$$\begin{aligned}
\sup_{b\in[-\delta_n,\delta_n]} \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b)\right| &= \max_{l=1\dots N}\sup_{b\in I_{nl}} \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b)\right| \\
&\le \max_{l=1\dots N}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b_l)\right| + \frac{1}{\sqrt{n}}\max_{l=1\dots N}\sum_{i=1}^n \sup_{b\in I_{nl}}\left|Z_{ni}(b) - Z_{ni}(b_l)\right| \\
&\le \max_{l=1\dots N}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b_l)\right| \\
&\quad + \sup_{t\in\mathbb{R}}\left|\Psi(t)\right| \max_{l=1\dots N}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde Z_{nil}\right| + 2\sqrt{n}Q_{nl}\right), \qquad (24)
\end{aligned}$$

where $Q_{nl} := \mathbb{P}\left\{\bar a + b_l < X_1 \le \bar a + \bar b_l\right\}$ and $\tilde Z_{nil} := \mathbf{1}\left\{\bar a + b_l < X_i \le \bar a + \bar b_l\right\} - Q_{nl}$. Notice that $|Z_{ni}(b)| \le \sup_{t\in\mathbb{R}}\left|\Psi(t)\right|$ and

$$\begin{aligned}
\sup_{b\in[-\delta_n,\delta_n]} \operatorname{Var}\left(Z_{ni}(b)\right) &\le \sup_{t\in\mathbb{R}}\Psi^2(t) \sup_{b\in[0,\delta_n]}\left(\mathbb{P}\{\bar a - b < X_1 \le \bar a + b\}\right) \\
&\le \sup_{t\in\mathbb{R}}\Psi^2(t) \sup_{x\in[\bar a-\delta_n,\bar a+\delta_n]} f(x)\,\delta_n.
\end{aligned}$$

Applying Bernstein's inequality (11), we obtain

$$\begin{aligned}
\sum_{n=1}^\infty \mathbb{P}\left\{\max_{l=1\dots N}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b_l)\right| > \varepsilon\right\} &\le \sum_{n=1}^\infty\sum_{l=1}^N \mathbb{P}\left\{\left|\sum_{i=1}^n Z_{ni}(b_l)\right| > \varepsilon\sqrt{n}\right\} \\
&\le O(1)\sum_{n=1}^\infty n\cdot\exp\left(-\frac{\varepsilon^2\sqrt{n}}{\kappa_4\left(\sqrt{\ln\ln n} + \varepsilon\right)}\right) < +\infty
\end{aligned}$$

for $\varepsilon > 0$ with constant $\kappa_4 > 0$, and hence

$$\max_{l=1\dots N}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{ni}(b_l)\right| = o(1)\ a.s. \qquad (25)$$

Analogously, one obtains

$$\max_{l=1\dots N}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde Z_{nil}\right| = o(1)\ a.s.$$

Notice that

$$\begin{aligned}
\max_{l=1\dots N} Q_{nl} &\le \max_{l=1\dots N}\mathbb{P}\left\{\bar a + b_l < X_1 \le \bar a + \bar b_l\right\} \\
&\le \frac{\delta_n}{n}\sup_{x\in[\bar a-\delta_n,\bar a+\delta_n]} f(t) = o(n^{-1/2}).
\end{aligned}$$

Taking (24) and (25) into account, we can conclude

$$\tilde\Delta_n = o_{\mathbb{P}}(n^{-1/2}).$$

Moreover,

$$\begin{aligned}
W_n &= \int_{\bar a}^{F_n^{-1}(\lfloor an\rfloor/n)} \bar\gamma(x)f(x)\,\mathrm{d}x + o_{\mathbb{P}}(n^{-1/2}) \\
&= (F_n^{-1}(\lfloor an\rfloor/n) - \bar a)\bar\gamma(\bar a)f(\bar a) + o_{\mathbb{P}}(n^{-1/2})
\end{aligned}$$

since $\bar\gamma$ is continuous at $\bar a$. An application of Proposition 10 completes the proof. □

Suppose that Assumption $\mathcal{A}_3$ is satisfied and $a \in I$ is fixed. We have the following decomposition of the term $\hat{\zeta}_n^{+-}(a) - \zeta_a^{+-}(C)$ whose asymptotic normality has to be shown:

$$\hat{\zeta}_n^{+-}(a) - \zeta_a^{+-}(C) = -(A_n + B_{n1} - B_{n2} - B_{n3} + D_n)\,\bar{\psi}^{-1}, \tag{26}$$

where $\check{\zeta}(a) = \mathbb{E}\psi(F_{\leq a}(X_i) - G_{\leq a}(Y_i))\mathbf{1}\left\{F(X_i) \leq a\right\} + \mathbb{E}\psi(F_{>a}(X_i) - \bar{G}_a(-Y_i))\,\mathbf{1}\left\{F(X_i) > a\right\}$,

$$\begin{aligned}
A_n &= \frac{1}{n}\sum_{i=1}^{n}\left(\psi(F_{\leq a}(X_i) - G_{\leq a}(Y_i))\mathbf{1}\left\{F(X_i) \leq a\right\} + \psi(F_{>a}(X_i) - \bar{G}_a(-Y_i))\mathbf{1}\left\{F(X_i) > a\right\} - \check{\zeta}(a),\right.\\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\left(\psi'(F_{\leq a}(X_i) - G_{\leq a}(Y_i))\right.\\
&\quad\quad \left(\tfrac{1}{a}\left(F_n(X_i) - F(X_i)\right) - \tilde{G}_{n,a}(Y_i) + G_{\leq a}(Y_i)\right)\mathbf{1}\left\{F(X_i) \leq a\right\} + \psi'(F_{>a}(X_i) - \bar{G}_a(-Y_i))\\
&\quad\quad \left(\tfrac{1}{1-a}\left(F_n(X_i) - F(X_i)\right) - \tilde{\bar{G}}_{n,a}(-Y_i) + \bar{G}_a(-Y_i)\right)\mathbf{1}\left\{F(X_i) > a\right\}\Big),
\end{aligned}$$

$$\begin{aligned}
|B_{n1}| &\leq O\left(\frac{1}{n}\right)\sum_{i=1}^{n}\Big(\left(\tfrac{1}{a}\left(F_n(X_i) - F(X_i)\right) - G_{n,a}(Y_i) + G_{\leq a}(Y_i)\right)^{1+\alpha}\mathbf{1}\left\{F(X_i) \leq a\right\}\\
&\quad + \left(\tfrac{1}{1-a}\left(F_n(X_i) - F(X_i)\right) - \bar{G}_{n,a}(-Y_i) + \bar{G}_a(-Y_i)\right)^{1+\alpha}\mathbf{1}\left\{F(X_i) > a\right\}\Big),\\
B_{n2} &= \frac{1}{n}\sum_{i=1}^{n}\psi'(F_{\leq a}(X_i) - G_{\leq a}(Y_i))\left(G_{n,a}(Y_i) - \tilde{G}_{n,a}(Y_i)\right)\mathbf{1}\left\{F(X_i) \leq a\right\},\\
B_{n3} &= \frac{1}{n}\sum_{i=1}^{n}\psi'(F_{>a}(X_i) - \bar{G}_a(-Y_i))\left(\bar{G}_{n,a}(-Y_i) - \tilde{\bar{G}}_{n,a}(-Y_i)\right)\mathbf{1}\left\{F(X_i) > a\right\},\\
D_n &= \frac{1}{n}\sum_{i=1}^{n}\left(\psi(\tfrac{1}{a}F_n(X_i) - G_{n,a}(Y_i)) - \psi(\tfrac{1}{1-a}\left(F_n(X_i) - a\right) - \bar{G}_{n,a}(-Y_i))\right)\\
&\quad \left(\mathbf{1}\left\{F_n(X_i) \leq a\right\} - \mathbf{1}\left\{F(X_i) \leq a\right\}\right)
\end{aligned}$$

(α is introduced in Assumption $\mathcal{A}_3$). Further by Lemma 8,

$$|B_{n1}| = O\left(\left(\frac{\ln\ln n}{n}\right)^{(1+\alpha)/2}\right) = o_{\mathbb{P}}(n^{-1/2}). \tag{27}$$

The next lemma deals with an asymptotic representation of D_n.

Lemma 13. *Let $\tilde{\psi}(x) = \mathbb{E}\left(\psi(1 - G_{\leq a}(Y_1)) - \psi(\bar{G}_a(-Y_1)) \mid X_1 = x\right)$. Assume that f and $\tilde{\psi}$ are continuous at $\bar{a} = F^{-1}(a)$ and $f(\bar{a}) > 0$. Then*

$$D_n = -\tilde{\psi}(\bar{a})\left(F_n(\bar{a}) - a\right) + o_{\mathbb{P}}(n^{-1/2}).$$

Proof. The term D_n is decomposed into three parts: $D_n = D_{n1} + D_{n2} + D_{n3}$,

$$\begin{aligned} D_{n1} &= \frac{1}{n}\sum_{i=1}^{n} \Big(\psi(\tfrac{1}{a}F_n(X_i) - G_{n,a}(Y_i)) - \psi(\tfrac{1}{1-a}\left(F_n(X_i) - a\right) - \bar{G}_{n,a}(-Y_i)) \\ &\quad -\psi(\tfrac{1}{a}F(X_i) - G_{\le a}(Y_i)) + \psi(\tfrac{1}{1-a}\left(F(X_i) - a\right) - \bar{G}_a(-Y_i))\Big) \\ &\quad \left(\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\left\{F(X_i) \le a\right\}\right), \\ D_{n2} &= \frac{1}{n}\sum_{i=1}^{n} \Big|\psi(\tfrac{1}{a}F(X_i) - G_{\le a}(Y_i)) - \psi(1 - G_{\le a}(Y_i)) \\ &\quad + \psi(\tfrac{1}{1-a}\left(F(X_i) - a\right) - \bar{G}_a(-Y_i)) - \psi(-\bar{G}_a(-Y_i))\Big| \\ &\quad \left(\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\left\{F(X_i) \le a\right\}\right), \\ D_{n3} &= \frac{1}{n}\sum_{i=1}^{n} \left(\psi(1 - G_{\le a}(Y_i)) - \psi(-\bar{G}_a(-Y_i))\right) \\ &\quad \left(\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\left\{F(X_i) \le a\right\}\right). \end{aligned}$$

In view of Lemma 8, we obtain

$$\begin{aligned} |D_{n1}| &\le O(1)\cdot \sup_{x,y\in\mathbb{R}} \left(\left|F_n(x) - F(x)\right| + \left|G_{n,a}(y) - G_{\le a}(y)\right| + \left|\bar{G}_{n,a}(y) - \bar{G}_a(y)\right|\right) \\ &\quad \cdot\frac{1}{n}\sum_{i=1}^{n} \left|\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\left\{F(X_i) \le a\right\}\right| \\ &= O\left(\frac{\ln\ln n}{n}\right) \ a.s. \end{aligned} \tag{28}$$

For $\omega : \bar{a} < X_i \le F_n^{-1}(\lfloor an\rfloor/n)$ or $F_n^{-1}(\lfloor an\rfloor/n) < X_i \le \bar{a}$,

$$\begin{aligned} \max_{i=1\ldots n}\left|F(X_i) - a\right| &\le \left|F(F_n^{-1}(\lfloor an\rfloor/n)) - F(\bar{a})\right| = \left(f(\bar{a}) + o_{\mathbb{P}}(1)\right)\left|F_n^{-1}(a) - \bar{a}\right| \\ &= \left|F_n(\bar{a}) - a\right|\left(1 + o_{\mathbb{P}}(1)\right) = o_{\mathbb{P}}(n^{-1/2}\sqrt{\ln\ln n}), \end{aligned}$$

where we used Proposition 10 and Lemma 8a). In view of Lemmas 8b) and by Lipschitz continuity of ψ, we have

$$\begin{aligned} |D_{n2}| &\le \frac{1}{n}\sum_{i=1}^{n} \left(\left|\tfrac{1}{a}F(X_i) - 1\right| + \left|\tfrac{1}{1-a}\left(F(X_i) - a\right)\right|\right)\left|\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\{F(X_i) \le a\}\right| \\ &\le O\left(\frac{1}{n}\right)\sum_{i=1}^{n}\left|F(X_i) - a\right|\left(\mathbf{1}\left\{\bar{a} < X_i \le F_n^{-1}(\tfrac{\lfloor an\rfloor}{n})\right\} + \mathbf{1}\{F_n^{-1}(\tfrac{\lfloor an\rfloor}{n}) < X_i \le \bar{a}\}\right) \\ &= o_{\mathbb{P}}(n^{-3/2}\sqrt{\ln\ln n})\sum_{i=1}^{n}\left|\mathbf{1}\left\{F_n(X_i) \le a\right\} - \mathbf{1}\{F(X_i) \le a\}\right| \\ &= o_{\mathbb{P}}(n^{-1/2}). \end{aligned} \tag{29}$$

Let $\Psi(y) := \psi(1 - G_{\le a}(y)) - \psi(-\bar{G}_a(-y))$. Applying Lemma 12, we have

$$D_{n3} = -\tilde{\psi}(\bar{a})(F_n(\bar{a}) - a) + o_{\mathbb{P}}(n^{-1/2}). \tag{30}$$

The lemma is a consequence of (28), (29) and (30). □

Proof of Theorem 4. Let $\bar{a} = F^{-1}(a)$. Using Lemma 11, we obtain

$$B_{n2} = -\frac{1}{na}\sum_{i=1}^{n} \psi'(\tfrac{1}{a}F(X_i) - G_{\le a}(Y_i))(F_n(\bar{a}) - a)\gamma(Y_i, \bar{a})\mathbf{1}\left\{F(X_i) \le a\right\} + o_{\mathbb{P}}(n^{-1/2})$$

and

$$B_{n3} = \frac{1}{n(1-a)}\sum_{i=1}^{n}\psi'(\tfrac{1}{1-a}\left(F(X_i)-a\right)-\bar{G}_a(-Y_i))(F_n(\bar{a})-a) \cdot \left(1-\gamma(Y_i,\bar{a})\right)\mathbf{1}\left\{F(X_i)>a\right\} \quad + o_{\mathbb{P}}(n^{-1/2}).$$

Using (26), (27) and Lemma 13, we can rewrite $\sqrt{n}\left(\hat{\zeta}_n^{+-}(a)-\zeta_a^{+-}(C)\right)$ as a U-statistic plus remaining term:

$$\sqrt{n}\left(\hat{\zeta}_n^{+-}(a)-\zeta_a^{+-}(C)\right) = -\sqrt{n}\bar{A}_n\bar{\psi}^{-1} + o_{\mathbb{P}}(1),$$

$$\begin{aligned}\bar{A}_n &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Psi(X_i,Y_i,X_j,Y_j).\\ &= \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\bar{\Psi}(X_i,Y_i,X_j,Y_j)\cdot\frac{n-1}{n} + o_{\mathbb{P}}(n^{-1/2}),\end{aligned}$$

where $\bar{\Psi}(x_1,y_1,x_2,y_2) = \Psi(x_1,y_1,x_2,y_2)+\Psi(x_2,y_2,x_1,y_1)$,

$$\begin{aligned}\Psi(x_1,y_1,x_2,y_2) = \;& \psi(F_{\le a}(x_1)-G_{\le a}(y_1))\mathbf{1}\left(F(x_1)\le a\right)\\ &+\psi(F_{>a}(x_1)-\bar{G}_a(-y_1))\mathbf{1}\left(F(x_1)>a\right)-\check{\zeta}(a)\\ &+\psi'(F_{\le a}(x_1)-G_{\le a}(y_1))\mathbf{1}\left(F(x_1)\le a\right)\\ &\quad\left(\tfrac{1}{a}\left(\mathbf{1}\left(x_2\le x_1\right)-F(x_1)\right)-\tfrac{1}{a}\mathbf{1}\left(y_2\le y_1,F(x_2)\le a\right)+G_{\le a}(y_1)\right)\\ &+\psi'(F_{>a}(x_1)-\bar{G}_a(-y_1))\mathbf{1}\left(F(x_1)>a\right)\\ &\quad\left(\tfrac{1}{1-a}\left(\mathbf{1}\left(x_2\le x_1\right)-F(x_1)\right)-\tfrac{1}{1-a}\mathbf{1}\left(-y_2\le -y_1,F(x_2)>a\right)+\bar{G}_a(-y_1)\right)\\ &+\tfrac{1}{a}\psi'(F_{\le a}(x_1)-G_{\le a}(y_1))\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right)\gamma(y_1,\bar{a})\mathbf{1}\left(F(x_1)\le a\right)\\ &-\tfrac{1}{1-a}\psi'(F_{>a}(x_1)-\bar{G}_a(-y_1))\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right)\left(1-\gamma(y_1,\bar{a})\right)\mathbf{1}\left(F(x_1)>a\right)\\ &-\tilde{\psi}(\bar{a})\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right).\end{aligned}$$

Further,

$$\begin{aligned}\mathbb{E}\left(\Psi(X_1,Y_1,x_2,y_2)\right) = &\int_{(-\infty,\bar{a}]\times\mathbb{R}}\psi'(F_{\le a}(x)-G_{\le a}(y))\left(\tfrac{1}{a}\left(\mathbf{1}\left(x_2\le x\right)-F(x)\right)\right.\\ &\left.-\tfrac{1}{a}\mathbf{1}\left(y_2\le y,F(x_2)\le a\right)+G_{\le a}(y)+\tfrac{1}{a}\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right)\gamma(y,\bar{a})\right)\,\mathrm{d}H(x,y)\\ &+\int_{[\bar{a},\infty)\times\mathbb{R}}\psi'(F_{>a}(x)-\bar{G}_a(-y))\\ &\left(\tfrac{1}{1-a}\left(\mathbf{1}\left(x_2\le x\right)-F(x)\right)-\tfrac{1}{1-a}\mathbf{1}\left(-y_2\le -y,F(x_2)>a\right)+\bar{G}_a(-y)\right.\\ &\left.-\tfrac{1}{1-a}\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right)\left(1-\gamma(y,\bar{a})\right)\right)\,\mathrm{d}H(x,y)-\tilde{\psi}(\bar{a})\left(\mathbf{1}\left(x_2\le\bar{a}\right)-a\right),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left(\Psi(x_1,y_1,X_1,Y_1)\right) = \;&\psi(F_{\le a}(x_1)-G_{\le a}(y_1))\mathbf{1}\left(F(x_1)\le a\right)\\ &+\psi(F_{>a}(x_1)-\bar{G}_a(-y_1))\mathbf{1}\left(F(x_1)>a\right)-\check{\zeta}(a).\end{aligned}$$

Now we apply the central limit theorem for U-statistics (see Theorem 5.5.1A in [11]) to obtain the theorem. □

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