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Pier Luigi Papini*

Bivariate copulas, norms and non-exchangeability

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Abstract: The present paper is related to the study of asymmetry for copulas by introducing functionals based on different norms for continuous variables. In particular, we discuss some facts concerning asymmetry and we point out some flaws occurring in the recent literature dealing with this matter.

Keywords: Copulas, Asymmetry, Measure of nonexchangeability

MSC: 60E05, 60G09; secondary 62H20

1 Introduction

Bivariate copulas (see for example Nelsen [11] for basic definitions and results) have been studied in details since more than half century. More recently (see Nelsen [12] and Klement and Mesiar [10]) a first measure of asymmetry has been considered; several recent papers deal with the largest asymmetry (in this sense) that some relevant families of copulas can reach (see for example Alvoni et al. [2]). For a general reference on asymmetry measures we refer for example to Genest and Nešlehová [9]. Concerning other recent papers on the subject, see for example [1, 3, 6, 8]. For a recent reference concerning copulas in general, we indicate the book by Durante and Sempi [7].

A possible set of natural axioms that a measure of asymmetry should satisfy has been indicated in Durante et al. [5]; in particular, a few measures of this type can be obtained by using some classical norms.

In Siburg and Stoimenov [14], an attempt to obtain in a new way, by any norm, a measure of asymmetry (not only for copulas, and not necessarily satisfying all the axioms indicated in Durante et al. [5]), has been done. Unfortunately, several facts indicated there are flawed.

Here we discuss the results in Siburg and Stoimenov [14]; we also indicate which additional properties of the norm imply some "good" properties for the corresponding measure of asymmetry defined there.

The plan of this paper is the following. In the next Section 2 we recall the definitions concerning copulas and asymmetry. In Section 3, we consider the functional connected with asymmetry, based on norms, introduced in Siburg and Stoimenov [14]; we indicate, by several examples, some mistakes contained there. In Section 4 we discuss some variations concerning the same functional, by using norms satisfying a few additional, simple properties. Finally, Section 5 contains a short discussion concerning the use of norms in the context of copulas.

*Corresponding Author: Pier Luigi Papini: Via Martucci, 19, 40136 Bologna, Italy, E-mail: plpapini@libero.it

2 Definitions and Notations

Set $I = [0, 1]$, so $I^2 = [0, 1] \times [0, 1]$. For the sake of simplicity, from now on, V will denote the vector space of all continuous real functions on I^2 . For $f \in V$, set

$$f^T : (x, y) \rightarrow f(y, x) \quad (2.1)$$

$$\widehat{f} : (x, y) \rightarrow x + y - 1 + f(1 - x, 1 - y). \quad (2.2)$$

Remark 1. Note that $\widehat{f^T} = (\widehat{f})^T$; for $\alpha, \beta \in R : (\alpha f + \beta g)^T = \alpha f^T + \beta g^T$; moreover, for $\alpha + \beta = 1 : \widehat{(\alpha f + \beta g)} = \alpha \widehat{f} + \beta \widehat{g}$.

A (bivariate) *copula* is an element of V satisfying:

$$C(x, 0) = C(0, y) = 0, \text{ for all } x, y \in I; \quad (2.3)$$

$$C(x, 1) = x; \quad C(1, y) = y, \text{ for all } x, y \in I; \quad (2.4)$$

for $0 \leq x \leq x' \leq 1, 0 \leq y \leq y' \leq 1$:

$$C(x', y') - C(x, y') \geq C(x', y) - C(x, y). \quad (2.5)$$

In particular, condition (2.5), usually called *2-increasingness*, together with (2.3) implies:

$$C(x, y) \text{ is increasing in each variable.} \quad (2.6)$$

A copula can be seen as the restriction to the unit square of a probability distribution function with uniform marginals on $[0, 1]$. We shall denote by \mathcal{C} the set of all copulas: this is a convex subset of V .

If $C \in \mathcal{C}$, then also C^T and \widehat{C} are in \mathcal{C} (they are called the *transpose*, and -respectively- the *survival copula* associated with C). A copula is said to be *symmetric* if $C = C^T$.

As known, the following (symmetric) copulas play an important role:

$$W(x, y) = \max\{x + y - 1, 0\}; \quad (2.7)$$

$$\Pi(x, y) = xy; \quad (2.8)$$

$$M(x, y) = \min\{x, y\}. \quad (2.9)$$

Recall that for any copula C we have

$$W(x, y) \leq C(x, y) \leq M(x, y). \quad (2.10)$$

According to Durante et al. [5], a *measure of non-exchangeability* for \mathcal{C} is a function $\mu: \mathcal{C} \rightarrow R^+ = \{x \in R : x \geq 0\}$ satisfying:

- (A1) there exists $k \in R^+$ such that $\mu(C) \leq k$ for all $C \in \mathcal{C}$;
- (A2) $\mu(C) = 0$ if and only if C is symmetric;
- (A3) $\mu(C) = \mu(C^T)$ for every $C \in \mathcal{C}$;
- (A4) $\mu(C) = \mu(\widehat{C})$ for every $C \in \mathcal{C}$;
- (A5) if $(C_n)_{n \in N}$ and C all belong to \mathcal{C} and $(C_n)_{n \in N}$ converges uniformly to C , then $\mu(C) = \lim_{n \rightarrow \infty} \mu(C_n)$.

As examples of such measures, the following ones were indicated in Durante et al. [5]:

$$\mu_p(C) := \left(\int_{I^2} |C(x, y) - C(y, x)|^p dx dy \right)^{1/p} \text{ if } 1 \leq p < \infty; \quad (2.11)$$

$$\mu_\infty(C) := \max\{|C(x, y) - C(y, x)| : (x, y) \in I^2\}. \quad (2.12)$$

By the way, μ_∞ is exactly the measure of asymmetry used in Nelsen [12] and in Klement and Mesiar [10]. Also, if we denote by $\|\cdot\|_p$ the classical L_p norms (in I^2), we have:

$$\mu_p(C) = \|C - C^T\|_p, \quad 1 \leq p \leq \infty. \quad (2.13)$$

Note that all p -norms ($1 \leq p \leq \infty$), applied to all functions f on V , satisfy the following properties:

- (N1) $\|f\| = \|f^T\|$ (for all $f \in V$);
- (N2) if $|f| \leq |g|$, then $\|f\| \leq \|g\|$ (lattice norms).

Moreover, for every copula C :

- (N3) $0 < \|C\|_p \leq 1$; $\|C\|_\infty = 1$;
- (N4) given two copulas C_1, C_2 we have $\|C_1 + C_2\|_1 = \|C_1\|_1 + \|C_2\|_1$; $\|C_1 + C_2\|_\infty = 2$. In particular: if $p \in \{1, \infty\}$, then $\|C + C^T\|_p = \|C\|_p + \|C^T\|_p$.

3 A functional and its properties

Concerning nonexchangeability, given a norm $\|\cdot\|$ in V , Siburg and Stoimenov [14] considered something different from (2.13).

For every $f \in V \setminus \{0\}$ (0 denoting the null element of V , i.e. the identically null function) set

$$\delta(f) := \frac{\|f + f^T\|^2 - \|f - f^T\|^2}{4\|f\|^2}. \quad (3.1)$$

Concerning δ , the following properties were indicated (see Siburg and Stoimenov [14, Theorem 2.4]):

- (P1) $\delta(V \setminus \{0\}) = [-1, 1]$;
- (P2) $\delta(f) = 1$ if and only if $f = f^T$;
- (P3) $\delta(f) = -1$ if and only if $f = -f^T$;
- (P4) δ is continuous;
- (P5) $\delta(\lambda f) = \delta(f)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$.

These properties are not completely natural. Also, we could extend the definition by setting $\delta(0) = 0$, so avoiding exclusions in a few cases.

While (P5) is trivially true, the other properties are doubtful, also when f is a copula, unless we require that the norm satisfies some additional properties. We discuss this by means of examples; in the first one we consider functions which are not copulas. The second one has been partly inspired by an example, called "exotic", in Durante et al. [5].

Example 3.1. Let $P = (2/3, 1/3)$. Consider in V (the vector space of continuous function in I^2) the following norms:

$$\|f\|_\alpha = \int_{I^2} |f(x, y)| \, dx dy + \alpha |f(P)|, \quad \alpha > 0. \quad (3.2)$$

Also, let $B(P, \epsilon)$ be the Euclidean ball centered at P , of radius ϵ ($\epsilon > 0$). Denote by δ_α the functional on $V \setminus \{0\}$ defined by setting $\|\cdot\|_\alpha$ in (3.1).

Define a sequence of functions $(f_n)_{n \in \mathbb{N}} \in V$, satisfying: $0 \leq f_n(x, y) \leq 1$; $f_n(1/3, 2/3) = 1$; $f_n(x, y) = 0$ in $I^2 \setminus B((\frac{1}{3}, \frac{2}{3}), \frac{1}{\sqrt{n}})$; $\int_{I^2} |f_n(x, y)| \, dx dy = 1/n$.

Clearly, for n large, $\|f_n\|_\alpha = 1/n$. Let $g \equiv 1$ in I^2 (so $g = g^T$); then $f_n + g \xrightarrow{n \rightarrow +\infty} g$.

Also:

$$\int_{I^2} |(f_n + g)(x, y)| \, dx dy = 1 + 1/n; \quad \|f_n + g\|_\alpha = 1 + 1/n + \alpha.$$

Moreover, still for n large,

$$\int_{I^2} |(f_n + g + (f_n + g)^T)(x, y)| \, dx dy = 2 + 2/n;$$

$$\int_{I^2} |(f_n + g - (f_n + g)^T)(x, y)| \, dx dy = 2/n;$$

$$\|f_n + g + (f_n + g)^T\|_\alpha = 2 + 2/n + 3\alpha; \quad \|f_n + g - (f_n + g)^T\|_\alpha = 2/n + \alpha.$$

Therefore

$$\delta_\alpha(f_n + g) = \frac{\|f_n + g + (f_n + g)^T\|_\alpha^2 - \|f_n + g - (f_n + g)^T\|_\alpha^2}{4\|f_n + g\|_\alpha^2} = \frac{(2 + 2/n + 3\alpha)^2 - (2/n + \alpha)^2}{4(1 + 1/n + \alpha)^2},$$

$$\text{so } \lim_{n \rightarrow \infty} \delta_\alpha(f_n + g) = \frac{(2 + 3\alpha)^2 - \alpha^2}{4(1 + \alpha)^2} \neq 1 = \delta_\alpha(g).$$

We could not deal with convergence to 0 since $\delta(0)$ is not defined. The previous example also shows the following facts: (P1) and (P4) are not true in general. Moreover, by choosing α in a suitable way, we can have for some n : $\delta_\alpha(f_n + g) = 1$ (but $f_n + g$ is not symmetric); so also (P2) is not true in general.

In the previous example, we dealt with functions in V which are not copulas. The following examples instead are based on copulas: they show that (also for them) (P3), as well (P1) and (P2), are not true in general.

In our next examples we shall consider a copula $K =: K(x, y)$ with the following properties:

$$K(2/3, 1/3) = K(1/3, 1/3) = 0; \quad K(2/3, 1/2) = K(1/3, 1/2) = 1/6;$$

$$K(1/3, 2/3) = K(2/3, 2/3) = 1/3 = \max\{|K(x, y) - K(y, x)| : x, y \in I\}.$$

A copula K with these properties has been considered for example in Nelsen [11], where the last equality ($\|K - K^T\|_\infty = 1/3$) was shown.

Example 3.2. Let again $P = (2/3, 1/3)$; consider in V the norms (satisfying (N2)):

$$\|f\| = \|f\|_\infty + a|f(P)| \quad (a > 0). \quad (3.3)$$

Let $\delta_a(f)$ be the corresponding functional, defined on $V \setminus \{0\}$ according to (3.1).

Now consider the copula K defined before this example. We have:

$$\|K + K^T\|_a = 2 + a/3; \quad \|K - K^T\|_a = 1/3 + a/3; \quad \|K\|_a = 1.$$

$$\text{Thus: } \delta_a(K) = \frac{\|K + K^T\|_a^2 - \|K - K^T\|_a^2}{4\|K\|_a^2} = \frac{(2 + a/3)^2 - (1/3 + a/3)^2}{4} = \frac{4 + (4a)/3 - 1/9 - (2a)/9}{4} = \frac{35 + 10a}{36}.$$

Therefore $\delta_a(K) > 1$ if $a > 1/10$. Also: if $a = 1/10$, then $\delta_a(K) = 1$ but $K \neq K^T$.

Thus, also for copulas, (P2) (as well as (P1)) is not true. Moreover the range of δ (also for the set of copulas) can be "very large".

Remark 2. Note that we have (since $\|K^T\|_a = 1 + a/3$):

$$\delta_a(K^T) = \frac{(2 + a/3)^2 - (1/3 + a/3)^2}{4(1 + a/3)^2} \xrightarrow{a \rightarrow +\infty} 0$$

Example 3.3. Set in V :

$$\|f\|_\beta = \|f\|_\infty + \beta|f(1/3, 2/3) - f(2/3, 1/2)| \quad (\beta > 0) \quad (3.4)$$

and let $\delta_\beta(f)$ be the corresponding functional on $V \setminus \{0\}$ (see (3.1)). Note that this norm does not satisfy (N1).

For the copula K we have:

$$\|K\|_\beta = 1 + \beta/6; \quad \|K^T\|_\beta = 1 + \beta/3; \quad (K - K^T)(1/3, 2/3) = (K + K^T)(1/3, 2/3) = 1/3;$$

$$(K + K^T)(2/3, 1/2) = 1/2; \quad (K - K^T)(2/3, 1/2) = -1/6, \text{ so } \|K + K^T\|_\beta = 2 + \beta/6; \quad \|K - K^T\|_\beta = 1/3 + \beta/2.$$

Therefore:

$$\delta_\beta(K) = \frac{\|K + K^T\|_\beta^2 - \|K - K^T\|_\beta^2}{4\|K\|_\beta^2} = \frac{(2 + \beta/6)^2 - (1/3 + \beta/2)^2}{4(1 + \beta/6)^2} = \frac{35 - 2\beta^2 + 3\beta}{(6 + \beta)^2} \xrightarrow{\beta \rightarrow +\infty} -2.$$

So, for a suitable value of β , we can have $\delta_\beta(K) = -1$ (but certainly $K \neq -K^T$); thus (P3) (as well as (P1)) is violated.

The same happens if in the denominator we put, instead of $4\|K\|_\beta^2$, for example:

$$4 \max\{\|K\|_\beta^2, \|K^T\|_\beta^2\}, \text{ or } 4\|K^T\|_\beta^2 \quad (\|K^T\|_\beta = 1 + \beta/3).$$

4 Some positive results

For any $f \neq 0$, the property $\|f\| = \|f^T\|$ is equivalent to $\delta(f) = \delta(f^T)$ (compare with Theorem 2.6 (i) in Siburg and Stoimenov [14]).

Assume that a norm satisfies (N1). Then we obtain for the corresponding δ (see (3.1)):

$$\delta(f) \leq \frac{\|f + f^T\|^2}{4\|f\|^2} = \left(\frac{\|f + f^T\|}{2\|f\|} \right)^2 \leq 1; \quad (4.1)$$

$$\delta(f) \geq \frac{-\|f - f^T\|^2}{4\|f\|^2} = -\left(\frac{\|f - f^T\|}{2\|f\|} \right)^2 \geq -1. \quad (4.2)$$

Moreover, in this case, $\delta(f) = 1 \Rightarrow \|f - f^T\| = 0 \Rightarrow f^T = f$ (and conversely);

$\delta(f) = -1 \Rightarrow \|f + f^T\| = 0 \Rightarrow f^T = -f$, and conversely (so this never happens for copulas).

Therefore, (P1), (P2) and (P3) hold. Moreover (under the assumption that (N1) holds) $\delta(C) = \delta(C^T)$ for every copula C .

But under the same assumption, we can also prove the following.

Proposition 4.1. *If a norm satisfies (N1), then δ (defined through (3.1)) satisfies (P4).*

Proof. Let $f_n \rightarrow f \neq 0$; then (according to (N1)) $\|f_n^T - f^T\| \rightarrow 0$; the sequences $\|f_n\|$, $\|f_n^T\|$, so also the sequences $\|f_n + f_n^T\|$ and $\|f_n - f_n^T\|$, are bounded. Moreover $\|f_n - f \pm (f_n^T - f^T)\| \rightarrow 0$. Then we have:

$$\begin{aligned} & \|f_n + f_n^T\|^2 - \|f + f^T\|^2 + \|f - f^T\|^2 - \|f_n - f_n^T\|^2 \\ &= (\|f_n + f_n^T\| + \|f + f^T\|)(\|f_n + f_n^T\| - \|f + f^T\|) + (\|f - f^T\| + \|f_n - f_n^T\|)(\|f - f^T\| - \|f_n - f_n^T\|) \\ &\leq (\|f_n + f_n^T\| + \|f + f^T\|)(\|f_n + f_n^T - f - f^T\|) + (\|f - f^T\| + \|f_n - f_n^T\|)(\|f - f^T - f_n + f_n^T\|). \end{aligned}$$

Both addends in the last term go to 0 for n going to ∞ , and this implies

$$\delta(f_n) - \delta(f) = \frac{\|f_n + f_n^T\|^2 - \|f_n - f_n^T\|^2}{4\|f_n\|^2} - \frac{\|f + f^T\|^2 - \|f - f^T\|^2}{4\|f\|^2} \xrightarrow{n \rightarrow +\infty} 0,$$

which proves the proposition. \square

If the norm does not satisfy $\|f\| = \|f^T\|$, then we can have (P1), for example, if we define (instead of using (3.1)):

$$\delta'(f) = \frac{\|f + f^T\|^2 - \|f - f^T\|^2}{2\|f\|^2 + 2\|f^T\|^2} \quad (4.3)$$

or we change in some other suitable way the denominator. It is simple to see that

$$-1 \leq \frac{\pm\|f \pm f^T\|^2}{4 \max\{\|f\|^2, \|f^T\|^2\}} \leq \frac{\|f \pm f^T\|^2}{2\|f\|^2 + 2\|f^T\|^2} \leq \frac{\|f \pm f^T\|^2}{(\|f\| + \|f^T\|)^2} \leq 1. \quad (4.4)$$

With such modification (P2) and (P3) hold. In fact (concerning δ' in (4.3)) we have

$$-1 \leq \frac{-(\|f\| + \|f^T\|)^2}{2\|f\|^2 + 2\|f^T\|^2} \leq \frac{-\|f - f^T\|^2}{2\|f\|^2 + 2\|f^T\|^2} \leq \frac{\|f + f^T\|^2 - \|f - f^T\|^2}{2\|f\|^2 + 2\|f^T\|^2} \leq 1.$$

Similar facts hold if we change the denominator in the other two ways ($4 \max\{\|f\|^2, \|f^T\|^2\}$, or $2(\|f\| + \|f^T\|)^2$).

But the same changes do not imply in general (P4). For example, consider Example 3.1; change the denominator to $(\|f\| + \|f^T\|)^2$ and denote by δ'_α the corresponding functional; with this change, concerning Example 3.1 we obtain (since $\|(f_n + g)^T\|_\alpha = 1 + 1/n + 2\alpha$):

$$\delta'_\alpha(f_n + g) = \frac{(2 + 2/n + 3\alpha)^2 - (2/n + \alpha)^2}{[(1 + 1/n + \alpha)^2 + (1 + 1/n + 2\alpha)]^2} \xrightarrow{n \rightarrow +\infty} \frac{(2 + 3\alpha)^2 - \alpha^2}{(2 + 3\alpha)^2} \neq 1 = \delta'_\alpha(g).$$

Example 4.2. Set in V :

$$\|f\|_\gamma = \|f\|_\infty + \gamma |f(1/3, 2/3) - f(2/3, 1/3)| \quad (\gamma > 0). \quad (4.5)$$

This norm satisfies (N1) (but not (N2)).

If we consider the copula K of Section 3, we have:

$$\|K\|_\gamma = \|K^T\|_\gamma = 1 + (\gamma/3); \quad (K - K^T)(1/3, 2/3) = (K + K^T)(1/3, 2/3) = (K + K^T)(2/3, 1/3) = 1/3; \\ (K - K^T)(2/3, 1/3) = -1/3.$$

Therefore $\|K + K^T\|_\gamma = 2$; $\|K - K^T\|_\gamma = 1/3 + (2/3)\gamma$, and then

$$\delta(K) = \frac{\|K + K^T\|_\gamma^2 - \|K - K^T\|_\gamma^2}{4\|K\|_\gamma^2} = \frac{4 - (1/3 + 2\gamma/3)^2}{4(1 + \gamma/3)^2} = \frac{35 - 4\gamma - 4\gamma^2}{4(3 + \gamma)^2} \xrightarrow{\gamma \rightarrow +\infty} -1.$$

The same considerations hold if at the denominator we put for example $2(\|K\|_\gamma^2 + \|K^T\|_\gamma^2)$, instead of $4\|K\|_\gamma^2$.

This shows that, if we consider all functionals δ derived by norms satisfying (N1), the union of ranges already for the set of copulas is $(-1, 1]$.

Remark 3. We observe the following fact. Let a norm satisfy (N2); note that, for any copula C , we have pointwise: $|C - C^T| \leq \max\{C, C^T\} \leq C + C^T$; then we have $0 \leq \delta(C)$ for all copulas. This estimate is sharp (see Remark 2).

Remark 4. Note that when (P1) holds, we can also "renormalize" δ so that its range become, for example, $[0, 1]$: however the value 0 would not correspond in general to f symmetric.

Concerning copulas, a better way to "renormalize" would be the following. Assume that a norm satisfies (N1) and (N2), so that $\delta(C) \in [0, 1]$ always (see Remark 3); then set $\mu'(C) = 1 - \delta(C)$. In this way $\mu'(C) \in [0, 1]$ always, and $\mu'(C) = 0$ if and only if $C = C^T$: so (A2) is satisfied.

For example, in this way, if the norm used is $\|\cdot\|_\infty$, then (since $\|C - C^T\|_\infty \leq 1/3$ always) the range of μ' would be $[0, 1/36]$.

Next example shows that in general $\delta(f) \neq \delta(\hat{f})$, also if f is a copula. A sufficient condition for the equality, was indicated (with a somehow sketched proof) in Siburg and Stoimenov [14, Theorem 2.6 (ii)]. But that condition, not necessary, does not apply to copulas.

Example 4.3. Set in V :

$$\|f\|_c = \|f\|_\infty + |f(2/3, 1/2) - f(1/2, 2/3)|. \quad (4.6)$$

This norm satisfies (N1) (but not (N2)).

Consider a copula C such that:

$$C(1/2, 1/3) = 0; \quad C(2/3, 1/3) = C(1/3, 1/2) = C(1/2, 1/2) = 1/6; \\ C(2/3, 1/2) = C(1/3, 2/3) = C(1/2, 2/3) = 1/3; \quad C(2/3, 2/3) = 1/2 \quad (\|C - C^T\|_\infty = 1/6).$$

We have:

$$\|C - C^T\|_c = \|C - C^T\|_\infty = 1/6; \quad \|C + C^T\|_c = \|C + C^T\|_\infty = 2; \quad \|C\|_c = 1. \\ \|\hat{C} - \hat{C}^T\|_c = \|\hat{C} - \hat{C}^T\|_\infty + |-1/6 - 1/6| = 1/6 + 1/3 = 1/2; \quad \|\hat{C} + \hat{C}^T\|_c = \|\hat{C} + \hat{C}^T\|_\infty = 2; \quad \|\hat{C}\|_c = 1 + 1/6.$$

$$\text{Therefore } \delta(C) = \frac{2^2 - (1/6)^2}{4} = \frac{143}{144}; \quad \delta(\hat{C}) = \frac{2^2 - (1/2)^2}{4(1 + 1/6)^2} = \frac{135}{196}.$$

Note that $\|C - C^T\|_c = 1/6 \neq \|\hat{C} - \hat{C}^T\|_c = 1/2$.

5 final discussion and concluding remarks

Concerning copulas, the use of norms seems to be not a straight way to deal with; a discussion was done in Darsow and Olsen [4], where also some problems were indicated. Later on, very few papers considered this

matter: see for example Siburg and Stoimenov [14]. We recall that, in particular, the use of the Sobolev scalar product for copulas was considered by Siburg and Stoimenov [13].

This is due to more reasons, we indicate some of them.

First, copulas do not form a subspace of V (nor a vector space), and the subspace they generate has a non evident structure (see Darsow and Olsen [4, Section 2]).

Second, we usually think of properties that the most used norms for the space of continuous functions have, but can be missed; for this reason we are led to think and hope for properties that only additional requirements on the norms imply. So, due to this, not all the norms are appropriate to obtain "nice" measures of asymmetry, also when limited to copulas. In particular, we have shown that in general the functional discussed in Section 3 is not so suitable to measure asymmetry of copulas: this attempt to use norms and extend the study of asymmetry to a larger class of functions needs special care.

Start from a norm and define symmetry, also in V , by using the simple formula (2.13); clearly (A2), (A3), (A5) (and (P5)) are satisfied (but not necessarily (A4): see Example 4.3). As we can understand, this is not true in general for δ : C and C^T play an asymmetric role in its definition.

We recall that the definition of δ was indicated by Genest and Nešlehová in [9, p.94], with the following sentence: "this measure, for the uniform norm, is equivalent to μ_∞ , so is not considered further here". Indeed, in that case $\delta = 1 - \frac{\mu_\infty^2}{4}$, but this equivalence is limited to a very particular situation. For example: when $C = C^T$ (so $\mu(C) = 0$), $\delta(C)$ can be both positive and negative (see Example 4.2).

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