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Cost-efficiency in multivariate Lévy models

Abstract: In this paper we determine lowest cost strategies for given payoff distributions called cost-efficient strategies in multivariate exponential Lévy models where the pricing is based on the multivariate Esscher martingale measure. This multivariate framework allows to deal with dependent price processes as arising in typical applications. Dependence of the components of the Lévy Process implies an influence even on the pricing of efficient versions of univariate payoffs. We state various relevant existence and uniqueness results for the Esscher parameter and determine cost efficient strategies in particular in the case of price processes driven by multivariate *NIG*- and *VG*-processes. From a monotonicity characterization of efficient payoffs we obtain that basket options are generally inefficient in Lévy markets when pricing is based on the Esscher measure. We determine efficient versions of the basket options in real market data and show that the proposed cost efficient strategies are also feasible from a numerical viewpoint. As a result we find that a considerable efficiency loss may arise when using the inefficient payoffs.

Keywords: cost-efficient strategies, multivariate Lévy models, multivariate Esscher transform, basket option

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1 Introduction

In this paper we study optimal investment decisions in incomplete markets where the prices of the risky assets are driven by multivariate Lévy processes. Apart from the pricing and hedging of options on a single asset, practically all financial applications require a multivariate model with dependence between the assets. The knowledge of the corresponding univariate marginals is not sufficient since it provides no information on the dependence structure which considerably influences the risks and returns of the value of the option. Thus, multidimensional models are capable to describe the actual financial states in a more appropriate and accurate manner. Moreover, an abundance of payoff function types such as the *Basket* option, *Worst-off* call, *Worst-off* put and their *Best-off* counterparts and many more can be treated with multivariate pricing models.

The concept of cost-efficient strategies has been introduced in [10, 11] and has been extended in a series of papers in Jouini and Kallal [22], Föllmer and Schied [15] in [32, 33], in [3, 4], as well as in Burgert and Rüschendorf [7] and others in a fairly general setting. The aim of the method of cost efficiency is to construct to a specified payoff distribution G a payoff $\underline{X_T}$ with payoff distribution G (w.r.t. the underlying probability measure P) which minimizes the price w.r.t. the pricing measure Q used in the market. G could be the distribution of a given option X_T. This approach thus improves concerning cost a given payoff or determines to a specified payoff distribution G a cheapest (cost efficient) payoff having this payoff distribution. [28] contains a discussion of various methods to specify payoff distributions in applications.

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The explicit form of cost-efficient strategies has been determined in the above mentioned papers mainly in the context of the Samuelson model. A detailed study of this concept for univariate exponential Lévy models was given in Hammerstein et al. [20] in the case that pricing is based on the Esscher martingale measure. In this paper also the potential gain and the hedging behaviour of cost efficient claims is investigated. As a result it turns out, that the cost-efficient payoff may lead to considerably reduced cost and compares also favourably concerning hedging behaviour as checked for real market data.

In typical cases cost efficient payoffs generate the payoff distribution by following the trend in the market. In particular they neglect possible hedging goals of investors but only aim to optimize the cost in order to reach a distributional goal of the investment. They are thus tools for law invariant investors but don't satisfy protection or securization purposes. In recent papers in [4, 5] and in [30] the method of cost efficiency has been extended to include state dependent constraints and thus to specify in which states income is requested.

The frame of the method of cost efficiency is the following. In a market model $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ with finite time horizon $[0, T]$ let $S = (S_t)_{0\le t\le T} \in \mathbb{R}^d$ be a market model for d stocks and $(Z_t)_{0\le t\le T}$ a pricing density for S rendering the discounted process $(e^{-rt}S_tZ_t)_{0\le t\le T}$ a P-martingale. The cost of a strategy with terminal payoff X_T then is given by

$$c(X_T) = E[e^{-rT}Z_TX_T]. \tag{1.1}$$

A basic and debatable assumption of the approach of cost efficient strategies is that the market participants agree on one and the same pricing measure Q. In an incomplete market this problem is not avoidable. Any no-arbitrage price corresponds to a chosen market measure or equivalently to a specific utility principle. Also the super hedging price, the empirical and risk minimizing pricing measures follows this principle and base their pricing on a worst case martingale measure, on 'minimal' martingale measures minimizing some hedging or risk functional. The assumption of a pricing measure Q allows as consequence to construct to any given payoff distribution G a cheapest (cost-efficient) payoff having this payoff distribution.

For a given payoff distribution G a strategy with terminal payoff $\underline{X}_T$ distributed with G (i.e. $\underline{X}_T \sim G$) is called *cost-efficient* if it minimizes the cost i.e.

$$c(\underline{X}_T) = \min_{X_T\sim G} c(X_T). \tag{1.2}$$

The strategy with payoff $\overline{X}_T \sim G$ is called *most-expensive* if

$$c(\overline{X}_T) = \max_{X_T\sim G} c(X_T). \tag{1.3}$$

The difference of the costs $\ell(X_T) = c(X_T) - c(\underline{X}_T)$ is called the *efficiency loss* of X_T.

The following result characterizes cost-efficient strategies in the general context described above (see e.g. [3, 4]).

Theorem 1.1. *Suppose that the state-price density Z_T has a continuous distribution function F_{Z_T}. Then $\underline{X}_T = G^{-1}(1 - F_{Z_T}(Z_T))$ is the cost-efficient strategy and $\overline{X}_T = G^{-1}(F_{Z_T}(Z_T))$ is the most-expensive way to achieve a payoff with given distribution function G. Moreover, for any payoff $X_T \sim G$, the lower and upper cost bounds are given by*

$$c(X_T) \ge E[e^{-rT}Z_T\underline{X}_T] = e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)G^{-1}(1-y)dy, \tag{1.4}$$

$$c(X_T) \le E[e^{-rT}Z_T\overline{X}_T] = e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)G^{-1}(y)dy, . \tag{1.5}$$

Furthermore, one obtains as consequence that a random payoff $X_T \sim G$ is cost-efficient if and only if X_T and Z_T are countermonotonic while $X_T \sim G$ is most-expensive if and only if X_T and Z_T are comonotonic. In state price models where $Z_T = h(S_T)$ (like in exponential Lévy models) path-dependent payoffs are not cost-efficient and can be improved by cost-efficient payoffs which are path-independent i.e. are of the form $X_T = g(S_T)$.

In this paper we apply the concept of cost-efficiency in the case of market models driven by multivariate Lévy processes in the case that pricing is based on the Esscher martingale measure. The Esscher transform has been introduced and motivated for contingent claim pricing in mathematical finance for Lévy processes in [16], [25], [12], [8], and [21] and has been extended to semimartingales and multivariate Lévy processes in [23], [13] and in [31]. The Esscher pricing principle thus is a well established pricing principle justified by a corresponding utility principle and by some inherent simplifications it leads to. We show in our paper that the determination of cost efficient strategies is doable in some standard classes of multivariate Lévy models under pricing by the Esscher pricing measure. We introduce in Section 2 the multivariate Esscher transform and describe some of its basic and delicate properties on the existence and uniqueness of the risk-neutral Esscher measure. In Section 3 we specify the construction of cost-efficient claims in Theorem 1.1 to the multivariate Lévy case. We find that generally basket options are inefficient. In Section 4 we introduce some multivariate normal mean variance mixture models in particular the *NIG* and the *VG* model and use them for modelling bivariate log-returns. We estimate the Lévy parameters from daily log-returns of German stock data and compute the Esscher parameters. As application in Section 5 we calculate to a given basket option the cost-efficient option and determine the efficiency loss for the real data sets as discussed above.

2 The Esscher transform and risk neutral Esscher measure

The notion of Esscher transformation as a change of measure was introduced by Gerber and Shiu [16] although the concept of Esscher transformation for Lévy processes had been used in finance before on a mathematically profound basis (see e.g. Madan and Milne [25]). Since then it became an established tool in financial and actuarial science. The Esscher measure provides the advantage that any Lévy process under the physical measure stays a Lévy process under the Esscher measure.

For $t \geq 0$ and $d \in \mathbb{N}$, let $S_t^{(i)} = S_0^{(i)} e^{L_t^{(i)}}$, $1 \leq i \leq d$ denote the price of the i-th risky asset and assume that $S_0^{(i)}$ is $\mathcal{F}_0$-measurable. Let $L^{(i)} := (L_t^{(i)})_{t\geq 0}$ and assume that $L := (L^{(1)}, \dots, L^{(d)})$ is a Lévy process with respect to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. Both $S_0^{(i)}$ and $L_t^{(i)}$ are real-valued. Recall that we consider strategies $(Y_t)_{0\leq t\leq T}$ on a finite trading period $[0, T]$. Then, apart from the cases where $L = (L_t)_{0\leq t\leq T}$ either is a Brownian motion or a Poisson process, such a Lévy market setting is incomplete. This means that the set of possible risk-neutral martingale measures is not a singleton, but typically has uncountably many elements. We therefore assume that the financial market is incomplete, but free of arbitrage, perfectly liquid and frictionless. To introduce the Esscher martingale measure, we need several properties of the moment generating function of *random vectors*.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be random vector with values in $\mathbb{R}^d$, $d \in \mathbb{N}$. Denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^d$. The moment generating function of X is given by

$$M_X(u) := E[e^{\langle u,X\rangle}], \quad u \in \mathbb{R}^d. \tag{2.1}$$

For $u_1, u_2 \in \mathbb{R}^d$ and any $\alpha \in (0, 1)$ holds using Hölder's unequality

$$\begin{aligned} M_X(\alpha u_1 + (1-\alpha)u_2) &= E[e^{\langle \alpha u_1 + (1-\alpha)u_2, X\rangle}] \\ &= E[e^{\langle \alpha u_1, X\rangle} \cdot e^{\langle (1-\alpha)u_2, X\rangle}] \\ &\leq (E[e^{\langle u_1, X\rangle}])^{\alpha} \cdot (E[e^{\langle u_2, X\rangle}])^{(1-\alpha)} \\ &= M_X(u_1)^{\alpha} \cdot M_X(u_2)^{(1-\alpha)} . \end{aligned}$$

Thus, $\log(M_X(u))$ is a convex function. As a consequence $M_X(u)$ is convex, since we can write the moment generating function $M_X(u) = \exp(\log(M_X(u))$ as a composition of two convex functions.

Lemma 2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be random vector with values in $\mathbb{R}^d$. Both $M_X(u)$ and the logarithm of the moment generating function $\mathfrak{L}_X(u) := \log(M_X(u))$ are convex functions.*

Now, consider a d-dimensional Lévy process $L = (L_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the *usual conditions*. Due to the stationarity and independence of the increments of Lévy processes we have the relation

$$M_{L_t}(u) = M_{L_1}(u)^t \text{ for all } u \in \mathbb{R}^d \text{ and } t \geq 0. \tag{2.2}$$

The following basic assumption on the Lévy process, which serves here as a driver for the price process, is made for the remainder of this paper. The notation of a *degenerate* Lévy process can be found in Sato [29, p. 165].

Assumption ($\mathbb{M}_d$) *The d-dimensional random variable L_1 is non-degenerate and possess a moment generating function $M_{L_1}(u) := E[e^{\langle u, L_1\rangle}]$ on some open interval $(a, b) := (a^{(1)}, b^{(1)}) \times \cdots \times (a^{(d)}, b^{(d)})$ such that $b^{(i)} - a^{(i)} > 1$ and $a^{(i)} < 0 < b^{(i)}$ for all $1 \leq i \leq d$.*

The latter condition will turn out to be necessary but not always sufficient for the existence of the risk-neutral Esscher measure.

Definition 2.2 (Esscher transform). *Let $(L_t)_{t\geq 0}$ be a d-dimensional Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We call* Esscher transform *any change of P to a locally equivalent measure Q^θ with a density process $Z_t = \frac{dQ^\theta}{dP}|_{\mathcal{F}_t} = Z_t^\theta$ of the form*

$$Z_t^\theta = \frac{e^{\langle \theta, L_t\rangle}}{M_{L_t}(\theta)}, \tag{2.3}$$

where M_{L_t} is the moment generating function of L_t, and $\theta \in (a, b)$.

We indicate by E_θ that the expectation is calculated with respect to Q^θ. The process $(Z_t^\theta)_{t\geq 0}$ is a density process for all $\theta \in (a, b)$. This measure preserves the Lévy property: $(L_t)_{t\geq 0}$ remains a Lévy process under the Esscher measure Q^θ. However, the discounted stock price process $(e^{-rt}S_t)_{t\geq 0}$ will not be a martingale under all Q^θ. A parameter $\bar\theta$ is called risk neutral Esscher parameter if $Q^{\bar\theta}$ is a martingale measure for S. $Q^{\bar\theta}$ then is called the Esscher martingale measure. The Esscher parameter $\bar\theta$ has to fulfil the following condition: For each $0 \leq i \leq d$, it must hold that $E^{\bar\theta}[S_t^{(i)}] < \infty$ and for all $0 \leq u \leq t \leq T$,

$$e^{-ru} S_u^{(i)} = E^{\bar\theta}[e^{-rt} S_t^{(i)} | \mathcal{F}_u]. \tag{2.4}$$

Due to the stationary and independent increments of a Lévy process $(L_t^{(i)})_{t\geq 0}$ we have:

$$E^{\bar\theta}[e^{-rt} S_t^{(i)} | \mathcal{F}_u] = e^{-ru} e^{L_u^{(i)}} E^{\bar\theta}[e^{-r(t-u)} S_{t-u}^{(i)}].$$

Thus, the discounted price process is a martingale under $Q^{\bar\theta}$ if and only if the equation $S_0^{(i)} = E^{\bar\theta}[e^{-rt} S_t^{(i)}]$ holds for all $t \geq 0$ and for $0 \leq i \leq d$. Or equivalently,

$$\begin{aligned} S_0^{(i)} &= E^{\bar\theta}[e^{-rt} S_t^{(i)}] = e^{-rt} S_0^{(i)} E\Big[\frac{e^{\langle \bar\theta, L_t\rangle}}{M_{L_t}(\bar\theta)} e^{L_t^{(i)}}\Big] = e^{-rt} S_0^{(i)} E\Big[\frac{e^{\langle \bar\theta + \mathbb{1}_i, L_t\rangle}}{M_{L_t}(\bar\theta)}\Big] \\ &= e^{-rt} S_0^{(i)} \Big(\frac{M_{L_t}(\bar\theta + \mathbb{1}_i)}{M_{L_t}(\bar\theta)}\Big) = e^{-rt} S_0^{(i)} \Big(\frac{M_{L_1}(\bar\theta + \mathbb{1}_i)}{M_{L_1}(\bar\theta)}\Big)^t, \end{aligned}$$

where $\mathbb{1}_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the i-th standard basis vector of $\mathbb{R}^d$. The above equation means that $\bar\theta \in (a, b)$ has to solve the system of equations

$$e^r = \frac{M_{L_1}(\bar\theta + \mathbb{1}_i)}{M_{L_1}(\bar\theta)}, \quad 1 \leq i \leq d. \tag{2.5}$$

This also explains why it is necessary to require M_{L_1} to be defined on an interval (a, b), where the length of each univariate interval $(a^{(i)}, b^{(i)})$ is greater than one. In summary, the following characterizes Esscher measures.

Lemma 2.3. *Let Assumption* ($\mathbb{M}_d$) *be fulfilled and suppose there is a parameter* $\bar{\theta}$ *such that* $M_{L_1}(\bar{\vartheta})$ *and* $M_{L_1}(\bar{\vartheta}+\mathbb{1}_i)$ *are finite,* $1 \leq i \leq d$, *and*

$$r = \mathfrak{L}_{L_1}(\bar{\vartheta}+\mathbb{1}_i) - \mathfrak{L}_{L_1}(\bar{\vartheta}) \tag{2.6}$$

holds for $1 \leq i \leq d$. *Then, for all* $T > 0$, *the discounted price process* $(e^{-rt}S_t)_{t\geq 0}$ *is a martingale under* $Q^{\bar{\theta}}$, *with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if and only if equation* (2.6) *holds true.* $\bar{\theta}$ *is called risk-neutral Esscher parameter.*

The Esscher parameters $\bar{\vartheta}^{(i)}$ of the univariate processes $L^{(i)}$ are identical to the components of the Esscher parameter $\bar{\theta}$ of the multivariate Lévy process L if the components of L are independent. In general, as in the examples considered in this paper, with dependent components of L they may be different. As consequence we get: If $\bar{\vartheta}^{(i)}$ are solutions of

$$r = \mathfrak{L}_{L_1^{(i)}}(\bar{\vartheta}+\mathbb{1}) - \mathfrak{L}_{L_1^{(i)}}(\bar{\vartheta}), \tag{2.7}$$

for $1 \leq i \leq d$, and if $\bar{\theta} = (\bar{\theta}^{(1)}, \dots, \bar{\theta}^{(d)})$ denotes a solution of the system of equations (2.6), then, $\bar{\vartheta}^{(i)} = \bar{\theta}^{(i)}$ for all i, that is, $\bar{\theta} = (\bar{\vartheta}^{(1)}, \dots, \bar{\vartheta}^{(d)})$ if $L^{(i)}$ and $L^{(j)}$, $i \neq j$ are independent. In dependent Lévy models, however, they may be different.

From the latter we see that pricing in the univariate Lévy setting differ from the multivariate case when dependence in the components is present. The inclusion of further dependent components in the market model may lead to lower prices of efficient versions of options depending only on one component of the market model compared to pricing in the single component model.

For illustration we consider an option on one asset with payoff $f(S_T^{(i)})$, $1 \leq i \leq d$. The cost in the univariate setting, that is, where only the Lévy process $L^{(i)}$ is present, is given by

$$c(f(S_T^{(i)})) = E\Big[\frac{e^{\bar{\vartheta}^{(i)}L_T^{(i)}}}{M_{L_T^{(i)}}(\bar{\vartheta}^{(i)})} f(S_T^{(i)})\Big],$$

whereas in the multivariate setting, that is, where $L = (L^{(1)}, \dots, L^{(d)})$ is the driving process, the cost is

$$c(f(S_T^{(i)})) = E\Big[\frac{e^{\bar{\vartheta}^{(i)}L_T^{(i)}} \cdot e^{\langle \bar{\vartheta}^{[i]}, L_T^{[i]}\rangle}}{E[e^{\bar{\vartheta}^{(i)}L_T^{(i)}} \cdot e^{\langle \bar{\vartheta}^{[i]}, L_T^{[i]}\rangle}]} f(S_T^{(i)})\Big],$$

where for $y \in \mathbb{R}^d$ the notation $y^{[i]}$ means $(y^{(1)}, \dots, y^{(i-1)}, y^{(i+1)}, \dots, y^{(d)}) \in \mathbb{R}^{d-1}$, $1 < i < d$ and $(y^{(2)}, \dots, y^{(d)})$ resp. $(y^{(1)}, \dots, y^{(d-1)})$ for $i = 1$ resp. $i = d$. The costs are equal if L has independent components.

As mentioned before, Assumption ($\mathbb{M}_d$) alone does not guarantee the existence of a solution $\bar{\theta}$. Theorem 2.6 provides a sufficient condition for existence and further shows that the solution, if existent, is unique. The uniqueness is based on the following strict convexity result (see e.g. Witting [34, Satz 1.164]).

Proposition 2.4 (Strict convexity of $\mathfrak{L}_\mu$). *Let* $\mu(\mathrm{d}x)$ *be a non-degenerate probability measure on* $(\mathbb{R}^d, \mathbb{B}^d)$ *which possesses a moment generating function* M_μ *in some open domain. Then* $H_u(\mathfrak{L}_\mu)$, *the Hessian of* $\mathfrak{L}_\mu$, *is positive definite. In particular,* $\mathfrak{L}_\mu$ *is strictly convex on the interior of its range of existence.*

Existence and uniqueness criteria for multivariate (exponential) Lévy processes have been studied in [23] and in [31] and for $d = 1$ in [27]. For models based on the stochastic exponential $S = S_0\,\mathcal{E}(X)$ a characterization of existence of an equivalent martingale measure is given in Tankov [31, Theorem 3]. Note that the stochastic exponential S is a local martingale if and only if X is a local martingale, assuming $S_0^i \neq 0$. In case X is a Lévy process this is equivalent to X being a martingale (even uniformly integrable on $[0, T]$). The proof of Tankov's theorem implies an existence result of an Esscher parameter if the underlying Lévy process has all exponential moments.

Proposition 2.5. *Let* (X, P) *be a d-dimensional Lévy process on* $[0, T]$ *having all exponential moments* $Ee^{\lambda X_1} < \infty, \forall \lambda \in \mathbb{R}^d$:
Then there exists a measure $Q_{\bar{\vartheta}} \sim P$ *with*

$$\frac{dQ_{\bar{\vartheta}}}{dP}(x) = \frac{\exp(\bar{\vartheta}\cdot x)}{M_X(\bar{\vartheta})} = Z_T^{\bar{\vartheta}}$$

such that X is a martingale w.r.t. $Q_{\overline{\vartheta}}$.

Any exponential Lévy model $S^{(i)} = S_0^i e^{L^{(i)}}$ can be represented as stochastic exponential model $S^{(i)} = S_0^i \mathcal{E}(X^{(i)})$ with some Lévy process $X^{(i)}$ and conversely. For a given stochastic exponential model $S_0^i \mathcal{E}(X^{(i)})$ define $Y^{(i)} = \ln \mathcal{E}(X^{(i)})$; then $\exp(Y^{(i)}) = \mathcal{E}(X^{(i)})$. For the converse direction $e^{X^{(i)}} = \mathcal{E}(Y^{(i)})$ implies that $Y^{(i)} = \mathcal{L}(e^{X^{(i)}})$ is the stochastic logarithm of $e^{X^{(i)}}$ (see Goll and Kallsen [17, Lemma 5.8]). The characteristics of $X^{(i)}$ are given explicitly in terms of the characteristics of $L^{(i)}$. Based on the convexity result in Proposition 2.4 the following existence and uniqueness result in Kallsen and Shiryaev [23, Theorems 4.4 and 4.5] implies existence and uniqueness of the Esscher measure under some regularity conditions.

Theorem 2.6. *Let condition* $(\mathbb{M}_d)$ *hold for the Lévy process* $X \in \mathbb{R}^d$ *and define* $Q_\theta = Z_T^\theta P$. *Then it holds:*

1) *The stochastic exponential processes* $S^i = S_o^i \mathcal{E}(X^{(i)})$ *are martingales if and only if the integral functions* $|x^{(i)} e^{\theta \cdot x} - h^i(x)| * \nu$ *are of finite variation,* $h^{(i)}$ *the cut off function used, and*

$$DM_X(\theta) = 0. \tag{2.8}$$

   Q_θ *then is called Esscher measure for S.*
2) *The Esscher measure is uniquely determined if it exists.*

Proposition 2.4 and Theorem 2.6 give some general conditions for existence and uniqueness of the Esscher measure. Condition (2.8) is a drift condition saying that the drift of X is zero w.r.t. Q_θ. The existence and uniqueness results can easily be transferred to the case of discounted models of the form $e^{-rt}S = e^{L_t - rt}$. Only the drift parameter has to be changed. In [36] it is shown that even in cases where an Esscher measure does not provide an equivalent martingale measure a mean correcting Esscher parameter can be chosen to reproduce the price of a European call option with respect to any risk neutral measure.

Remark 2.7. *A direct approach to solve equations (2.8) leads to consider* $\delta_j(u) = \mathfrak{L}_{L_1}(u + \mathbb{1}_j) - \mathfrak{L}_{L_1}(u)$ *and* $\delta(u) = (\delta_j(u))_{1 \le j \le d}$. *Then by Proposition 2.4* $H_n(\mathfrak{L}_{L_1})$, *the Hessian of* $\mathfrak{L}_{L_1}$, *is positive definite and one obtains*

$$\begin{aligned} 0 &< \int_0^1 \left\langle \mathbb{1}_j, H_u(\mathfrak{L}_{L_1}(u + \mathbb{1}_j t))\mathbb{1}_j \right\rangle dt \\ &= \left\langle \mathbb{1}_j, \int_0^1 H_u(\mathfrak{L}_{L_1}(u + \mathbb{1}_j t))\mathbb{1}_j \, dt \right\rangle \\ &= \left\langle \mathbb{1}_j, \nabla\left(\mathfrak{L}_{L_1}(u + \mathbb{1}_j) - \mathfrak{L}_{L_1}(u)\right) \right\rangle \\ &= \frac{\partial}{\partial u_j} \delta_j(u) \end{aligned}$$

i.e. δ_j *are strictly increasing in u. In* $d = 1, 2$ *it leads under the assumption that* $\lim_{u \downarrow a} M_{L_1}(u) = \lim_{u \uparrow b} M_{L_1}(u) = \infty$ *by some simple geometric arguments to the existence of a unique solution of the equations:*

$$\delta_j(u) = c\,, \quad 1 \le j \le d\,.$$

For the general case however one has to rely either on an iterative construction or on more general results in Hodge theory as used to prove existence of solutions of log-Likelihood equations in [26].

By the uniqueness result in Theorem 2.6 we can now define in a formal way the risk-neutral Esscher measure under Assumption $(\mathbb{M}_d)$.

Definition 2.8 (Esscher martingale measure). *The unique* $\bar{\theta} \in \mathbb{R}^d$ *such that the process* $(e^{-rT} S_t)_{t \ge 0}$ *is a martingale with respect to* $Q^{\bar{\theta}}$ *is called the Esscher parameter and* $Q^{\bar{\vartheta}}$ *is called the* Esscher martingale measure *or* risk-neutral Esscher measure.

Remark 2.9. *An alternative way to prove the uniqueness of the risk-neutral Esscher measure is to prove uniqueness of the minimal entropy martingale measure and to establish that a risk-neutral Esscher measure if it exists is given by the minimum entropy martingale measure (see e.g. [18] and Esche and Schweizer [14, Theorem B]).*

3 Cost bounds in multivariate Lévy models

In this section we specialize the general construction result for cost-efficient payoffs in Theorem 1.1 to the case of multivariate Lévy models. The formulas for the cost bounds are given in terms of the Lévy process themselves (instead of the market models).

Proposition 3.1 (Cost-efficient payoffs in multivariate Lévy models)**.** *Let $(L_t)_{t\geq 0}$ be a multivariate Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that the risk-neutral Esscher parameter $\bar{\theta}$ exists.*

Then $\underline{X}_T = G^{-1}(1 - F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, L_T\rangle))$ is the cost-efficient strategy and $\overline{X}_T = G^{-1}(F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, L_T\rangle))$ is the most-expensive way to achieve a payoff with payoff distribution G. Moreover, for any payoff $X_T \sim G$, the lower and upper cost bounds are given by,

$$\begin{aligned} c(X_T) \geq c(\underline{X}_T) &= e^{-rT} \int \frac{e^{\langle\bar{\theta},y\rangle}}{M_{L_T}(\bar{\theta})} G^{-1}(1 - F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, y\rangle))\, \mathrm{d}P^{L_T}(y) \\ c(X_T) \leq c(\overline{X}_T) &= e^{-rT} \int \frac{e^{\langle\bar{\theta},y\rangle}}{M_{L_T}(\bar{\theta})} G^{-1}(F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, y\rangle))\, \mathrm{d}P^{L_T}(y). \end{aligned} \tag{3.1}$$

Proof. Observe that

$$F_{Z_T}(y) = P(Z_T \leq y) = P(\langle\bar{\theta}, L_T\rangle) \leq \ln(y \cdot M_{L_T}(\bar{\theta}))) = F_{\langle\bar{\theta},L_T\rangle}(\ln(y \cdot M_{L_T}(\bar{\theta}))),$$

and, hence $1 - F_{Z_T}(Z_T) = 1 - F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, L_T\rangle)$ almost surely. Thus, the statement follows by applying Theorem 1.1 to $\underline{X}_T = G^{-1}(1 - F_{Z_T}(Z_T)) = G^{-1}(1 - F_{\langle\bar{\theta},L_T\rangle}(\langle\bar{\theta}, L_T\rangle))$. The most-expensive part is similar. □

As consequence of the latter result we obtain

Corollary 3.2 (Characterization of cost-efficiency)**.** *Under the assumptions of Proposition 3.1 it holds:*

1. *A strategy with terminal payoff X_T is cost-efficient if and only if X_T is a decreasing function in $\langle\bar{\theta}, L_T\rangle$.*
2. *A strategy with terminal payoff X_T is most-expensive if and only if X_T is an increasing function in $\langle\bar{\theta}, L_T\rangle$.*

Remark 3.3. *Corollary 3.2 implies that strategies with payoffs of the form $X_T = f(\langle a, L_T\rangle) \sim G$ are cost-efficient if*

$$f \text{ is decreasing and } a = t \cdot \bar{\theta} \text{ for some } t > 0, \tag{3.2}$$

while X_T is most-expensive if

$$f \text{ is increasing and } a = t \cdot \bar{\theta} \text{ for some } t > 0, \tag{3.3}$$

In the particular cases $\bar{\theta}^{(i)} > 0$ for all i resp. $\bar{\theta}^{(i)} < 0$ for all i we obtain a direct connection of cost-efficiency to monotonic behaviour in L_T.

Corollary 3.4. *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of* (2.5) *exists.*

1. *If $\bar{\theta}^{(i)} < 0$ for all $1 \leq i \leq d$, then a cost-efficient payoff $X_T \sim G$ is componentwise increasing in L_T.*
2. *If $\bar{\theta}^{(i)} > 0$ for all $1 \leq i \leq d$, then a cost-efficient payoff $X_T \sim G$ is componentwise decreasing in L_T.*

For the most-expensive strategy, the reverse holds true.

Proof. Let all components of the risk-neutral Esscher parameter $\bar{\theta}$ have a negative sign and let $X_T \sim G$ be a cost-efficient payoff. Then, due to Proposition 3.1 resp. Corollary 3.2 $\underline{X}_T = G^{-1}(1 - F_{\langle\bar{\theta}, L_T\rangle}(\langle\bar{\theta}, L_T\rangle))$ is decreasing in $\langle\bar{\theta}, L_T\rangle$. Moreover, since $\bar{\theta}^{(i)} < 0$ for all $1 \le i \le d$ the function $h(L_T) = \langle\bar{\theta}, L_T\rangle$ is componentwise decreasing in L_T. Thus, the strategy $\underline{X}_T$ is componentwise increasing in L_T. The other cases can be shown analogously. □

Corollary 3.4 allows in the cases where $\bar{\theta}^{(i)} < 0$ or $\bar{\theta}^{(i)} > 0$ for all $1 \le i \le d$ to identify inefficient payoffs from its monotonic behaviour in the coordinates of L_T.

Example 3.5 (Basket options are inefficient)**.** *Form Corollary 3.2 we find in particular that basket options $X_T = (\alpha S_T^{(1)} + \beta S_T^{(2)} - K)_+$ are neither efficient nor most-expensive. For $\alpha < 0 < \beta$ or $\beta < 0 < \alpha$ this is a consequence of Corollary 3.4. In general this is a consequence of the fact that $h(x_1, x_2) = a\exp(x_1) + b\exp(x_2)$ is not constant on any line $\{x : \langle\bar{\theta}, x\rangle = t\}$ and thus h can not be represented as a function of the form $f(\langle\bar{\theta}, x\rangle)$. Thus X_T by Corollary 3.2 can not be cost efficient nor most expensive. In Section 5 we determine cost-efficient improvements of basket options in some specific multivariate Lévy models.*

4 Multivariate Lévy processes and application to real market data

In this section we recall some properties of multivariate normal mixture models its densities and moment generating functions as needed for the computation of the risk-neutral Esscher parameters for some class of Lévy models. For two sets of real market data we give a statistical analysis in terms of three different multivariate Lévy models the *NIG*, the *VG* and the normal model.

Normal mean variance mixture models

Normal mean variance mixtures are valuable models for analysing data from a variety of heavy-tailed and skew empirical distributions. They have been used a lot in the more recent literature for financial data but also in various other areas. Detailed expositions are given in Barndorff-Nielsen [1], Blæsild [6] and Barndorff-Nielsen et al. [2]. Some recent developments in particular for dependence modelling are given in [24].

An $\mathbb{R}^d$-valued random variable X is said to have *multivariate normal mean-variance mixture* distribution if

$$X \overset{\mathrm{d}}{=} \mu + Z\beta + \sqrt{Z}AW, \tag{4.1}$$

where $\mu, \beta \in \mathbb{R}^d$, A is a real-valued $d \times d$ matrix such that $\Delta := AA^\top$ is positive definite, W is a standard normal distributed random vector ($W \sim N_d(0, \mathrm{I}_d)$) and $Z \sim F_Z$ is a real-valued, non-negative random variable independent of W. An equivalent definition is the following:

A probability measure Q on $(\mathbb{R}^d, \mathbb{B}^d)$ is said to be a multivariate normal mean-variance mixture if

$$\mathrm{Q}(\mathrm{d}x) = \int_{\mathbb{R}_+} N_d(\mu + y\beta, y\Delta)(\mathrm{d}x) F_Z(\mathrm{d}y), \tag{4.2}$$

where the mixing distribution F_Z is a probability measure on $(\mathbb{R}_+, \mathbb{B}_+)$. A practical short hand notation of equation (4.2) is $\mathrm{F} = N_d(\mu + y\beta, y\Delta) \circ F_Z$.

Multivariate generalized hyperbolic distributions are defined as normal mean-variance mixtures with Generalized inverse Gaussian (*GIG*) mixing distributions:

$$GH_d(\lambda, \alpha, \beta, \delta, \mu, \Delta) = N_d(\mu + y\Delta\beta, y\Delta) \circ GIG(\lambda, \delta\sqrt{\alpha^2 - \langle\beta, \Delta\beta\rangle}), \tag{4.3}$$

where it is usually assumed without loss of generality that $\det(\Delta) = 1$, which we shall do in the following. Due to the parameter restrictions of *GIG* distributions, the other *GH* parameters have to fulfil the constraints

$\lambda \in \mathbb{R},\ \alpha, \delta \in \mathbb{R}_+,\ \beta, \mu \in \mathbb{R}^d$ and

$$\begin{aligned} &\delta \geq 0,\ 0 \leq \sqrt{\langle \beta, \Delta\beta \rangle} < \alpha,\ \text{if } \lambda > 0 \\ &\delta > 0,\ 0 \leq \sqrt{\langle \beta, \Delta\beta \rangle} < \alpha,\ \text{if } \lambda = 0 \\ &\delta > 0,\ 0 \leq \sqrt{\langle \beta, \Delta\beta \rangle} \leq \alpha,\ \text{if } \lambda < 0. \end{aligned} \tag{4.4}$$

The meaning and influence of the parameters is similar as in the univariate case. The representation in (4.1) entails that the infinite divisibility of the mixing Generalized inverse Gaussian distributions transfers to the GH_d distribution. In consequence there exists a Lévy process $(L_t)_{t\geq 0}$ with $\mathcal{L}(L_1) = GH_d(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ (see e.g. Sato [29, Theorem 7.10 (iii)]). The following properties of GH_d distributions and in particular of *NIG* and *VG* distributions are given in Hammerstein [19].

If $\delta > 0$ and $\sqrt{\langle \beta, \Delta\beta \rangle} < \alpha$, then the density of $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ can be derived from (4.3):

$$\begin{aligned} d_{GH_d(\lambda,\alpha,\beta,\delta,\mu,\Delta)}(x) &= \int_0^\infty d_{N_d(\mu+y\Delta\beta, y\Delta)}(x)\, d_{GIG(\lambda,\delta\sqrt{\alpha^2-\langle\beta,\Delta\beta\rangle})}(y)\,\mathrm{d}y \\ &= e^{\langle \beta, x-\mu\rangle} \frac{(\alpha^2 - \langle \beta, \Delta\beta\rangle)^{\frac{\lambda}{2}}}{(2\pi)^{\frac{d}{2}} \alpha^{\lambda-\frac{d}{2}} \delta^\lambda} \left(\langle x-\mu, \Delta^{-1}(x-\mu)\rangle + \delta^2\right)^{\frac{\lambda-\frac{d}{2}}{2}} \frac{K_{\lambda-\frac{d}{2}}(\alpha\sqrt{\langle x-\mu, \Delta^{-1}(x-\mu)\rangle + \delta^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \langle\beta, \Delta\beta\rangle})}. \end{aligned}$$

The moment generating function of a multivariate generalized hyperbolic distribution is given in the following proposition.

Proposition 4.1. *If in equation* (4.3) *the* $GH_d(\lambda, \alpha, \beta, \delta, \mu, \Delta)$ *parameters fulfil the constraints in* (4.4), *then its moment generating function is given by*

$$M_{GH_d(\lambda,\alpha,\beta,\delta,\mu,\Delta)}(u) = e^{\langle u, \mu\rangle} \left(\frac{\alpha^2 - \langle \beta, \Delta\beta\rangle}{\alpha^2 - \langle \beta+u, \Delta(\beta+u)\rangle} \right)^{\frac{\lambda}{2}} \cdot \frac{K_\lambda(\delta\sqrt{\alpha^2 - \langle \beta+u, \Delta(\beta+u)\rangle})}{K_\lambda(\delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle})}. \tag{4.5}$$

The densities of the multivariate analogues of the *NIG* and *VG* then have a representation given in the next lemma.

Lemma 4.2. *With* $\lambda = -\frac{1}{2}$ *the multivariate normal inverse Gaussian distribution* $NIG_d(\alpha, \beta, \delta, \mu, \Delta)$ *possesses the density*

$$d_{NIG(\alpha,\beta,\delta,\mu,\Delta)}(x) = \sqrt{\frac{2}{\pi}} \frac{\delta \alpha^{\frac{d+1}{2}} e^{\delta\sqrt{\alpha^2 - \langle\beta,\Delta\beta\rangle}}}{(2\pi)^{\frac{d}{2}}} \left(\langle x-\mu, \Delta^{-1}(x-\mu)\rangle + \delta^2\right)^{-\frac{d+1}{4}} K_{\frac{d+1}{2}}\left(\alpha\sqrt{\langle x-\mu, \Delta^{-1}(x-\mu)\rangle + \delta^2}\right) e^{\langle\beta, x-\mu\rangle}.$$

The density $d_{VG(\lambda,\alpha,\beta,\mu,\Delta)}(x)$ *of the multivariate Variance-Gamma distribution, a limiting case of the GH distribution, can be derived by letting* $\delta \to 0$. *If* $\lambda > 0$, *then*

$$d_{VG(\lambda,\alpha,\beta,\mu,\Delta)}(x) = \frac{(\alpha^2 - \langle \beta, \Delta\beta\rangle)^\lambda}{(2\pi)^{\frac{d}{2}} \alpha^{\lambda-\frac{d}{2}} 2^{\lambda-1}\Gamma(\lambda)} \left(\langle x-\mu, \Delta^{-1}(x-\mu)\rangle\right)^{\frac{(\lambda-\frac{d}{2})}{2}} K_{\lambda-\frac{d}{2}}\left(\alpha\sqrt{\langle x-\mu, \Delta^{-1}(x-\mu)\rangle}\right) e^{\langle\beta, x-\mu\rangle}.$$

We briefly recall the multivariate Samuelson model which serves as a benchmark model in this context. The driving Lévy process is given by

$$L_t^{(i)} = (\mu^{(i)} - \frac{{\sigma^{(i)}}^2}{2})t + \sigma^{(i)} B_t^{(i)},\ t > 0$$

for $1 \leq i \leq d$, where $(B_t^{(i)})_{t\geq 0}$ is a standard Brownian motion under the physical measure P, $\mu^{(i)}$ is the drift and $\sigma^{(i)}$ the volatility parameter. Thus, each asset price process fulfills the stochastic differential equation

$$\mathrm{d}S_t^{(i)} = \mu^{(i)} S_t^{(i)} + \sigma^{(i)} S_t^{(i)}\, \mathrm{d}B_t^{(i)},$$

with the processes $B_t^{(i)}$ being correlated such that $E[\mathrm{d}B_t^{(i)}\, \mathrm{d}B_t^{(j)}] = \rho_{ij}\,\mathrm{d}t$ where $\rho_{ii} = 1$. The law of the multivariate Lévy process L is determined by a multivariate normal distribution with a drift vector $\tilde{\mu} \in \mathbb{R}^d$, where

$\tilde{\mu}^{(i)} = (\mu^{(i)} - \frac{\sigma^{(i)2}}{2})$ and a positive-definite $d \times d$ covariance matrix $\Sigma = \left(\mathrm{Cov}(L_1^{(i)}, L_1^{(j)})\right)$, $1 \leq i, j \leq d$, that is, $\mathcal{L}(L_1) = N_d(\tilde{\mu}, \Sigma)$ such that $\sigma_{ij} = \sigma^{(i)}\rho_{ij}\sigma^{(j)}$, with density

$$d_{L_t}(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2}\left\langle (x-\tilde{\mu}), \Sigma^{-1}(x-\tilde{\mu})\right\rangle}, \; t > 0$$

The moment generating function of L_1 is equal to

$$M_{N_d(\tilde{\mu},\Sigma)}(u) = e^{\langle u,\tilde{\mu}\rangle + \frac{1}{2}\langle u,\Sigma u\rangle}. \tag{4.6}$$

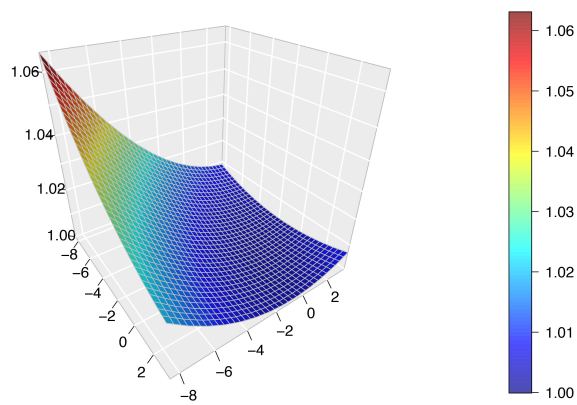

Figure 1: Moment generating function for a bivariate NIG process. The parameters used to derive the moment generating function are listed in Table 1.

Application to real market data

In this subsection we illustrate an application of some multivariate Lévy processes to the analysis of real market data. We consider German stock price data for Allianz and Volkswagen and for E.ON and Thyssen Krupp from May 28, 2010 to September 28, 2012. That is, we consider the Lévy process $L^{(\mathrm{A,VW})} = (L_t^{\mathrm{Allianz}}, L_t^{\mathrm{VW}})_{0 \leq t \leq T}$ in order to model the daily log-returns of Allianz and Volkswagen in a bivariate Lévy model, and analogously $L^{(E.ON,TK)}$ for E.ON and Thyssen Krupp (see Figure 2). Table 1 contains the estimated parameters from daily log-returns of Allianz and Volkswagen for the bivariate *NIG*, *VG*, and the Samuelson model. The interest rate used to calculate the Esscher parameter $\bar{\theta}$ in the last column is the continuously compounded 1-Month-Euribor rate of October 1, 2012, which is $r = 4.2027 \cdot 10^{-6}$; note that this is the continuously compounded daily rate which we need to do daily calculations and used as well as for daily rebalancing for hedging purposes for one-dimensional options. This explains the extremely small value. The annualized Euribor rates at that time point are in the order 10^{-3} instead.

For the determination of the Esscher parameter we numerically solved the determining system of equations (2.5) using the estimated parameters. The alternative way proposed by Theorem 2.6 is to establish existence of the Esscher parameter first by determining the associated stochastic exponential model, which is also based on the estimated parameters. Then check the $(\mathbb{M}_d)$ condition and the finite variation condition. All of these seem to be doable. Then finally solve numerically equation (2.8) in order to obtain the Esscher parameter. This alternative seems however to be more involved.

Figure 1 gives the moment generating function for a bivariate *NIG* process $L^{(\mathrm{E.ON,TK})}$, which models the daily log-returns of the E.ON and Thyssen Krupp stock prices from May 28, 2010 to September 28, 2012. See Table 1 for the estimated parameters used for the computations.

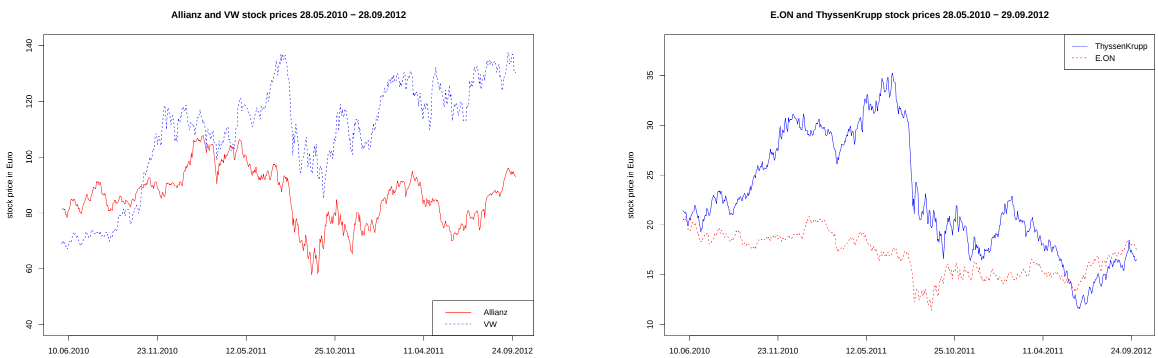

Figure 2: *Left:* Daily closing prices of Allianz and Volkswagen used for parameter estimation. *Right:* Daily closing prices of E.ON and Thyssen Krupp used for parameter estimation.

An application of the bivariate *NIG* model to data of Allianz and Volkswagen is given in Figure 3. For the statistical fitting of the model we used the estimated parameters from Table 1. The histogram of the daily log-returns and the model fit for Allianz and Volkswagen is presented in Figure 3.

With the estimated parameters and the formulas for the moment generating functions in Proposition 4.1 it is possible to solve numerically equation (2.5) i.e. to determine the Esscher parameters (see Table 1).

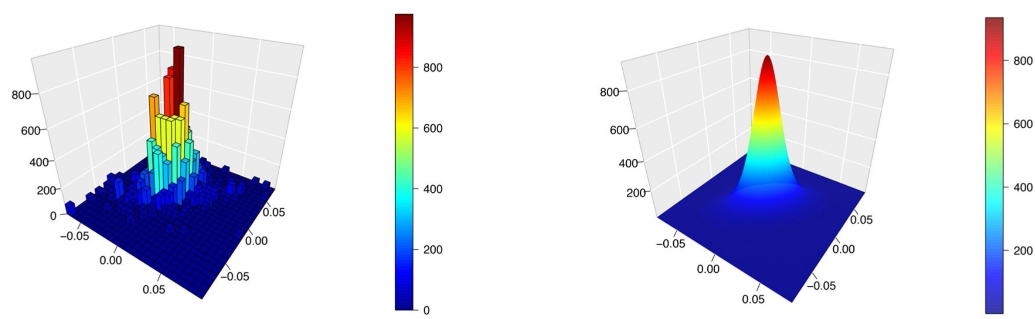

Figure 3: *Left:* Histogram for the daily log-returns of Allianz and Volkswagen from May 28, 2010 to September 28, 2012. *Right:* Fitted bivariate NIG density curve for Allianz and Volkswagen log-returns. The parameters used to derive the density are listed in Table 1.

Although the moment generating function of NIG_d and VG_d has an analytical representation, an analytic expression for the Esscher parameter $\bar{\theta}$ is not available. For the multivariate normal distribution an explicit expression for $\bar{\theta}$ is given in Gerber and Shiu [16, Section 7].

As pointed out in [20] in the univariate setting the sign of $\overline{\theta}$ describes a drift; a negative sign a positive drift and a positive sign a negative drift. The size of $|\bar{\theta}|$ reflects the magnitude of the drift of the price process and thus can be regarded as a measure for the strength of the market trend.

In the multivariate setting we have the following observation. From the more pronounced (positive) trend in the Allianz and VW data than in the E.ON and Thyssen Krupp data we can expect that the potential savings in the Allianz and Volkswagen case are higher than for the E.ON and Thyssen Krupp case. Note that the dependence between the stocks implies that the Esscher parameters in the joint model as in Table 1 are different form the parameters in the single models as considered in [20]. For example this dependence implies that in the joint model Allianz gets a slightly positive Esscher parameter, indicating a mild relative negative drift in

Table 1: Estimated parameters from daily log-returns of Allianz and Volkswagen and E.ON and Thyssen Krupp for the bivariate NIG, VG, and the Samuelson model.

$L^{(A,VW)}$	λ	α	β	δ	μ	Δ	$\bar{\theta}$
NIG	−0.5	51.3819	$\begin{pmatrix} 3.1651 \\ -3.4936 \end{pmatrix}$	0.01809	$\begin{pmatrix} -0.000149 \\ 0.001942 \end{pmatrix}$	$\begin{pmatrix} 1.097855 & 0.693566 \\ 0.693566 & 1.349024 \end{pmatrix}$	$\begin{pmatrix} 0.937069 \\ -3.169199 \end{pmatrix}$
VG	1.5844	96.67	$\begin{pmatrix} 4.6788 \\ -5.7154 \end{pmatrix}$	0.0	$\begin{pmatrix} -0.000169 \\ 0.002553 \end{pmatrix}$	$\begin{pmatrix} 1.112008 & 0.709215 \\ 0.709215 & 1.351597 \end{pmatrix}$	$\begin{pmatrix} 1.025704 \\ -3.299607 \end{pmatrix}$
Normal	$\mu = \begin{pmatrix} 0.000428 \\ 0.001287 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0.0004105 & 0.0002615 \\ 0.0002615 & 0.0004675 \end{pmatrix}$						$\begin{pmatrix} 1.114694 \\ -3.368318 \end{pmatrix}$
$L^{(E.ON,TK)}$	λ	α	β	δ	μ	Δ	$\bar{\theta}$
NIG	−0.5	50.7124	$\begin{pmatrix} 0.146985 \\ -1.883107 \end{pmatrix}$	0.01858	$\begin{pmatrix} 0.00019196 \\ 0.00073098 \end{pmatrix}$	$\begin{pmatrix} 0.901532 & 0.751105 \\ 0.751105 & 1.734999 \end{pmatrix}$	$\begin{pmatrix} 0.323143 \\ 0.037110 \end{pmatrix}$
VG	1.4653	90.4023	$\begin{pmatrix} -0.43541 \\ -0.50398 \end{pmatrix}$	0.0	$\begin{pmatrix} -3.6075e{-11} \\ 2.9021e{-11} \end{pmatrix}$	$\begin{pmatrix} 0.912754 & 0.752004 \\ 0.752004 & 1.715151 \end{pmatrix}$	$\begin{pmatrix} 0.308843 \\ 0.066307 \end{pmatrix}$
Normal	$\mu = \begin{pmatrix} -0.0001002 \\ -0.0001280 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0.000356 & 0.000283 \\ 0.000283 & 0.000599 \end{pmatrix}$						$\begin{pmatrix} 0.188492 \\ 0.131720 \end{pmatrix}$

the joint model, while it has a mild positive drift in the individual model. As consequence this implies that in the joint market it is possible to make use of the higher drift in the Volkswagen market and its correlation to the Allianz market to obtain better (i.e. cheaper) constructions and improvements of options based on the Allianz stock alone.

5 Application to basket options

As an example for the determination of efficient options in case $d = 2$ we consider the long basket option. As shown in Section 3 these basket options themselves are not efficient in general. A basket option (on two assets) is a weighted sum of $S^{(1)}$ and $S^{(2)}$, for the underlying $S = (S^{(1)}, S^{(2)})$. This *exotic* option with strike $K > 0$, weights $w_1, w_2 \in \mathbb{R}$ and maturity $T > 0$ has the payoff

$$X_T^{\text{ba}} = (w_1 S_T^{(1)} + w_2 S_T^{(2)} - K)_+.$$

The bivariate payoff function equals

$$\omega^{\text{ba}}(y) = (\langle \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix} \right), y \rangle - K)_+.$$

Denote $S_T^w = w_1 S_T^{(1)} + w_2 S_T^{(2)}$ and observe for $x > 0$ that

$$G_{\text{ba}}(x) = P((S_T^w - K)_+ \le x) = P((S_T^w - K \le x, S_T^w > K) + P(S_T^w \le K) = F_{S_T^w}(K + x),$$

that is,

$$G_{\text{ba}}(x) = \begin{cases} F_{S_T^w}(K + x), & x > 0, \\ 0, & x \le 0. \end{cases} \tag{5.1}$$

Its generalized inverse is given by

$$G_{\text{ba}}^{-1}(y) = (F_{S_T^w}^{-1}(y) - K)_+, \; y \in (0, 1). \tag{5.2}$$

Applying Proposition 3.1 the cost-efficient payoff that generates the same distribution G_{ba} as the basket option is therefore given by the following proposition.

Proposition 5.1 (Cost-efficient basket option). *The cost-efficient payoff of the basket option X_T^{ba} is given by*

$$\underline{X}_T^{\text{ba}} = G_{\text{ba}}^{-1}(1 - F_{\langle \bar{\theta}, L_T \rangle}(\langle \bar{\theta}, L_T \rangle)) = (F_{S_T^w}^{-1}(1 - F_{\langle \bar{\theta}, L_T \rangle}(\langle \bar{\theta}, L_T \rangle)) - K)_+. \tag{5.3}$$

Its payoff function is given by

$$\underline{\omega}^{\text{ba}}(y) = (F_{S_T^w}^{-1}(1 - F_{\langle \bar{\theta}, L_T \rangle}(\langle \bar{\theta}, \log(y) - \log(S_0) \rangle)) - K)_+,$$

where $S_0 = (S_0^i)_{1 \le i \le 2}$ and the logarithm is applied componentwise.

As example we consider the standard basket option with weights $w_1 = w_2 = 0.5$ and strike $K = 15$ for the E.ON, Thyssen Krupp and the Allianz, Volkswagen data. This payoff is symmetrically increasing in rising markets. For $S_T^{(1)} + S_T^{(2)} \leq 30$ the outcome is zero, which means that such an option rewards the writer when at least one of the assets $S_T^{(i)}$ is high while the other asset decreases at most at the same level (compare Figure 4). On

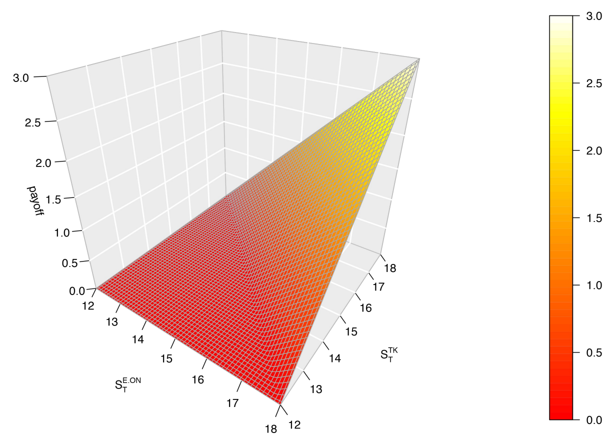

Figure 4: Standard payoff of a long basket option for E.ON and Thyssen Krupp with weights $w_1 = w_2 = 0.5$, maturity $T = 23$ days and strike $K = 15$.

the contrary, the corresponding cost-efficient payoff $\underline{X}_T^{\text{ba}}$ of the basket option shows a reverse behaviour. This is consistent with Corollary 3.4 since the risk-neutral Esscher parameter $\bar{\theta} = \binom{0.323143}{0.037110}$ is componentwise positive. Figure 5 displays the efficient payoff $\underline{X}_T^{\text{ba}}$ of the optimal long basket option on E.ON and Thyssen Krupp stocks with strike $K = 15$ and maturity $T = 23$ days for the *NIG* model. Similar calculations are done for the Allianz, Volkswagen data.

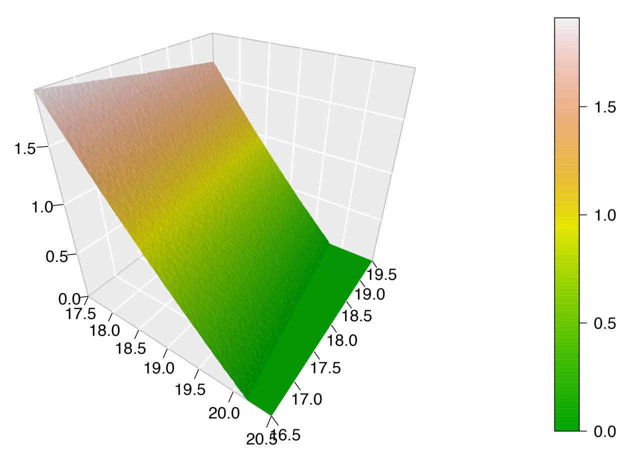

Figure 5: Optimal payoff of a long basket option for E.ON and Thyssen Krupp with weights $w_1 = w_2 = 0.5$, maturity $T = 23$ days and strike $K = 15$ in the NIG model.

All computations are based on the estimated parameters given in Table 1. The initial stock prices $S_0^{(i)}$ are the closing prices at October 1, 2012, and the time to maturity is chosen to be $T = 23$ trading days, meaning that the long basket options mature on November 1, 2012. The chosen initial stock prices equal $S_0^{\text{A}} = 93.42$, $S_0^{\text{VW}} =$

Table 2: Comparison of the cost of a standard long basket option with its corresponding cost-efficient counterpart on Allianz and Volkswagen as well as E.ON and Thyssen Krupp.

$L^{(A,VW)}$	$c(X_T^{ba})$	$c(\underline{X}_T^{ba})$	**Efficiency loss in %**
NIG	5.04	4.06	19.47
VG	5.00	4.00	20.00
Normal	5.09	3.97	22.06
$L^{(E.ON,TK)}$	$c(X_T^{ba})$	$c(\underline{X}_T^{ba})$	**Efficiency loss in %**
NIG	2.161	2.061	4.65
VG	2.158	2.052	4.92
Normal	2.160	2.086	3.40

130.55, $S_0^{\text{E.ON}} = 17.48$ and $S_0^{\text{TK}} = 16.73$. The weights are $w_1 = w_2 = 0.5$. The strike for Allianz and Volkswagen is $K = 110$, whereas for E.ON and Thyssen Krupp it is $K = 15$.

In Table 2 the prices for the long basket option and its cost-efficient counterpart as well as the efficiency loss for Allianz and Volkswagen and for E.ON and Thyssen Krupp in all three bivariate Lévy models as discussed in Section 4 are listed. As a result for the Allianz and Volkswagen case a substantial efficiency loss is observed for basket options while in the E.ON and Thyssen Krupp case the efficiency loss is more moderate. As shown in Hammerstein et al. [20, Proposition 2.3] for the one dimensional case a greater size of $|\bar{\theta}|$ leads to a higher efficiency loss. This effect can be seen from Tables 1 and 2 in our two dimensional examples as well. Thus, we expect that an analogous result also holds true in the multivariate setting in greater generality when dependent components are present.

6 Numerical issues

In order determine the risk-neutral Esscher parameter i.e. to solve the system of non-linear equations as in (2.5) we use numerical methods provided by the R program. In particular, the package `nleqslv` provides two algorithms for solving systems of non-linear equations with either a *Broyden* or a full *Newton* method. For further information we refer to Dennis and Schnabel [9] and the related documentaries.

For evaluation of multidimensional integrals over hypercubes we used the package `cubature`. The calculation of standard prices $c(X_T^{ba})$ needs about 10.1 seconds. Its absolute error lies in the region of 10^{-6}. The computational time becomes better if suitable starting values and hypercubes are chosen. For the cost-efficient versions $c(\underline{X}_T^{ba})$ the calculation is more involved and needs significantly more time. The running time varies from 270 to 1.560 seconds depending on the particular Lévy model, accuracy, starting value, and hypercube chosen if the number of simulations (of the bivariate Lévy process) is at most 10^5. For a higher number of simulations, and thus a more accurate calculation, the time of calculation of such prices can grow substantially.

7 Conclusion

In this paper we adapt and develop the techniques necessary to determine cost-efficient payoffs in the case of multivariate exponential Lévy asset models when pricing is based on the Esscher martingale measure. We show that all calculations are doable for certain classes of multivariate Lévy models as *NIG*, *VG* or in the normal case. As application we determine cost-efficient payoffs generating the same payoff distribution as the inefficient basket options when pricing with the Esscher pricing measure. We describe the influence and effect of dependence between the components of the Lévy model to the pricing of the cost efficient payoffs in the example of basket options which implies that the relative trend of a stock may switch in the joint model and lead to greater improvements compared to the construction of efficient claims in one dimensional models. As a result we obtain that the efficiency loss can be considerable indicating that the use of cost-efficient payoffs may be profitable. It is expected, that as in the one-dimensional Lévy case also in the multivariate case cost efficient options behave favourably concerning the hedging behaviour but this still has to be explored. In the one-dimensional case it has been shown in [20] and [35] concerning Δ-hedging as well as the basic options. Extensions of the cost efficiency method to empirical pricing measures and a heuristic approach to the choice of the payoff distribution have been given in a recent paper in [28] in the case of one-dimensional Lévy processes. An extension to the multivariate case is subject of subsequent work.

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