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On capital allocation for stochastic arrangement increasing actuarial risks

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Abstract: This paper studies the increasing convex ordering of the optimal discounted capital allocations for stochastic arrangement increasing risks with stochastic arrangement decreasing occurrence times. The application to optimal allocation of policy limits is presented as an illustration as well.

Keywords: Coverage limits; Discount rate; Increasing convex order; Loss function; Utility function

1 Introduction

In the literature of actuarial science, the capital allocation has attracted great attention from researchers, and related work could be found in [5], [17], [8], [21], [6], and references therein. Recently, [22] proposed a general loss function in the study of capital allocation for independent or comonotonic risks.

Consider an insurer exposed to multiple random risks $\boldsymbol{X} = (X_1, \cdots, X_n)$, which may come from not only policy holders in one or more types of insurances, but also the investment of premium. Let $\boldsymbol{T} = (T_1, \cdots, T_n)$ be the corresponding occurrence times of the corresponding risks. The *aggregate discounted loss* is then

$$\sum_{i=1}^{n} e^{-\delta T_i} X_i,$$

where $\delta > 0$ is the *discount rate*. Assume that the insurer wishes to allocate a total amount of capital $\hbar$ to the risks $\boldsymbol{X}$. Denote $\mathcal{A}_\hbar$ all admissible allocation vectors $\boldsymbol{d} = (d_1, \cdots, d_n)$ such that $\sum_{i=1}^n d_i = \hbar$ and $d_i \geq 0$, for all $i = 1, \cdots, n$. Motivated by [22], one reasonable criterion of reducing the loss is to set the capital d_i to X_i as close as possible in terms of some appropriate distance measure, $i = 1, \cdots, n$. Thus, the insurer attains the *total discounted loss*

$$\sum_{i=1}^{n} e^{-\delta T_i} \phi(X_i - d_i),$$

where ϕ is some *loss function*. In general, the insurer attempts to allocate the capital amount $\boldsymbol{d}$ to risks $\boldsymbol{X}$ so that the above total discounted loss is minimized. Such an optimization problem can be summarized as follows:

$$\begin{cases} \min\limits_{\boldsymbol{d} \in \mathcal{A}_\hbar} \mathbb{E}\left[u\left(\sum\limits_{i=1}^{n} e^{-\delta T_i} \phi(X_i - d_i)\right)\right], \\ \text{where } u(x) \text{ is increasing and convex, and } \boldsymbol{X} \text{ is independent of } \boldsymbol{T}. \end{cases} \tag{1}$$

As a direct consequence of the theory presented in this paper, we will get the ordering of the optimal capital allocations of (1).

It should be remarked that [22] investigated the case of $\delta = 0$ for mutually independent or comonotonic risks. Because the assumption of independence among risks is hardly realistic and the comonotonicity among

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risks corresponds to the extreme case, it is of interest to introduce more general dependence structure to concerned risks and thus make the model more flexible in practice. Recently, for multiple risks [2] took the first to introduce the *stochastic arrangement increasing* (SAI) (see Definition 2.3) property, which, as a generalization of the arrangement increasing function, is rather convenient in risk management due to integrating monotonicity with dependence. Subsequently, [3] and [23] addressed more detailed properties of SAI along with some applications to insurance and economics. In this study, we once again put focus on SAI risks with *stochastic arrangement decreasing* (SAD) occurrence times.

From the viewpoint of the risk-averse investors, this study considers the total discounted loss for SAI risks with SAD occurrence times. We mainly investigate how the capital allocation strategy impact capital allocation in the sense of the increasing convex order. The remaining of this paper rolls out as follows: Section 2 reviews some basic concepts and recalls several facts, which are useful in formulating our main theoretical results. In Section 3 we present two technical lemmas to be utilized in developing our main theoretical results. Section 4 develops the increasing convex order on the total discounted loss. To illustrate the present results, we also address one application to optimal allocations of coverage limits in Section 5. The proofs of the two technical lemmas are deterred to the appendix.

Throughout this note, the terms *increasing* and *decreasing* stand for non-decreasing and non-increasing, respectively, and all expectations are implicitly assumed to be finite whenever they appear.

2 Preliminaries

For ease of reference, we review some important notions including concerned stochastic orders, AI (AD) function, SAI (SAD) and comonotonicity.

Let X and Y be two random variables with probability density (or mass) functions f and g, and survival functions $\bar{F}$ and $\bar{G}$ respectively.

Definition 2.1. A random variable X is said to be smaller than the other one Y in the

(i) *usual stochastic* order, denoted by $X \le_{\rm st} Y$, if $\bar{F}(t) \le \bar{G}(t)$ for all t or, equivalently, if $\mathrm{E}[h(X)] \le \mathrm{E}[h(Y)]$ for all increasing functions h;
(ii) *likelihood ratio* order, denoted by $X \le_{\rm lr} Y$, if $g(t)/f(t)$ is increasing in t for which the ratio is well defined;
(iii) *increasing convex* order, denoted by $X \le_{\rm icx} Y$, if $\mathrm{E}[h(X)] \le \mathrm{E}[h(Y)]$ for all increasing convex functions h for which the expectations exist.

The following chain of implications is well-known,

$$X \le_{\rm lr} Y \Longrightarrow X \le_{\rm st} Y \Longrightarrow X \le_{\rm icx} Y.$$

For more on stochastic orders, we refer readers to [16], [19], and [9].

In [7], a bivariate function $g(x, y)$ is said to be *arrangement increasing* (AI) if $g(x, y) \ge g(y, x)$ for $x \le y$. Afterward, this notion was generalized to its multivariate version by [1]. Denote $(\pi(1), \cdots, \pi(n))$ a permutation of $\{1, \cdots, n\}$ and $\pi(\boldsymbol{x}) = (x_{\pi(1)}, \cdots, x_{\pi(n)})$. For any $1 \le i \ne j \le n$, let

$$\pi_{ij}(1, \cdots, n) = (\pi_{ij}(1), \cdots, \pi_{ij}(n))$$

with $\pi_{ij}(i) = j$, $\pi_{ij}(j) = i$ and $\pi_{ij}(k) = k$, $k /\in \{i, j\}$. A real-valued function $g(\boldsymbol{x})$ defined on $\mathbb{R}^n$ is said to be *arrangement increasing* (AI) or *arrangement decreasing* (AD) if

$$(x_i - x_j)\left[g(\boldsymbol{x}) - g(\pi_{ij}(\boldsymbol{x}))\right] \le (\ge) 0,$$

for any pair (i, j) such that $1 \le i < j \le n$. Subsequently, [20] related the bivariate AI function to the joint likelihood ratio order between two dependent random variables.

Definition 2.2. For a random vector (X, Y) on $\mathbb{R}^2$, X is said to be smaller than Y in the sense of the *joint likelihood ratio order*, denoted by $X \le_{\text{lr:j}} Y$, if $\text{E}[g(X, Y)] \ge \text{E}[g(Y, X)]$ for any AI function $g(x, y)$.

Recently, [2, 3] introduced the following characterization for the monotonicity of mutually dependent random variables, extending the joint likelihood ratio order to multiple random variables.

Definition 2.3. A random vector $\boldsymbol{X} = (X_1, \cdots, X_n)$ is said to be *stochastic arrangement increasing* (SAI) if $\text{E}[g(\boldsymbol{X})] \ge \text{E}[g(\pi_{ij}(\boldsymbol{X}))]$ for any AI function g and all pair (i, j) such that $1 \le i < j \le n$.

As a dual, $\boldsymbol{X}$ is said to be *stochastic arrangement decreasing* (SAD) if $(X_n, \cdots, X_1)$ is SAI. According to [7] and [15], each of the following conditions is sufficient to the SAI property of $(X_1, \cdots, X_n)$:
- they are independent and identically distributed;
- they are independent and $X_1 \le_{\text{lr}} \cdots \le_{\text{lr}} X_n$;
- they are of exchangeable joint distribution;
- they are comonotonic and $X_1 \le_{\text{st}} \cdots \le_{\text{st}} X_n$.

In [18], random variables $X_1, \cdots, X_n$ are said to be AI (AD) if their joint density $f(\boldsymbol{x})$ is AI (AD). Actually, the SAI property of an absolutely continuous random vector can be characterized by the AI joint probability density.

Also let us recall the following two useful facts related to the above notions. One is a characterization first pointed out in [20] and further remarked in [2].

Lemma 2.4. An absolutely continuous random vector is SAI (SAD) if and only if the corresponding probability density function is AI (AD).

The other can be proved in a similar manner to Proposition 3.3(iii) of [2].

Lemma 2.5. If $(X_1, \cdots, X_n)$ is SAD and h is decreasing, then $(h(X_1), \cdots, h(X_n))$ is SAI.

For more on SAI and various its weak versions, we refer readers to [2, 3], [12], [24], and [10].

3 Two technical lemmas

Before proceeding to the theoretical results, let us first build the following two technical lemmas, which facilitate the proofs of the main results in the sequel. For the sake of smoothness, we defer their proofs to Appendix.

For real vectors $\boldsymbol{w} = (w_1, w_2)$, $\boldsymbol{a} = (a_1, a_2)$ on $\mathbb{R}^2$ and real functions ϕ, g on $\mathbb{R}$, let $\pi(\boldsymbol{a}) = (a_2, a_1)$ and denote

$$\begin{aligned} \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) &= w_1\phi(x_1 - a_1) + w_2\phi(x_2 - a_2), \\ \Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) &= g(\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w})) - g(\eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w})). \end{aligned}$$

Lemma 3.1. If $g(x)$ and $\phi(x)$ are both increasing and convex, then,

(i) $\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) \ge 0$, and
(ii) $\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) + \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) \ge 0$,

for $x_2 \ge x_1$, $a_2 \ge a_1$ and $w_2 \ge w_1 \ge 0$.

For real vectors $\boldsymbol{w} = (w_1, w_2)$, $\boldsymbol{a} = (a_1, a_2)$, a random vector (X_1, X_2) on $\mathbb{R}^2$ and real functions ϕ, g on $\mathbb{R}$, denote

$$\begin{aligned} \zeta(w_1, w_2; \boldsymbol{a}) &= \mathrm{E}\big[g\big(w_1\phi(X_1 - a_1) + w_2\phi(X_2 - a_2)\big)\big], \\ \Delta\zeta(w_1, w_2; \boldsymbol{a}) &= \zeta(w_1, w_2; \pi(\boldsymbol{a})) - \zeta(w_1, w_2; \boldsymbol{a}). \end{aligned}$$

Lemma 3.2. If $\phi(x)$, $g(x)$ are increasing and convex, and (X_1, X_2) is absolutely continuous and SAI, then,

(i) $\Delta\zeta(w_1, w_2; \boldsymbol{a}) \geq 0$, and
(ii) $\Delta\zeta(w_1, w_2; \boldsymbol{a}) + \Delta\zeta(w_2, w_1; \boldsymbol{a}) \geq 0$,

for $a_2 \geq a_1$ and $w_2 \geq w_1 \geq 0$.

4 Main results

Now, we are ready to present the main results as well as one application to the optimal capital allocations to SAI risks.

Denote $d_{(1)} \leq d_{(2)} \leq \cdots \leq d_{(n)}$ the increasing arrangement of $(d_1, \cdots, d_n)$.

Theorem 4.1. If $(X_1, \cdots, X_n)$ and $(W_1, \cdots, W_n)$ are both SAI and independent with each other, then, for any $(d_1, \cdots, d_n) \in \mathbb{R}^n$ and increasing and convex function ϕ,

$$\sum_{i=1}^{n} W_i\phi\big(X_i - d_i\big) \geq_{\text{icx}} \sum_{i=1}^{n} W_i\phi\big(X_i - d_{(i)}\big).$$

Proof. We only prove the case of $n = 2$ and $d_1 \leq d_2$. The proof for the case of $n > 2$ is quite similar and hence omitted here for briefness.

Let us use the notations in Lemmas 3.1 and 3.2. Denote $h(w_1, w_2)$ the probability density of (W_1, W_2). Assume that $g(x)$ is increasing and convex. Since (W_1, W_2) is SAI, based upon Lemmas 2.4 and 3.2, we have

$$\begin{aligned} & \mathrm{E}\big[g\big(W_1\phi(X_1 - d_2) + W_2\phi(X_2 - d_1)\big)\big] - \mathrm{E}\big[g\big(W_1\phi(X_1 - d_1) + W_2\phi(X_2 - d_2)\big)\big] \\ = \;& \mathrm{E}\big[\Delta\zeta(W_1, W_2; d_1, d_2)\big] \\ = \;& \iint_{w_1 \leq w_2} \Delta\zeta(w_1, w_2; d_1, d_2)h(w_1, w_2)\mathrm{d}w_1\mathrm{d}w_2 + \iint_{w_2 \leq w_1} \Delta\zeta(w_1, w_2; d_1, d_2)h(w_1, w_2)\mathrm{d}w_1\mathrm{d}w_2 \\ = \;& \iint_{w_1 \leq w_2} \big[\Delta\zeta(w_1, w_2; d_1, d_2)h(w_1, w_2) + \Delta\zeta(w_2, w_1; d_1, d_2)h(w_2, w_1)\big]\mathrm{d}w_1\mathrm{d}w_2 \\ \geq \;& \iint_{w_1 \leq w_2} \big[\Delta\zeta(w_1, w_2; d_1, d_2) + \Delta\zeta(w_2, w_1; d_1, d_2)\big]h(w_2, w_1)\mathrm{d}w_1\mathrm{d}w_2 \\ \geq \;& 0. \end{aligned}$$

As a consequence, it holds that

$$\mathrm{E}\big[g\big(W_1\phi(X_1 - d_2) + W_2\phi(X_2 - d_1)\big)\big] \geq \mathrm{E}\big[g\big(W_1\phi(X_1 - d_1) + W_2\phi(X_2 - d_2)\big)\big],$$

and the desired increasing convex order follows immediately from the arbitrariness of the increasing and convex g. □

According to Lemma 2.5, $(e^{-\delta T_1}, \cdots, e^{-\delta T_n})$ is SAI whenever $(T_1, \cdots, T_n)$ is SAD. As an immediate consequence of Theorem 4.1, we come up with the following proposition.

Proposition 4.2. Suppose $(X_1, \cdots, X_n)$ is SAI, $(T_1, \cdots, T_n)$ is SAD and they are independent with each other. If $\phi(x)$ is increasing and convex, then the optimal solution $\boldsymbol{d}^*$ of (1) satisfies $d_i^* \le d_j^*$ for any (i, j) such that $1 \le i < j \le n$.

While [22] had a discussion on the case of the discount rate $\delta = 0$ and convex loss function ϕ, we focus on the case of $\delta > 0$ and increasing and convex function ϕ in this study. So, one may naturally wonder whether the increasing property or the convexity of the loss function ϕ can be dropped off in Theorem 4.1. To answer this question we address the necessary aspects of these two properties through the following numerical examples on bivariate random risks (X_1, X_2).

For the sake of convenience, we denote

$$h_\phi(d_1, d_2) = \mathrm{E}\big[W_1\phi(X_1 - d_2) + W_2\phi(X_2 - d_1)\big] - \mathrm{E}\big[W_1\phi(X_1 - d_1) + W_2\phi(X_2 - d_2)\big]. \quad (2)$$

In the coming Examples 4.3, 4.4 and 4.5, we illustrate $h(d_1, d_2) < 0$ for some specific loss function ϕ's, and this refutes

$$W_1\phi(X_1 - d_2) + W_2\phi(X_2 - d_1) \ge_{\text{icx}} W_1\phi(X_1 - d_1) + W_2\phi(X_2 - d_2), \quad \text{for } d_1 \le d_2.$$

One referee points out that the decreasing and convex function $\phi(x) = -x$ reverses the increasing and convex order in Theorem 4.1 and hence the increasing ϕ is obviously necessary in Proposition 4.2. The following numerical examples further sharply confirm that the increasing property of the loss function ϕ can not be dropped off.

Example 4.3. Let independent random variables X_1, X_2 be of a common normal distribution $\mathcal{N}(0, 1)$ and independent random variables W_1, W_2 be of uniform distributions $\mathcal{U}(0, 1)$ and $\mathcal{U}(0, 2)$, respectively. Assume that X_1, X_2, W_1, W_2 are mutually independent. Trivially, (X_1, X_2) is SAI.

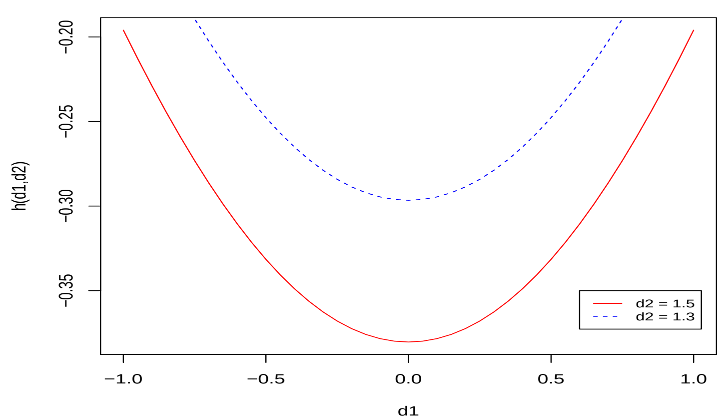

Figure 1: $h_\phi(d_1, d_2)$ for fixed d_2 for normal r.v.s

Since (W_1, W_2) has the probability density

$$\begin{aligned} f(w_1, w_2) &= 0.5 \times I(0 < w_1 < 1)I(0 < w_2 < 2) \\ &\ge 0.5 \times I(0 < w_2 < 1)I(0 < w_1 < 2) = f(w_2, w_1), \end{aligned}$$

for all $w_1 \le w_2$, we conclude that (W_1, W_2) is SAI. Therefore, the conditions assumed for (X_1, X_2) and (W_1, W_2) in Theorem 4.1 are all fulfilled.

Consider now a convex function $\phi(x) = |x|$ yet decreasing in $x \in (-\infty, 0]$. Note that $h_\phi(d_1, d_2) = 0.5\ell(d_1) - 0.5\ell(d_2)$ with $\ell(x) = \sqrt{2/\pi}\exp\{-x^2/2\} - x[1 - 2\Phi(x)]$, where $\Phi(x)$ is the cumulative distribution function of $\mathcal{N}(0, 1)$. As is seen in Figure 1, $h_\phi(d_1, d_2) < 0$ holds for $d_2 = 1.5$ and $d_2 = 1.3$ whenever $d_1 \in [-1, 1]$. This refutes $W_1|X_1 - d_2| + W_2|X_2 - d_1| \geq_{\text{icx}} W_1|X_1 - d_1| + W_2|X_2 - d_2|$, for $d_1 \leq d_2$. □

Next example mainly concerns with random risks with Pareto distribution, which is oftentimes employed to model the claim amount in fire and auto mobile insurance and hence is of important interest in insurance practice. For more details about Pareto distribution and its applications in economics and insurance, we refer readers to [14].

Example 4.4. Let X_1, X_2 be i.i.d random variables having Pareto probability density function $f(x) = 3x^{-4}I(x > 1)$. Assume that (W_1, W_2) is independent of (X_1, X_2) and W_i has uniform distribution $\mathcal{U}(0, i)$, $i = 1, 2$, independently. As per the discussion in Example 4.3, (X_1, X_2) and (W_1, W_2) are both SAI.

Note that, for a convex loss function $\phi(x) = x^2$ yet decreasing for $x \leq 0$,

$$h_\phi(d_1, d_2) = \frac{1}{2}(d_1 - d_2)(d_1 + d_2 - 3).$$

For any (d_1, d_2) with $d_1 < d_2$ and $d_1 + d_2 > 3$, it verifies $h_\phi(d_1, d_2) < 0$ and hence invalidates $W_1(X_1 - d_2)^2 + W_2(X_2 - d_1)^2 \geq_{\text{icx}} W_1(X_1 - d_1)^2 + W_2(X_2 - d_2)^2$, for $d_1 \leq d_2$. □

Further, Example 4.5 below remarks that the convexity assumed for the loss function ϕ is also necessary.

Example 4.5. Assume that $X_1 \sim \mathcal{N}(0, 1)$, $X_2 \sim \mathcal{N}(1, 1)$, $W_1 \sim \mathcal{U}(0, 1)$ and $W_2 \sim \mathcal{U}(0, 2)$ are mutually independent. As per the above example, (W_1, W_2) is SAI. According to [3], (X_1, X_2) is also SAI.

It is plain that $\phi(x) = x^3$ is increasing but not convex. Let $s(d) = 0.5d(3 - d)^2$. Then we have $h_\phi(d_1, d_2) = s(d_2) - s(d_1)$. In light of $s'(d) = 1.5(3 - d)(1 - d)$, we conclude that $s(d)$ is decreasing for any $1 < d < 3$. As a result, it holds that $h_\phi(d_1, d_2) \leq 0$ for any $1 < d_2 < 3$, and this negates $W_1(X_1 - d_2)^3 + W_2(X_2 - d_1)^3 \geq_{\text{icx}} W_1(X_1 - d_1)^3 + W_2(X_2 - d_2)^3$, for $d_1 \leq d_2$. □

5 An application to allocation of coverage limits

The coverage limits are usually applied to insurance risks to avoid potential loss due to heavier right tails in insurance industry. Sometimes, in order to attract customers the insurer grants a total amount of coverage limit ℓ and the policyholder can allocate coverage limits $\boldsymbol{l} = (l_1, \cdots, l_n)$ to risks $\boldsymbol{X} = (X_1, \cdots, X_n)$ covered by a policy according to their own will. For example, [4] pointed out that the compensation package of some big company includes the 'Flexible Spending Account Programme', which allows employees allocate pre-tax income to specific expenses such as health care, medical cost and dependent care etc.

Let $\delta > 0$ be the discount rate and $\boldsymbol{T} = (T_1, \cdots, T_n)$ be the vector of occurrence times of those risks. Denote $\mathcal{A}_\ell$ all admissible allocation vectors such that $\sum_{i=1}^n l_i = \ell$ and $l_i \geq 0$ for all $i = 1, \cdots, n$. Then, for any $\boldsymbol{l} \in \mathcal{A}_\ell$, the policyholder gets the total potential loss

$$\sum_{i=1}^n e^{-\delta T_i}\big(X_i - (X_i \wedge l_i)\big) = \sum_{i=1}^n e^{-\delta T_i}(X_i - l_i)_+,$$

where $x \wedge l = \min\{x, l\}$ and $(x - l)_+ = \max\{x - l, 0\}$. So, it is of interest for the policyholder to consider the following optimization problem based on the utility theory,

$$\begin{cases} \min_{\boldsymbol{l} \in \mathcal{A}_\ell} \mathbb{E}\left[u\left(\sum_{i=1}^n e^{-\delta T_i}(X_i - l_i)_+\right)\right], \\ \text{where } u \text{ is increasing and convex, and } \boldsymbol{X} \text{ is independent of } \boldsymbol{T}. \end{cases} \tag{3}$$

Denote $\boldsymbol{l}^\star = (l_1^\star, \cdots, l_n^\star)$ the solution to the above problem. [13] was among the first to show in the context of comonotonic $\boldsymbol{X}$ with mutually independent $\boldsymbol{T}$ that $l_i^\star \le l_j^\star$ whenever $T_j \le_{\mathrm{lr}} T_i$ and $X_i \le_{\mathrm{st}} X_j$, for any $1 \le i \neq j \le n$. Subsequently, in the context of the comonotonic severity $\boldsymbol{X}$ with $\boldsymbol{T}$ having some Archimedean copula, [11] further proved that it is least favorable for the risk-averse policyholder to allocate a smaller coverage limit to the loss with higher severity and frequency.

Note that (i) as per Lemma 2.5, $\left(e^{-\delta T_1}, \cdots, e^{-\delta T_n}\right)$ is SAI whenever $(T_1, \cdots, T_n)$ is SAD, and (ii) $(x-d)_+$ is increasing and convex. As a direct consequence of Theorem 4.1, we reach Proposition 5.1, which is exactly Theorem 6.5 of [2], $(T_1, \cdots, T_n)$ is SAD if and only if $(e^{-T_1}, \cdots, e^{-T_n})$ is SAI.

Proposition 5.1. Suppose that $(X_1, \cdots, X_n)$ is SAI, $(T_1, \cdots, T_n)$ is SAD and they are independent with each other. The solution $\boldsymbol{l}^\star$ of (3) satisfies $l_i^\star \le l_j^\star$ for $1 \le i < j \le n$.

According to Proposition 5.1, *the optimal allocations always assign larger coverage limit to larger risk*. Evidentally, Proposition 5.1 serves as a nice generalization of the ordering result due to [13].

In this study, we pay attention to the optimal allocations to SAI risks $(X_1, \cdots, X_n)$ associated with SAD occurrence times $(T_1, \cdots, T_n)$, which is implicitly assumed through concerned SAI random variables $(W_1, \cdots, W_n)$ in Theorem 4.1. At the end, we remark one insightful comment from the other referee: In general, the allocation d_i should take into account both the risk X_i and corresponding occurrence time T_i, $i = 1, \cdots, n$. With no doubt the allocation problem in the general context is of both theoretical and practical interest and hence deserves future research in this line.

Appendix

Proof of Lemma 3.1

(i) Since ϕ is increasing and convex, for any $a_1 \le a_2$ and $x_1 \le x_2$, it holds that

$$\phi(x_2 - a_1) - \phi(x_2 - a_2) \ge \phi(x_1 - a_1) - \phi(x_1 - a_2) \ge 0,$$

and hence, for any $w_2 \ge w_1 \ge 0$,

$$w_1\phi(x_1 - a_2) + w_2\phi(x_2 - a_1) \ge w_1\phi(x_1 - a_1) + w_2\phi(x_2 - a_2).$$

That is, $\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w})$. Taking the increasing g into account we reach $\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) \ge 0$ for any $x_2 \ge x_1$.

(ii) Due to the convexity of ϕ, it holds that

$$\begin{aligned} & \eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) + \eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w}) - \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) - \eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) \\ = \;& (w_2 - w_1)[\phi(x_2 - a_1) - \phi(x_2 - a_2) + \phi(x_1 - a_1) - \phi(x_1 - a_2)] \ge 0, \end{aligned}$$

yielding

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) + \eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) + \eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}). \tag{4}$$

On the other hand, the increasing ϕ also implies

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) - \eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) = (w_2 - w_1)[\phi(y - a_1) - \phi(x - a_2)] \ge 0.$$

According to (i), we have $\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w})$, and this invokes

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \max\{\eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}), \eta_1(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})\}. \tag{5}$$

In combination with (4) and (5), we conclude by the convexity of g that

$$\begin{aligned} & \Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) + \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) \\ = \;& g(\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w})) - g(\eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w})) + g(\eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w})) - g(\eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})) \\ \ge \;& 0, \qquad \text{for } x_1 \le x_2. \end{aligned}$$

This completes the proof. □

Proof of Lemma 3.2

Denote $f(x_1, x_2)$ the probability density of (X_1, X_2). Since (X_1, X_2) is SAI, it follows from Lemma 2.4 that $f(x_1, x_2) \ge f(x_2, x_1)$ for $x_1 \le x_2$. Now, let us proceed with notations in Lemma 3.1.

(i) Owing to Lemma 3.1 we have

$$\begin{aligned}
&\Delta\zeta(w_1, w_2; a_1, a_2)\\
=\ & \iint_{\mathbb{R}^2} \Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2\\
=\ & \iint_{x_1 \le x_2} \Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 + \iint_{x_2 \le x_1} \Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2\\
=\ & \iint_{x_1 \le x_2} \left[\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) f(x_1, x_2) + \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) f(x_2, x_1)\right] \mathrm{d}x_1 \mathrm{d}x_2\\
\ge\ & \iint_{x_1 \le x_2} [\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) + \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})] f(x_2, x_1) \mathrm{d}x_1 \mathrm{d}x_2\\
\ge\ & 0, \qquad \text{for all } a_1 \le a_2 \text{ and } w_2 \ge w_1 \ge 0.
\end{aligned}$$

(ii) Since ϕ is increasing and convex, for any $a_1 \le a_2$ and $x_1 \le x_2$, it holds that

$$\phi(x_2 - a_1) - \phi(x_1 - a_1) \ge \phi(x_2 - a_2) - \phi(x_1 - a_2) \ge 0, \tag{6}$$

and hence, for any $0 \le w_1 \le w_2$,

$$w_1\phi(x_1 - a_2) + w_2\phi(x_2 - a_1) \ge w_1\phi(x_2 - a_2) + w_2\phi(x_1 - a_1).$$

That is,

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w}).$$

In combination with Lemma 3.1 (i) we have

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}),$$

and this invokes

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) \ge \max\{\eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}), \eta_1(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w})\}. \tag{7}$$

Moreover, by (6) we have, for $x_1 \le x_2$,

$$\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w}) + \eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}) \ge \eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) + \eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w}). \tag{8}$$

In combination with (7) and (8) we further get

$$\begin{aligned}
&\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) - \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})\\
=\ & g(\eta(x_1, x_2; \pi(\boldsymbol{a}), \boldsymbol{w})) - g(\eta(x_1, x_2; \boldsymbol{a}, \boldsymbol{w})) - g(\eta(x_2, x_1; \pi(\boldsymbol{a}), \boldsymbol{w})) + g(\eta(x_2, x_1; \boldsymbol{a}, \boldsymbol{w}))\\
\ge\ & 0, \qquad \text{for } x_1 \le x_2.
\end{aligned} \tag{9}$$

Therefore, it holds that

$$\begin{aligned}
&\Delta\zeta(w_1, w_2; a_1, a_2) + \Delta\zeta(w_2, w_1; a_1, a_2)\\
=\ & \iint_{\mathbb{R}^2} [\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) - \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})] f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2\\
=\ & \iint_{x_1 \le x_2} [\Delta g(x_1, x_2; \boldsymbol{a}, \boldsymbol{w}) - \Delta g(x_2, x_1; \boldsymbol{a}, \boldsymbol{w})][f(x_1, x_2) - f(x_2, x_1)] \mathrm{d}x_1 \mathrm{d}x_2\\
\ge\ & 0, \qquad \text{for } a_2 \ge a_1 \text{ and } w_2 \ge w_1 \ge 0,
\end{aligned}$$

where the last inequality stems from Lemma 2.4 and (9). □

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