

## Research Article

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# Quantile of a Mixture with Application to Model Risk Assessment

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**Abstract:** We provide an explicit expression for the quantile of a mixture of two random variables. The result is useful for finding bounds on the Value-at-Risk of risky portfolios when only partial dependence information is available. This paper complements the work of [4].

**Keywords:** Model Risk; Rearrangement Algorithm; Mixture; Model Risk

**MSC:** 60E05, 60E15

## 1 Introduction

Consider a mixture  $S = \mathbb{I}X + (1 - \mathbb{I})Y$  where  $\mathbb{I}$  is a Bernoulli distributed random variable with parameter  $q$  and where the components  $X$  and  $Y$  are independent of  $\mathbb{I}$ . In this paper, we aim at finding an explicit expression for the quantiles of  $S$  as a function of the quantiles of the variables  $X$  and  $Y$ . Here, the quantile at level  $p$  ( $0 < p < 1$ ) of a given distribution  $F$  is defined<sup>1</sup> as

$$F^{-1}(p) = \inf \{x \in \mathbb{R} \mid F(x) \geq p\}. \quad (1)$$

By convention,  $\inf\{\emptyset\} = \infty$  and  $\inf\{\mathbb{R}\} = -\infty$ , so that the quantile is properly defined by (1) for all  $p \in [0, 1]$ . In a risk management context, one often considers  $X$  (distributed with  $F_X$ ) as a loss variable in which case  $F_X^{-1}(p)$  can be broadly interpreted as the maximum loss (“Value-at-Risk”) one can observe with  $p$ -confidence.

Expressing  $F_S^{-1}(\cdot)$  in terms of  $F_X^{-1}(\cdot)$  and  $F_Y^{-1}(\cdot)$  can sometimes be relatively straightforward. For example, when  $F_X$  and  $F_Y$  are strictly increasing with unbounded support<sup>2</sup>,

$$F_S^{-1}(p) = F_X^{-1}(\alpha_*) = F_Y^{-1}(\beta_*), \quad (2)$$

in which  $0 < \alpha_* < 1$  and  $0 < \beta_* < 1$  are uniquely defined and satisfy  $q\alpha_* + (1 - q)\beta_* = p$ . However, formula (2) does not cover the general case. We need a more general version of it, which requires a careful examination of all possible cases for the distributions  $F_X$  and  $F_Y$  (Theorem 1). This result is of some probabilistic interest,

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<sup>1</sup> Note that the definition of quantile is not unique. Here, we consider the lower quantile, which is the prevalent way to define quantiles in a risk management context; See e.g., Section 4.4 and Definition A.20 in [7] as well as the works of [1] and [11].

<sup>2</sup> Note that under similar conditions on the distribution functions the two-dimensional case can be extended to  $n$  dimensions in a straightforward way; it is actually sufficient that the distributions of the components involved in the mixture are invertible and have a common support. Another notable case dealing with distributions that have a disjoint support was treated in [5]. In this paper we focus further on providing a general formula for the quantile of a mixture in the two-dimensional case.



as it makes it possible to reduce the dimensionality of the problem of assessing a quantile of a mixture. It also has a clear application in risk management that we further explain as follows:

In the risk assessment of high dimensional portfolios  $\mathbf{X} := (X_1, X_2, \dots, X_d)$  the variable of interest is typically the portfolio sum  $\sum_{i=1}^d X_i$  and the risk measure used in the industry is a quantile. From Sklar's theorem, the evaluation of quantiles of  $\sum_{i=1}^d X_i$  is at most a numerical issue once the marginal distributions of the variables  $X_i$  as well as their dependence (copula) is completely specified. Unfortunately, estimating copulas is a difficult task and the assessment of  $\mathbf{X}$  is thus prone to model misspecification. [4] assume that candidate models are consistent with the following distributional properties of  $\mathbf{X}$ :

- (i) the marginal distribution  $F_{X_i}$  of  $X_i$  are known.
- (ii) the distribution of  $\mathbf{X} \mid \{\mathbf{X} \in \mathcal{F}\}$  is known for some  $\mathcal{F} \subset \mathbb{R}^d$ .
- (iii) the probability  $p_{\mathcal{F}} := P(\mathbf{X} \in \mathcal{F})$  is known.

In other words, the joint distribution of  $\mathbf{X}$  is only fully specified on the subarea  $\mathcal{F}$  of  $\mathbb{R}^d$  and the quantiles of  $\sum_{i=1}^d X_i$  cannot be computed (unless  $p_{\mathcal{F}} = 1$ ). It is then of interest to find among all possible distributional models for  $\mathbf{X}$  the one that yields the highest (resp. lowest) possible outcome for the desired quantile of  $\sum_{i=1}^d X_i$ . The maximum and minimum possible values can be obtained using a mixture representation. Specifically, consider the indicator variable  $\mathbb{I}$  corresponding to the event " $\mathbf{X} \in \mathcal{F}$ "

$$\mathbb{I} := \mathbb{1}_{\mathbf{X} \in \mathcal{F}}, \quad (3)$$

It is then clear that for any choice of joint distribution for  $\mathbf{X}$  that is consistent with properties (i), (ii) and (iii), there exists a multivariate vector  $(Z_1, Z_2, \dots, Z_d)$  that we can take independent of  $\mathbb{I}$ , such that the portfolio sum can be represented as a mixture

$$\sum_{i=1}^d X_i =_d \mathbb{I} \sum_{i=1}^d X_i + (1 - \mathbb{I}) \sum_{i=1}^d Z_i, \quad (4)$$

where " $=_d$ " denotes equality in distribution. Here, for  $i = 1, 2, \dots, d$ ,

$$F_{Z_i}(z) = F_{X_i \mid \mathbf{X} \in \mathcal{U}}(z), \quad (5)$$

where  $\mathcal{U} = \mathbb{R}^d \setminus \mathcal{F}$ . From the properties (i), (ii) and (iii) it follows that the marginal distributions  $F_{Z_i}$  of  $Z_i$  are known, but the copula of  $(Z_1, Z_2, \dots, Z_d)$  remains unspecified. In the given context, we thus effectively aim at finding copulas that yield maximum and minimum value for quantiles of a mixture like in (4). In this regard, an explicit formula for the quantile of a mixture is useful, as it avoids that one has to resort to lengthy (nested) simulations; see Section 3 and the discussion that follows Proposition 2 in particular.

The explicit computation of the quantile of a mixture is presented in Section 2. Its application to model risk assessment is developed in Section 3.

## 2 Quantile of a mixture

We formulate the following result for the quantile of a mixture.

**Theorem 1** (Quantile of a mixture). *Consider a sum  $S = \mathbb{I}X + (1 - \mathbb{I})Y$ , where  $\mathbb{I}$  is a Bernoulli distributed random variable with parameter  $q$  and where the components  $X$  and  $Y$  are independent of  $\mathbb{I}$  and have cdf  $F_X$  and  $F_Y$ , respectively. Define  $\alpha_* \in [0, 1]$  by*

$$\alpha_* := \inf \left\{ \alpha \in (0, 1) \mid \exists \beta \in (0, 1) \left\{ \begin{array}{l} q\alpha + (1 - q)\beta = p \\ F_X^{-1}(\alpha) \geq F_Y^{-1}(\beta) \end{array} \right\} \right\}$$

and let  $\beta_* = \frac{p-q\alpha_*}{1-q} \in [0, 1]$ . Then, for  $p \in (0, 1)$ ,

$$s_p := F_S^{-1}(p) = \max \left\{ F_X^{-1}(\alpha_*), F_Y^{-1}(\beta_*) \right\}. \quad (6)$$

This maximum can be computed explicitly by distinguishing along the four following cases for  $F_X(\cdot)$  and for  $F_Y(\cdot)$ :

Case (1)  $F_X$  is continuous at  $s_p$  and for all  $z < s_p$ ,  $F_X(z) < F_X(s_p)$

Case (2)  $F_X$  is continuous at  $s_p$  and there exists  $z < s_p$  such that  $F_X(z) = F_X(s_p)$

Case (3)  $F_X$  is discontinuous at  $s_p$  and for all  $z < s_p$ ,  $F_X(z) < F_X(s_p^-)$

Case (4)  $F_X$  is discontinuous at  $s_p$  and there exists  $z < s_p$  such that  $F_X(z) = F_X(s_p^-)$

Case (a)  $F_Y$  is continuous at  $s_p$  and for all  $z < s_p$ ,  $F_Y(z) < F_Y(s_p)$

Case (b)  $F_Y$  is continuous at  $s_p$  and there exists  $z < s_p$  such that  $F_Y(z) = F_Y(s_p)$

Case (c)  $F_Y$  is discontinuous at  $s_p$  and for all  $z < s_p$ ,  $F_Y(z) < F_Y(s_p^-)$

Case (d)  $F_Y$  is discontinuous at  $s_p$  and there exists  $z < s_p$  such that  $F_Y(z) = F_Y(s_p^-)$

We have summarized the computations of  $s_p$  in Table 1 for the sixteen possible combinations.

**Table 1:** Summary of all cases for the quantiles of a mixture where  $s_p = F_S^{-1}(p)$ . In all cases,  $\alpha_*$  is defined by (7) and  $\beta_* = \frac{p-q\alpha_*}{1-q} \leq F_Y(s_p)$ ,  $\alpha_* = \frac{p-(1-q)\beta_*}{q} \geq F_X(s_p)$ .

	(a)	(b)	(c)	(d)
(1)	$\alpha_* = F_X(s_p)$ $\beta_* = F_Y(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$	$\alpha_* = F_X(s_p)$ $\beta_* = F_Y(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $> F_Y^{-1}(\beta_*)$	$\alpha_* = F_X(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$	$\alpha_* = F_X(s_p)$ if $F_S(s_p^-) < p$ , $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$ if $F_S(s_p^-) = p$ , $s_p = F_X^{-1}(\alpha_*)$ $> F_Y^{-1}(\beta_*)$
(2)	$\alpha_* = F_X(s_p)$ $\beta_* = F_Y(s_p)$ $s_p = F_Y^{-1}(\beta_*)$ $> F_X^{-1}(\alpha_*)$	Impossible	$\alpha_* = F_X(s_p)$ $s_p = F_Y^{-1}(\beta_*)$ $> F_X^{-1}(\alpha_*)$	$\alpha_* = F_X(s_p)$ $s_p = F_Y^{-1}(\beta_*)$ $> F_X^{-1}(\alpha_*)$
(3)	$\beta_* = F_Y(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$	$\beta_* = F_Y(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $> F_Y^{-1}(\beta_*)$	$s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$	if $F_S(s_p^-) < p$ , $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$ if $F_S(s_p^-) = p$ , $s_p = F_X^{-1}(\alpha_*)$ $> F_Y^{-1}(\beta_*)$
(4)	$\beta_* = F_Y(s_p)$ if $F_S(s_p^-) < p$ , $s_p = F_Y^{-1}(\beta_*)$ $= F_X^{-1}(\alpha_*)$ if $F_S(s_p^-) = p$ , $s_p = F_Y^{-1}(\beta_*)$ $> F_X^{-1}(\alpha_*)$	$\beta_* = F_Y(s_p)$ $s_p = F_X^{-1}(\alpha_*)$ $> F_Y^{-1}(\beta_*)$	if $F_S(s_p^-) < p$ , $s_p = F_X^{-1}(\alpha_*)$ $= F_Y^{-1}(\beta_*)$ if $F_S(s_p^-) = p$ , $s_p = F_Y^{-1}(\beta_*)$ $> F_X^{-1}(\alpha_*)$	Impossible

*Proof.* Since  $X$  and  $Y$  are independent of  $\mathbb{I}$  we find for the distribution of  $S = \mathbb{I}X + (1 - \mathbb{I})Y$ ,

$$F_S(x) = qF_X(x) + (1 - q)F_Y(x) \quad x \in \mathbb{R}.$$

Let  $p \in (0, 1)$  and denote  $F_S^{-1}(p)$  by  $s_p$ ,

$$s_p = \inf \{x \in \mathbb{R} \mid qF_X(x) + (1 - q)F_Y(x) \geq p\}.$$

In what follows, when considering  $\alpha, \beta \in (0, 1)$  we always assume that they satisfy  $q\alpha + (1 - q)\beta = p$ . Note that we define  $\alpha_*$  as

$$\alpha_* := \inf \left\{ \alpha \in (0, 1) \mid \exists \beta \in (0, 1) / q\alpha + (1 - q)\beta = p \text{ and } F_X^{-1}(\alpha) \geq F_Y^{-1}(\beta) \right\} \quad (7)$$

and  $\beta_* = \frac{p - q\alpha_*}{1 - q}$ . The proof consists in verifying that  $s_p$  can always be expressed as

$$s_p = \max \left\{ F_X^{-1}(\alpha_*), F_Y^{-1}(\beta_*) \right\}. \quad (8)$$

From Table 1, it is clear that (8) is proved. Let us now make the calculations case by case to prove Table 1.

Case 1:  $F_X$  is continuous at  $s_p$  and for all  $z < s_p$ ,  $F_X(z) < F_X(s_p)$

In this case we always have that  $s_p = F_X^{-1}(F_X(s_p))$ . Hence, we only need to show that  $\alpha_* = F_X(s_p)$  (i.e.  $\beta_* = \frac{p - qF_X(s_p)}{1 - q}$ ) and that  $s_p = F_X^{-1}(\alpha_*) \geq F_Y^{-1}(\beta_*)$  as in this case (8) will obviously hold.

Since  $F_S^{-1}(p) = s_p$  then  $F_S(s_p^-) = qF_X(s_p^-) + (1 - q)F_Y(s_p^-) \leq p \leq F_S(s_p) = qF_X(s_p) + (1 - q)F_Y(s_p)$ . Thus, by continuity of  $F_X$ ,  $qF_X(s_p) + (1 - q)F_Y(s_p^-) \leq p \leq qF_X(s_p) + (1 - q)F_Y(s_p)$ . Thus,

$$F_Y(s_p^-) \leq \frac{p - qF_X(s_p)}{1 - q} \leq F_Y(s_p) \quad (9)$$

(1a):  $F_Y$  is continuous at  $s_p$  and for all  $z < s_p$ ,  $F_Y(z) < F_Y(s_p)$ . Then,  $s_p = F_Y^{-1}(F_Y(s_p))$ . It is also clear that for  $\alpha < F_X(s_p)$  and thus  $\beta > F_Y(s_p)$ , one has that  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha_*$ , one has  $\alpha_* = F_X(s_p)$ ,  $\beta_* = F_Y(s_p)$  and  $s_p = F_X^{-1}(\alpha_*) = F_Y^{-1}(\beta_*)$ .

(1b):  $F_Y$  is continuous at  $s_p$  and there exists  $z < s_p$ ,  $F_Y(z) = F_Y(s_p)$  (thus,  $F_Y$  is constant on the interval  $(z, s_p)$ ). Then,  $F_Y^{-1}(F_Y(s_p)) < s_p = F_X^{-1}(F_X(s_p))$ . However, for  $\alpha < F_X(s_p)$  and thus  $\beta > F_Y(s_p)$ , one has that  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha_*$ ,  $\alpha_* = F_X(s_p)$ ,  $\beta_* = F_Y(s_p)$  and  $s_p = F_X^{-1}(\alpha_*) > F_Y^{-1}(\beta_*)$ . Thus,  $s_p = F_X^{-1}(\alpha_*) > F_Y^{-1}(\beta_*)$ .

(1c):  $F_Y$  has a discontinuity at  $s_p$  and for all  $z < s_p$ ,  $F_Y(z) < F_Y(s_p^-)$ . From (9), in this case,  $F_Y^{-1}\left(\frac{p - qF_X(s_p)}{1 - q}\right) = s_p$ . For  $\alpha < F_X(s_p)$  and thus  $\beta > \frac{p - qF_X(s_p)}{1 - q}$ ,  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha_*$ ,  $\alpha_* = F_X(s_p)$ ,  $\beta_* = \frac{p - qF_X(s_p)}{1 - q}$  and  $s_p = F_X^{-1}(\alpha_*) = F_Y^{-1}(\beta_*)$ .

(1d):  $F_Y$  has a discontinuity at  $s_p$  and there exists  $z < s_p$ ,  $F_Y(z) = F_Y(s_p^-)$  so that  $F_Y$  is constant on some interval  $(r, s_p)$  with  $r < s_p$ . From (9),

$$F_Y^{-1}\left(\frac{p - qF_X(s_p)}{1 - q}\right) \leq s_p.$$

If  $\frac{p - qF_X(s_p)}{1 - q} > F_Y(s_p^-)$  (or equivalently,  $F_S(s_p^-) < p$ ), then  $F_Y^{-1}\left(\frac{p - qF_X(s_p)}{1 - q}\right) = F_X^{-1}(F_X(s_p)) = s_p$ . Clearly, for  $\alpha < F_X(s_p)$  and thus  $\beta > \frac{p - qF_X(s_p)}{1 - q}$ , one has that  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha_*$ , one has  $\alpha_* = F_X(s_p)$ ,  $\beta_* = \frac{p - qF_X(s_p)}{1 - q}$  and  $s_p = F_X^{-1}(\alpha_*) = F_Y^{-1}(\beta_*)$ . If  $\frac{p - qF_X(s_p)}{1 - q} = F_Y(s_p^-)$  (or equivalently,  $F_S(s_p^-) = p$ ), then this implies that  $F_Y^{-1}\left(\frac{p - qF_X(s_p)}{1 - q}\right) < s_p$ . When  $\alpha < F_X(s_p)$  thus  $\beta > \frac{p - qF_X(s_p)}{1 - q}$  one has that  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta) \leq s_p$ . Hence, as per definition of  $\alpha_*$ , one has  $\alpha_* = F_X(s_p)$ ,  $\beta_* = \frac{p - qF_X(s_p)}{1 - q}$  and  $s_p = F_X^{-1}(\alpha_*) > F_Y^{-1}(\beta_*)$ .

Case 2:  $F_X$  is continuous at  $s_p$  and there is a  $z < s_p$ ,  $F_X(z) = F_X(s_p)$  ( $F_X(\cdot)$  is constant on  $(z, s_p)$ )

(2a): this case can be obtained from (1b) by changing the role of  $X$  and  $Y$ .

(2b):  $F_Y$  is continuous at  $s_p$  and there exists  $z < s_p$ ,  $F_Y(z) = F_Y(s_p)$ . Thus  $F_Y$  is constant on some interval  $(r, s_p)$  with  $r < s_p$ . Hence,  $F_S^{-1}(p) \leq \min(z, z) < s_p$  which contradicts the definition of  $s_p = F_S^{-1}(p)$ . The case (2b) is impossible.

(2c):  $F_Y$  is discontinuous at  $s_p$  and for all  $z < s_p$ ,  $F_Y(z) < F_Y(s_p^-)$ . From (9), in this case,  $F_Y^{-1}\left(\frac{p-qF_X(s_p)}{1-q}\right) = s_p > F_X^{-1}(F_X(s_p))$ . However, for all  $\alpha > F_X(s_p)$  and thus  $\beta < \frac{p-qF_X(s_p)}{1-q}$  it holds that  $F_X^{-1}(\alpha) > F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha^*$ ,  $\alpha^* = F_X(s_p)$ ,  $\beta^* = \frac{p-qF_X(s_p)}{1-q}$  and  $s_p = F_Y^{-1}(\beta^*) > F_X^{-1}(\alpha^*)$ .

(2d):  $F_Y$  is discontinuous at  $s_p$  and there exists  $z < s_p$ ,  $F_Y(z) = F_Y(s_p^-)$ . From (9),  $F_Y^{-1}\left(\frac{p-qF_X(s_p)}{1-q}\right) \leq s_p$ . If  $\frac{p-qF_X(s_p)}{1-q} > F_Y(s_p^-)$  (or equivalently,  $F_S(s_p^-) < p$ ), then  $F_Y^{-1}\left(\frac{p-qF_X(s_p)}{1-q}\right) = s_p > F_X^{-1}(F_X(s_p))$ . For  $\alpha > F_X(s_p)$  and thus  $\beta < \frac{p-qF_X(s_p)}{1-q}$  one has that  $F_X^{-1}(\alpha) > F_Y^{-1}(\beta)$ . Hence, as per definition of  $\alpha^*$ ,  $\alpha^* = F_X(s_p)$ ,  $\beta^* = \frac{p-qF_X(s_p)}{1-q}$  and  $F_X^{-1}(\alpha^*) < F_Y^{-1}(\beta^*) = s_p$ . The case that  $\frac{p-qF_X(s_p)}{1-q} = F_Y(s_p^-)$  is excluded as it implies that  $F_S^{-1}(p) < s_p$  should hold (similar to the case (2b)) which is a contradiction with the definition of  $s_p$ .

Case 3:  $F_X$  has a discontinuity at  $s_p$  and for all  $z < s_p$ ,  $F_X(z) < F_X(s_p^-)$

In this case,  $s_p = F_X^{-1}(F_X(s_p))$ . This situation is merely identical to previous cases.

(3a): it is the same as (1c) by changing the role of  $X$  and  $Y$ .

(3b): it is the same as (2d) by changing the role of  $X$  and  $Y$ .

(3c): Observe that  $F_X^{-1}(\alpha) = s_p$  for all  $F_X(s_p^-) \leq \alpha \leq F_X(s_p)$  and also that  $F_Y^{-1}(\beta) = s_p$  for all  $F_Y(s_p^-) \leq \beta \leq F_Y(s_p)$ .

We also know that  $F_S(s_p^-) \leq p \leq F_S(s_p)$  hence there exists  $F_X(s_p^-) \leq \alpha_1 \leq F_X(s_p)$  and  $F_Y(s_p^-) \leq \beta_1 \leq F_Y(s_p)$  so that  $q\alpha_1 + (1-q)\beta_1 = p$  and  $F_X^{-1}(\alpha_1) = F_Y^{-1}(\beta_1) = s_p$ . Therefore,  $F_X^{-1}(\alpha^*) = F_Y^{-1}(\beta^*) = s_p$ .

(3d): Observe that  $F_X^{-1}(\alpha) = s_p$  for all  $F_X(s_p^-) \leq \alpha \leq F_X(s_p)$  and also that  $F_Y^{-1}(\beta) = s_p$  for all  $F_Y(s_p^-) < \beta \leq F_Y(s_p)$ .

We also know that  $F_S(s_p^-) \leq p \leq F_S(s_p)$  and there are two possibilities:

In the case when  $F_S(s_p^-) < p$ , then there exists  $\alpha_1 \in (F_X(s_p^-), F_X(s_p))$  and  $\beta_1 \in (F_Y(s_p^-), F_Y(s_p))$  so that  $q\alpha_1 + (1-q)\beta_1 = p$  and  $F_X^{-1}(\alpha_1) = F_Y^{-1}(\beta_1) = s_p$ . Therefore,  $F_X^{-1}(\alpha^*) = F_Y^{-1}(\beta^*) = s_p$ .

In the case when  $F_S(s_p^-) = p$ , then  $qF_X(s_p^-) + (1-q)F_Y(s_p^-) = p$  and one has that  $F_X^{-1}(F_X(s_p^-)) > F_Y^{-1}(F_Y(s_p^-))$ , while for  $\alpha < F_X(s_p^-)$  and  $\beta > F_Y(s_p^-)$  one has that  $F_X^{-1}(\alpha) < F_Y^{-1}(\beta)$ . Hence,  $\alpha^* = F_X(s_p^-)$ ,  $F_Y(s_p^-) = \beta^*$  and  $s_p = F_X^{-1}(\alpha^*) > F_Y^{-1}(\beta^*)$ .

Case 4:  $F_X$  has a discontinuity at  $s_p$  and there exists  $z < s_p$ ,  $F_X(z) = F_X(s_p^-)$

By changing the role of  $X$  and  $Y$  we have that the case (4a) corresponds to (1d), the case (4b) corresponds to (2d) and the case (4c) corresponds to (3d). Finally the case of (4d) is treated as follows. In the case (4d), both  $F_X$  and  $F_Y$  are discontinuous at  $s_p$ , and there exists  $z_1$  and  $z_2$  such that  $F_X(z_1) = F_X(s_p)$  and  $F_Y(z_2) = F_Y(s_p)$  so that  $F_X$  is constant on  $(z_1, s_p)$  and  $F_Y$  is constant on  $(z_2, s_p)$ . Then  $F_S^{-1}(p) \leq \min(z_1, z_2) < s_p$  which contradicts the definition of  $s_p = F_S^{-1}(p)$ . This case is thus impossible.  $\square$

It is clear that in many cases  $F_X^{-1}(\alpha^*) = F_Y^{-1}(\beta^*)$ . For example, by inspection of Table 1 we find it is sufficient for  $F_X$  and  $F_Y$  to be strictly increasing with unbounded support (Case (1a)).

### 3 Application: Bounds on Quantiles of Portfolios

Let  $\mathbf{X} := (X_1, X_2, \dots, X_d)$  be some random vector of interest having finite mean and defined on an atomless probability space. In what follows we interpret  $\mathbf{X}$  as a portfolio of risks that a financial institution is exposed to. Its distribution is not fully known but complies with properties (i), (ii) and (iii) for a given  $\mathcal{F} \subset \mathbb{R}$ . Hence,

$S := \sum_{i=1}^d X_i$  can be represented as a mixture of the type (4) and its risk assessment is intimately connected with the analysis of extreme dependence among  $Z_i$  in (4).

A special role in this analysis is played by the comonotonic dependence, i.e., when all  $Z_i$  are increasing in each other. For this particular dependence we denote  $Z_i$  by  $Z_i^c$ . Formally, we write

$$Z_i^c = F_{Z_i}^{-1}(U), \quad i = 1, 2, \dots, d, \quad (10)$$

for some uniformly distributed random variable  $U$  that we take independent of  $\mathbb{I}$ .

To assess the risk of  $S$ , it is standard to compute  $F_S^{-1}(p)$  for some  $0 < p < 1$  that is typically close to 1 (e.g.,  $p = 0.995$  as in Solvency III and Basel II regulation). In this context, a quantile is typically called a VaR. Precisely, we denote, by  $\text{VaR}_p(S)$  the VaR of  $S$  at level  $p$ ,

$$\text{VaR}_p(S) = F_S^{-1}(p).$$

In the further analysis of VaR bounds on  $S$ , two other measures of risk are useful.  $\text{TVaR}_p(S)$  denotes the Tail Value-at-Risk (TVaR) at level  $p$ , i.e.,

$$\text{TVaR}_p(S) = \frac{1}{1-p} \int_p^1 \text{VaR}_u(S) du, \quad p \in (0, 1).$$

Observe that  $p \rightarrow \text{TVaR}_p$  is continuous. We define  $\text{TVaR}_1(S) = \lim_{p \nearrow 1} \text{TVaR}_p(S)$ .  $\text{TVaR}_p$  is a weighted average of all upper VaRs from probability level  $p$  onwards. Similarly, we can define the left Tail Value-at-Risk (LTVaR) at level  $p$  as the average of the VaRs below  $p$ , i.e.  $\text{LTVaR}_p(S) = \frac{1}{p} \int_0^p \text{VaR}_u(S) du$  and  $\text{LTVaR}_0(S) = \text{LTVaR}_{p \searrow 0}(S)$ . In what follows the best-possible lower bound for the VaR of the aggregate risk  $S$  is denoted by  $\varrho_{\mathcal{F}}^-$  and the upper bound is denoted by  $\varrho_{\mathcal{F}}^+$ , where  $\mathcal{F} \subset \mathbb{R}$  is the subset on which the joint distribution of  $\mathbf{X}$  is known. In general, it is difficult to obtain explicit expressions for  $\varrho_{\mathcal{F}}^-$  and  $\varrho_{\mathcal{F}}^+$ .

In their Proposition 4.1, [4] provide the following VaR bounds on  $\sum_{i=1}^d X_i$  (and thus bounds on  $\varrho_{\mathcal{F}}^-$  and  $\varrho_{\mathcal{F}}^+$ ):

**Proposition 2** (VaR Bounds for  $\sum_{i=1}^d X_i$ ). *Let  $\mathbf{X}$  be a random vector that satisfies properties (i), (ii) and (iii), and let  $\mathbb{I}$ ,  $(Z_1^c, Z_2^c, \dots, Z_d^c)$  and  $U$  be defined as in (3) and (10). Define the variables  $L_i$  and  $H_i$  as*

$$L_i = \text{LTVaR}_U(Z_i^c) \text{ and } H_i = \text{TVaR}_U(Z_i^c).$$

Furthermore, define

$$m_p := \text{VaR}_p \left( \mathbb{I} \sum_{i=1}^d X_i + (1 - \mathbb{I}) \sum_{i=1}^d L_i \right), \quad M_p := \text{VaR}_p \left( \mathbb{I} \sum_{i=1}^d X_i + (1 - \mathbb{I}) \sum_{i=1}^d H_i \right).$$

The best-possible bounds  $\varrho_{\mathcal{F}}^-$  and  $\varrho_{\mathcal{F}}^+$  for the VaR of the aggregate risk satisfy

$$\varrho_{\mathcal{F}}^- \geq m_p \text{ and } \varrho_{\mathcal{F}}^+ \leq M_p. \quad (11)$$

At first, the role of the variables  $H_i$  and  $L_i$  may seem odd. However, note that the variables  $Z_i$  that appear in the general mixture (4) can also be expressed as  $Z_i = \text{VaR}_{U_i}(Z_i)$  for some uniformly distributed random variable  $U_i$ . Clearly, the VaR of  $\sum_{i=1}^d \text{VaR}_{U_i}(Z_i)$  is bounded by its TVaR. Furthermore, TVaR is maximized in the case of a comonotonic dependence and VaR and TVaR are additive, we thus obtain that the VaR of  $\sum_{i=1}^d \text{VaR}_{U_i}(Z_i)$  is bounded by the VaR of the comonotonic sum  $\sum_{i=1}^d H_i$ . When there is full uncertainty, i.e., when  $\mathcal{U} = \mathbb{R}^d$ , then  $\mathbb{I} = 0$ , and we recover the VaR bounds of the portfolio, as provided in Theorem 2.1 of the [2].

In general the bounds  $m_p$  and  $M_p$  are not known in analytic form and their numerical evaluation is not straightforward to do. Specifically, while it is easy to simulate possible realizations for  $(Z_1^c, Z_2^c, \dots, Z_d^c)$  the realizations for  $(L_1, L_2, \dots, L_d)$  and  $(H_1, H_2, \dots, H_d)$  do not follow immediately, which leads to nested simulations when computing  $m_p$  and  $M_p$ . Indeed, the simulation of a single realization for  $L_i$  (or for  $H_i$ ) requires,

for each simulated value  $u$  of the uniformly distributed variable  $U$ , a large number of draws from the variable  $Z_i$  in order to estimate LTVaR and TVaR at the level  $U = u$ . In this respect, the formula for the quantile of a mixture is convenient, as it allows to develop an alternative formulation of the VaR bounds. This alternative formulation makes use of an auxiliary variable  $T$ ,

$$T := F_{\sum_i X_i | (X_1, X_2, \dots, X_d) \in \mathcal{F}}^{-1}(U) \quad (12)$$

for the same uniform random variable  $U$  used in the definition of  $Z_i^c$  in (10). Hence,  $T$  is a random variable independent of  $\mathbb{I}$  with distribution  $F_{\sum_i X_i | (X_1, X_2, \dots, X_d) \in \mathcal{F}}(x)$ . We formulate the following proposition.

**Proposition 3** (Alternative formulation of the VaR Bounds). *Let  $(X_1, X_2, \dots, X_d)$  be a random vector that satisfies properties (i), (ii) and (iii), and let  $\mathbb{I}$ ,  $(Z_1^c, Z_2^c, \dots, Z_d^c)$  and  $T$  be defined as in (3), (10) and (12). Recall that  $p_{\mathcal{F}} = P(\mathbb{I} = 1)$ . Define*

$$\alpha_* := \inf \left\{ \alpha \in (\alpha_1, \alpha_2) \mid \text{VaR}_{\alpha}(T) \geq \text{TVaR}_{\frac{p-p_{\mathcal{F}}\alpha}{1-p_{\mathcal{F}}}} \left( \sum_{i=1}^d Z_i^c \right) \right\},$$

where  $\alpha_1 = \max \left\{ 0, \frac{p+p_{\mathcal{F}}-1}{p_{\mathcal{F}}} \right\}$  and  $\alpha_2 = \min \left\{ 1, \frac{p}{p_{\mathcal{F}}} \right\}$ . Then, for  $p \in (0, 1)$ ,

$$M_p = \max \left\{ \text{VaR}_{\alpha_*}(T), \text{TVaR}_{\beta_*} \left( \sum_{i=1}^d Z_i^c \right) \right\}, \quad (13)$$

where  $\beta_* = \frac{p-p_{\mathcal{F}}\alpha_*}{1-p_{\mathcal{F}}}$ . Specifically,

$$M_p = \begin{cases} \text{TVaR}_{\frac{p-p_{\mathcal{F}}\alpha_*}{1-p_{\mathcal{F}}}} \left( \sum_{i=1}^d Z_i^c \right) & \text{if } \frac{p+p_{\mathcal{F}}-1}{p_{\mathcal{F}}} < \alpha_* < \frac{p}{p_{\mathcal{F}}} \\ \text{VaR}_{\alpha_*}(T) & \text{if } \alpha_* = \frac{p}{p_{\mathcal{F}}} \\ \max \left\{ \text{VaR}_{\alpha_*}(T), \text{TVaR}_{\frac{p-p_{\mathcal{F}}\alpha_*}{1-p_{\mathcal{F}}}} \left( \sum_{i=1}^d Z_i^c \right) \right\} & \text{if } \alpha_* = \frac{p+p_{\mathcal{F}}-1}{p_{\mathcal{F}}}. \end{cases} \quad (14)$$

The expression for the lower bound  $m_p$  is obtained by replacing, in the above statements, “TVaR” with “LTVaR”.

*Proof.* The proof follows as a direct application of Theorem 1. Consider  $X = T$  with distribution  $F_X$ , and  $Y = \sum_{i=1}^d \text{TVaR}_U(Z_i) (= \sum_{i=1}^d \text{TVaR}_U(Z_i^c))$  with distribution  $F_Y$ . From Theorem 1,

$$M_p = \max \{ \text{VaR}_{\alpha_*}(T), \text{TVaR}_{\beta_*}(Y) \}. \quad (15)$$

It is clear that the cdf  $F_Y$  of  $Y$  is continuous and strictly increasing on its support. First, let  $0 < F_Y(M_p) < 1$ . By inspection of the table displayed in Theorem 1, we are in the situation of the cases (1a), (2a), (3a) and (4a). We observe that  $\beta_* = F_Y(M_p)$  and

$$M_p = \text{VaR}_{\beta_*}(Y) = \text{VaR}_{\beta_*} \left( \text{TVaR}_U \left( \sum_{i=1}^d Z_i^c \right) \right) = \text{TVaR}_{\beta_*} \left( \sum_{i=1}^d Z_i^c \right).$$

Thus, for all  $0 < \beta_* < 1$ , or, equivalently,  $\frac{p+p_{\mathcal{F}}-1}{p_{\mathcal{F}}} < \alpha_* < \frac{p}{p_{\mathcal{F}}}$ , it holds that  $M_p = \text{TVaR}_{\beta_*}(\sum_i Z_i^c)$ . Second, when  $F_Y(M_p) = 0$ , we are always in the cases (1b), (3b) and (4b) so that  $M_p = \text{VaR}_{\alpha_*}(T)$  with  $\beta_* = 0$  and  $\alpha_* = \frac{p}{p_{\mathcal{F}}}$ . (this is also clear from the fact that in this case  $\text{VaR}_0(Y) = -\infty$ ). When  $F_Y(M_p) = 1$ , we are either in the cases (1b), (3b) and (4b) or in the cases (1a), (2a), (3a) and (4a). In the first situation, it follows from inspection of the table again that  $M_p = \text{VaR}_{\alpha_*}(T)$  with  $\alpha_* = \frac{p+p_{\mathcal{F}}-1}{p_{\mathcal{F}}}$  and note that  $\text{VaR}_{\alpha_*}(T) > \text{VaR}_1(Y) (= \text{TVaR}_1(\sum_i Z_i^c))$ . In the second situation,  $\beta_* = 1$  and  $M_p = \text{TVaR}_1(\sum_i Z_i^c) \geq \text{VaR}_{\alpha_*}(T)$ . The proof of the expression for  $m_p$  follows by applying Theorem 1 again, where we now take  $Y$  as  $Y = \sum_{i=1}^d \text{LTVaR}_U(Z_i^c)$ .  $\square$

Proposition 3 may look more complicated than Proposition 2; however, it is now easier to use Monte Carlo simulations for estimating VaR bounds because nested simulations can be avoided.



**Remark 4** (Best-possible bounds). In finite dimensions, the inequalities in (11) are typically strict so that the bounds  $m_p$  and  $M_p$  are not best-possible in general. Indeed, it is not straightforward to find a vector  $(Z_1, Z_2, \dots, Z_d)$  with given marginal distributions as in (5) such that for a given  $0 < p < 1$ ,  $\mathbb{I} \sum_{i=1}^d X_i + (1 - \mathbb{I}) \sum_{i=1}^d Z_i$  and  $\mathbb{I} \sum_{i=1}^d X_i + (1 - \mathbb{I}) \sum_{i=1}^d H_i$  have the same  $p$ -quantile. This situation would be obtained when the vector  $(Z_1, Z_2, \dots, Z_d)$  is such that its sum  $\sum_{i=1}^d Z_i$  has a flat quantile function on the appropriate interval  $(\beta_*, 1)$ . The literature refers to this situation as “joint mixability” (for a homogeneous portfolio this concept is known as “complete mixability”), a concept that can essentially be traced back to a paper of [8] and has been extensively studied in a series of papers including [15], and [13, 14]. In these papers it is shown, among other results, that in several theoretical cases of interest one can construct a dependence among the risks that lead to mixability.

**Remark 5** (High dimensions). A large class of distributions exhibits asymptotic mixability implying that in high-dimensional problems the bounds  $m_p$  and  $M_p$  that are stated in Proposition 2 and Proposition 3 are expected to be approximately best-possible; see e.g., [14] and [12].

**Remark 6** (Rearrangement Algorithm). [4] show that the Rearrangement Algorithm (RA) of [6] can be conditionally applied to obtain approximations of the best-possible VaR bounds. There is numerical evidence that these approximations typically yield results that do not differ a lot from the bounds we investigate here; see [2] and [3] for illustrations.

The following example illustrates Proposition 3 with a multivariate Student’s  $t$  distribution as a benchmark model.

### 3.1 Example (Multivariate Student’s $t$ distribution)

We consider a random vector  $\mathbf{X}$  with standard Student’s  $t$  distributed marginals that follows a multivariate standard Student’s  $t$  distribution on a trusted area  $\mathcal{F}$ ; see also [4]. Specifically, the density of  $\mathbf{X}$  on  $\mathcal{F}$  is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((d+v)/2)}{(v\pi)^{d/2} \Gamma(v/2) \sqrt{|\mathbf{R}|}} \left(1 + \frac{\mathbf{x}^T \mathbf{R}^{-1} \mathbf{x}}{v}\right)^{-(d+v)/2}.$$

Here,  $v$  is the number of degrees of freedom and  $|\mathbf{R}|$  is the determinant of the correlation matrix  $\mathbf{R}$  satisfying  $\mathbf{R}_{i,j} = \rho$  ( $-1/(d-1) < \rho < 1$ ) for all pairs  $(X_i, X_j)$  with  $i \neq j$  (homogeneous portfolio); see also [9]. When  $v > 2$ , the covariance matrix  $\Sigma$  exists and is given by  $\Sigma = \frac{v}{v-2} \mathbf{R}$ . As for the subset  $\mathcal{F}$  we take the ellipsoid

$$\mathcal{F} = \mathcal{C}_{p_{\mathcal{F}}, \rho, v, d} := \left\{ \mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \Sigma^{-1} \mathbf{x}^t \leq c(p_{\mathcal{F}}) \right\}, \quad (16)$$

where  $c(p_{\mathcal{F}})$  is the appropriate cutoff value<sup>3</sup> corresponding to  $P(\mathbf{X} \in \mathcal{F}) = p_{\mathcal{F}}$ . We further consider a portfolio of  $d = 20$  risks and consider a multivariate Student’s  $t$  distribution with  $v = 10$  degrees of freedom. The VaR bounds reported in Table 2 were obtained within a few minutes, using 3,000,000 Monte Carlo simulations.<sup>4</sup>

We make the following observations. First, model risk is clearly present even when the dependence is “mostly” known (i.e.,  $p_{\mathcal{F}}$  is large). Furthermore, the precise degree of model error depends highly on the level of the probability  $p$  that is used to assess  $\text{VaR}_p$ . Let us consider the benchmark model with  $\rho = 0$  (the risks are uncorrelated and standard Student’s  $t$  distributed) and  $p_{\mathcal{F}} = 1$  (no uncertainty). We find that  $\text{VaR}_{95\%} \left( \sum_{i=1}^{20} X_i \right) = 8.1$  and, similarly,  $\text{VaR}_{99.5\%} \left( \sum_{i=1}^{20} X_i \right) = 14.2$ ,  $\text{VaR}_{99.95\%} \left( \sum_{i=1}^{20} X_i \right) = 20.7$ . However, if  $p_{\mathcal{F}} = 98\%$ , then  $p_{\mathcal{U}} = 2\%$ , and the benchmark model might overestimate the 95%–VaR by  $(8.1-7.9)/7.9=2.5\%$

<sup>3</sup> To determine  $c(p_{\mathcal{F}})$ , one can use the fact that the scaled squared Mahalanobis distance  $\frac{1}{d} \mathbf{X} \mathbf{R}^{-1} \mathbf{X}^t$  follows a  $F$ -distribution with parameters  $d$  and  $v$  (i.e.,  $\frac{1}{d} \mathbf{X} \mathbf{R}^{-1} \mathbf{X}^t \sim F(d, v)$ ).

<sup>4</sup> When no information on the dependence is available ( $p_{\mathcal{F}} = 0\%$ ), the upper and lower bounds stated in Proposition 2 reduce to  $A = \sum_{i=1}^d \text{TVaR}_p(X_i)$  and  $B = \sum_{i=1}^d \text{LTVaR}_p(X_i)$  (see [2]) and can be computed exactly ([10]).



**Table 2:**  $\text{VaR}_{95\%}$ ,  $\text{VaR}_{99.5\%}$  and  $\text{VaR}_{99.95\%}$  of  $\sum_{i=1}^{20} X_i$  are reported in the absence of uncertainty (i.e.,  $\mathcal{U} = \emptyset$ ). VaR bounds are then are reported as pairs  $(m_p, M_p)$  for various probability levels  $p_{\mathcal{F}}$  and correlation coefficients  $\rho$ . All digits reported in the table are significant in that the standard deviation of each number estimated in the table is smaller than 0.01.

$\mathcal{F} = \mathcal{C}_{p_{\mathcal{F}}, \rho, v, d}$		$\mathcal{U} = \emptyset$				$\mathcal{U} = \mathbb{R}^d$
		$p_{\mathcal{F}} = 100\%$	$p_{\mathcal{F}} = 98\%$	$p_{\mathcal{F}} = 80\%$	$p_{\mathcal{F}} = 20\%$	$p_{\mathcal{F}} = 0\%$
$p=95\%$	$\rho = 0$	8.1	(7.9, 9.0)	(6.6, 40.3)	(2.2, 48.1)	(-2.5, 48.2)
	$\rho = 0.1$	13.8	(13.4, 15.1)	(11.3, 40.4)	(3.6, 48.1)	(-2.5, 48.2)
	$\rho = 0.5$	26.3	(25.4, 27.8)	(21.4, 40.8)	(7.0, 48.0)	(-2.6, 48.2)
$p=99.5\%$	$\rho = 0$	14.2	(13.4, 56.6)	(11.0, 75.2)	(6.2, 75.7)	(-0.4, 75.7)
	$\rho = 0.1$	24.2	(22.8, 56.5)	(18.7, 75.1)	(10.5, 75.7)	(-0.4, 75.7)
	$\rho = 0.5$	45.9	(43.4, 58.5)	(35.7, 75.0)	(19.9, 75.5)	(-0.4, 75.7)
$p=99.95\%$	$\rho = 0$	20.7	(18.2, 103.3)	(14.2, 106)	(8.6, 106)	(-0.1, 106)
	$\rho = 0.1$	34.8	(30.7, 102.4)	(24.2, 106)	(14.5, 106)	(-0.1, 106)
	$\rho = 0.5$	66.1	(58.4, 103.0)	(46.2, 106)	(27.7, 106)	(-0.1, 106)

or underestimate it by  $(9-8.1)/9 = 10\%$ . However, when using the 99.5%–VaR, the degree of underestimation may rise to  $(56.6-14.2)/56.6 = 75\%$ , whereas the degree of overestimation is equal only to  $(14.2-13.4)/13.4 = 6.0\%$ . Hence, the risk of underestimation is sharply increasing in the probability level that is used to assess VaR. Finally, note that when very high probability levels are used in VaR calculations ( $p = 99.95\%$ ; see the last three rows in Table 2), the constrained upper bounds are very close to the unconstrained upper bound, even when there is almost no uncertainty on the dependence ( $p_{\mathcal{F}} = 98\%$ ). The bounds computed by [6] are thus nearly the best possible bounds, even though it seems that the multivariate model is known at a very high confidence level as  $\mathcal{F}$  nearly contains all  $\mathbb{R}^d$ . This implies that any effort to fit a multivariate model accurately will not reduce the model risk on the assessment of Value-at-Risk at very high confidence levels.

## 4 Final remarks

In this paper, we provide an explicit expression for the quantile of a mixture of *two* random variables and provide an application to finding VaR bounds of risky portfolios when only partial dependence information is available. We leave it to future research to extend the results to the general  $n$ -dimensional case.

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