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Research Article

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Approximation results in Orlicz spaces by modified sampling Kantorovich operators

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Abstract: This paper deals with the modified sampling Kantorovich operators. We start by presenting the primary notations of Orlicz spaces and fundamental auxiliary results. To show modular convergence of the corresponding operators in Orlicz spaces, we obtain well definiteness in Orlicz spaces and norm convergence in the space of continuous functions with compact support. In the last section, we present some examples of ρ-kernels which satisfy the corresponding assumptions and present some graphical representations.

Keywords: sampling Kantorovich operators; modified sampling operators; Orlicz spaces; modular convergence

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1 Introduction

Bernstein polynomials play a significant role in approximation theory, primarily because they provide a constructive proof of the classical Weierstrass approximation theorem (see [1]). In order to enhance the rate of convergence to the desired function and reduce the approximation error, King [2] proposed a modification to the traditional Bernstein polynomials for functions $f \in C[0,1]$. This approach involves a sequence of continuous functions (r_n) defined on [0,1], where each $r_n(x)$ satisfies $0 \le r_n(x) \le 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Later, in [3], the authors introduced another variation of the Bernstein operators for $f \in C[0,1]$ as follows:

$$\left(B_n\left(f \circ \rho^{-1}\right)\right)(\rho(x)) = \sum_{k=0}^{n} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}(\rho(x))^k(1-\rho(x))^{n-k}$$

where a particular function $\rho : [0,1] \to \mathbb{R}$ is utilized under certain appropriate conditions.

On the other hand, to establish an approximation scheme valid over the entire real line, Butzer and his research group in Aachen introduced the concept of generalized sampling operators, defined by

$$\left(G_w^{\chi} f\right)(x) := \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \;\; x \in \mathbb{R}, w > 0,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ denotes a specific function that fulfills certain conditions, and $f : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function on $\mathbb{R}$. A broad range of works has examined these generalized sampling operators and

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their variants; see [4–25]. To deal with the approximation of functions that are not necessarily continuous, Bardaro et al. [26] developed an L^1-based version of these sampling operators by substituting the pointwise samples $f\left(\frac{k}{w}\right)$ with the average values $w\int_{k/w}^{(k+1)/w} f(u)\mathrm{d}u$. This led to the formulation of sampling Kantorovich operators, expressed as

$$\left(K_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k)\, w \int_{k/w}^{(k+1)/w} f(u)\mathrm{d}u, \;\; x\in\mathbb{R},$$

where f is a locally integrable function and χ serves as a kernel function. The sampling Kantorovich operators have been extensively investigated across various functional settings; see [27–33].

In a very recent study [34], the authors proposed a modified version of the generalized sampling series, given by

$$\begin{aligned}\left(G_w^{\chi,\rho} f\right)(x) &= \left[G_w^{\chi}\left(f\circ\rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k\in\mathbb{Z}} \left(f\circ\rho^{-1}\right)\left(\frac{k}{w}\right)\chi(w\rho(x)-k), \quad x\in\mathbb{R},\; w>0,\end{aligned}$$

where χ is an appropriately chosen kernel function. In their work, they explored the approximation capabilities of this operator in both the space of continuous functions and in weighted spaces of functions. Moreover, by employing a Voronovskaja-type result (see [34, Theorem 5]), they compared the approximation efficiency of the classical generalized sampling operators with that of their newly defined operators and demonstrated that the latter offers improved approximation under suitable conditions.

Inspired by the effectiveness of this new framework and the widespread use of sampling Kantorovich operators, the same group of authors extended their approach in [35] by introducing a modified version of the sampling Kantorovich operators, defined as

$$\begin{aligned}\left(K_w^{\chi,\rho} f\right)(x) &= \left[K_w^{\chi}\left(f\circ\rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k\in\mathbb{Z}} \chi(w\rho(x)-k)\, w \int_{k/w}^{(k+1)/w} \left(f\circ\rho^{-1}\right)(u)\mathrm{d}u, \quad x\in\mathbb{R},\; w>0,\end{aligned}$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is a suitable kernel and $f\circ\rho^{-1}:\mathbb{R}\to\mathbb{R}$ is a locally integrable function. They analyzed the approximation features of these operators in both standard continuous function spaces and weighted spaces of continuous functions.

This paper is structured as follows: Section 2 outlines the essential notations and preliminaries. Section 3 provides a summary of the operators along with some fundamental results. In Section 4, we focus on investigating the approximation behavior of the corresponding operators within Orlicz spaces. Final section is devoted to present some examples of ρ-kernels which satisfy corresponding assumptions and present some graphical representations.

2 Basic notations and preliminaries

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ to denote the sets of positive integers, integers, and real numbers, respectively. The set of all non-negative real numbers is denoted by $\mathbb{R}_0^+$. The space $C(\mathbb{R})$ refers to the set of all continuous functions defined on $\mathbb{R}$, which are not necessarily bounded. The subset $CB(\mathbb{R})$ consists of all functions in $C(\mathbb{R})$ that are bounded, equipped with the norm $\|f\| := \sup_{x\in\mathbb{R}}|f(x)|$. Additionally, we denote by $UC(\mathbb{R})$ the space of functions in $CB(\mathbb{R})$ that are uniformly continuous, and by $C_c(\mathbb{R})$ the space of functions with compact support. The symbol $M(\mathbb{R})$ stands for the set of all (Lebesgue) measurable real or complex functions, while $L^\infty(\mathbb{R})$ represents the space of essentially bounded measurable functions on $\mathbb{R}$.

Let $\rho: \mathbb{R} \to \mathbb{R}$ be a strictly increasing function that satisfies the following conditions:

(ρ_1) ρ belongs to $C(\mathbb{R})$;
(ρ_2) $\rho(0) = 0$ and $\lim_{x\to\pm\infty}\rho(x) = \pm\infty$;
(ρ_3) $\rho'(x) > 1$ holds for every $x \in \mathbb{R}$
(ρ_4) $\rho' \in L^\infty(\mathbb{R})$.

Remark 1. It is possible to find many examples of ρ functions which satisfy the conditions $(\rho_1) - (\rho_4)$. For example, we state $\rho(x) = \alpha x$ for all $\alpha > 1$ and $\rho(x) = 2x + \tanh\, x$.

2.1 Context of Orlicz spaces

Let $\eta: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex η-function, that is, a function satisfying the following conditions:

i. $\eta(0) = 0$, $\eta(u) > 0$ for all $u > 0$, and η is non-decreasing on $\mathbb{R}_0^+$;
ii. η is convex on $\mathbb{R}_0^+$.

We now define the modular functional

$$I^\eta[f] := \int_{\mathbb{R}} \eta(|f(x)|)\,\mathrm{d}x \qquad (f \in M(\mathbb{R})).$$

It is well known that I^η is a convex modular on the space $M(\mathbb{R})$ (see [36–38]), and it generates the Orlicz space defined by

$$L^\eta(\mathbb{R}) := \{f \in M(\mathbb{R}) : I^\eta[\lambda f] < +\infty \text{ for some } \lambda > 0\}.$$

The Orlicz space $L^\eta(\mathbb{R})$ is a vector space, and its subspace

$$E^\eta(\mathbb{R}) := \{f \in M(\mathbb{R}) : I^\eta[\lambda f] < +\infty \text{ for every } \lambda > 0\}$$

is called the space of all finite elements of $L^\eta(\mathbb{R})$. In general, $E^\eta(\mathbb{R})$ is a proper subspace of $L^\eta(\mathbb{R})$; however, the two spaces coincide if and only if the so-called Δ_2-condition is satisfied, that is,

$$\eta(2u) \le M\eta(u), \quad u \in \mathbb{R}_0^+$$

for some constant $M > 0$. Examples of η-functions satisfying the Δ_2-condition can be found in [26,36,37].

In the context of the $L^\eta(\mathbb{R})$ space, we can consider two distinct notions of convergence for net of functions. The first, named "modular convergence" describes the asymptotic behavior of a sequence $(f_w)_{w>0}$ within $L^\eta(\mathbb{R})$ approaching a limit function f through the modular functional I^η. Namely, $(f_w)_{w>0}$ is said to modularly convergent to f if for some $\lambda > 0$,

$$\lim_{w\to+\infty} I^\eta[\lambda(f_w - f)] = 0$$

holds. This notion induces a topology in $L^\eta(\mathbb{R})$, known as the "modular topology".

Furthermore, another concept of convergence, called "norm convergence" can be established by introducing a norm on $L^\eta(\mathbb{R})$, referred as the Luxemburg norm. This norm, denoted by $\|\cdot\|_\eta$, is defined as

$$\|f\|_\eta := \inf\{\lambda > 0 : I^\eta[f/\lambda] \le 1\}.$$

As a result, norm convergence represents a stronger notion of convergence than modular convergence in $L^\eta(\mathbb{R})$ (convergence in the sense of the norm $\|\cdot\|_\eta$ generated by η of a sequence (f_n) to f implies its modular convergence to f). Easily, for a net of functions $(f_w)_{w>0}$ in $L^\eta(\mathbb{R})$, the convergence $\lim_{w\to+\infty}\|f_w - f\|_\eta = 0$ is equivalent to the condition $\lim_{w\to+\infty} I^\eta[\lambda(f_w - f)] = 0$ for all $\lambda > 0$. However, these two notions of convergence become coincide only if the function η satisfies the Δ_2 condition. For further details concerning these spaces, see, e.g., [36–40].

3 Construction of the operators and some basic results

For any $\beta \geq 0$, absolute moment of order β associated with ρ (or simply ρ-absolute moment) is given by

$$M_\beta^\rho(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(\rho(u)-k)||k-\rho(u)|^\beta,\ u \in \mathbb{R}.$$

Definition 1. Throughout the paper, a function $\chi : \mathbb{R} \to \mathbb{R}$ will be called a kernel associated with ρ (or simply ρ-kernel) if it satisfies the following assumptions:

(χ1) $\chi \in C(\mathbb{R})$;

(χ2) for every $u \in \mathbb{R}$

$$m_0^\rho(\chi, u) = \sum_{k\in\mathbb{Z}} \chi(\rho(u)-k) = 1;$$

(χ3) for $\beta > 0$, ρ-absolute moment of χ is finite, that is

$$M_\beta^\rho(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(\rho(u)-k)||k-\rho(u)|^\beta < +\infty.$$

By ψ, we will denote the class of all functions satisfying the assumptions (χ1), (χ2) and (χ3). Now, we recall a family of Kantorovich type sampling operators so called modified sampling Kantorovich series given by

$$\begin{aligned}\left(K_w^{\chi,\rho} f\right)(x) &= \left[K_w^\chi\left(f\circ\rho^{-1}\right)\right](\rho(x)) \\ &:= \sum_{k\in\mathbb{Z}} \chi(w\rho(x)-k) w \int_{k/w}^{(k+1)/w} \left(f\circ\rho^{-1}\right)(u)\mathrm{d}u,\ x\in\mathbb{R}, w>0\end{aligned} \tag{3.1}$$

for $\chi \in \psi$ and locally integrable functions $f \circ \rho^{-1}$.

Now, we state a remark which was proved in [26] for classical moments of kernel functions but it is trivial to adapt to ρ-moments of kernel functions:

Remark 2. For all functions χ belongs to ψ, we have

i. $M_0^\rho(\chi) < +\infty$;

ii. for every $\delta > 0$

$$\lim_{w\to+\infty} \sum_{|k-w\rho(x)|\geq w\delta} |\chi(w\rho(x)-k)| = 0$$

uniformly with respect to $x \in \mathbb{R}$;

iii. For every $\delta > 0$ and $\varepsilon > 0$, there exists a constant $M > 0$ such that

$$\int_{|\rho(x)|>M} w|\chi(w\rho(x)-k)|\mathrm{d}x < \varepsilon$$

for every sufficiently large $w > 0$ and $k/w \in [-\delta, \delta]$.

We also note that, for $\nu, \gamma > 0$ with $\nu < \gamma$, $M_\gamma^\rho(\chi) < +\infty$ implies $M_\nu^\rho(\chi) < +\infty$. When χ has compact support, we immediately have that $M_\gamma^\rho(\chi) < +\infty$ for every $\gamma > 0$, see [41].

Remark 3 ([35]).

i. The operator (3.1) is well-defined if, for example, $f \in L^{\infty}(\mathbb{R})$.

ii. Let $\chi \in \psi$ be a ρ-kernel. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded function, then

$$\lim_{w \to \infty} \left(K_w^{\chi,\rho} f \right)(t) = f(t)$$

holds at each continuity point $t \in \mathbb{R}$ of f. Moreover, if $f \circ \rho^{-1} \in UC(\mathbb{R})$, then we have

$$\lim_{w \to +\infty} \| K_w^{\chi,\rho} f - f \|_{\infty} = 0.$$

The proof of above items could be found in the corresponding reference.

Lemma 1. *Let $f \in L^{\eta}(\mathbb{R})$. Then, we have $f \circ \rho^{-1} \in L^{\eta}(\mathbb{R})$.*

Proof. By the assumption $f \in L^{\eta}(\mathbb{R})$, we know that for some $\lambda > 0$, $I^{\eta}[\lambda f] < +\infty$. We aim to show that

$$I^{\eta}\left[\lambda\left(f \circ \rho^{-1}\right)\right] = \int_{\mathbb{R}} \eta\left(\left|\lambda f(\rho^{-1}(x))\right|\right) \mathrm{d}x < +\infty.$$

Using change of variables $x = \rho(t)$, we can write

$$I^{\eta}\left[\lambda\left(f \circ \rho^{-1}\right)\right] = \int_{\mathbb{R}} \eta\left(|\lambda f(t)|\right) \rho'(t) \mathrm{d}t.$$

Since $\rho' \in L^{\infty}(\mathbb{R})$, let $\tilde{C} := \|\rho'\|_{\infty} < +\infty$. Then, by assumption

$$I^{\eta}\left[\lambda\left(f \circ \rho^{-1}\right)\right] \le \tilde{C} \int_{\mathbb{R}} \eta(|\lambda f(t)|) \mathrm{d}t = \tilde{C} I^{\eta}[\lambda f] < +\infty.$$

Thus, we have $f \circ \rho^{-1} \in L^{\eta}(\mathbb{R})$. □

Lemma 2. *Let $\chi \in \psi$ be a ρ-kernel. Then, $\chi \in L^1(\mathbb{R})$.*

Proof. As in [6, (2.3)], we have with $a := \rho^{-1}(0)$ and $b := \rho^{-1}(1)$

$$\begin{aligned} \int_{\mathbb{R}} |\chi(u)| \, \mathrm{d}u &= \sum_{k \in \mathbb{Z}} \int_0^1 |\chi(u-k)| \, \mathrm{d}u = \sum_{k \in \mathbb{Z}} \int_a^b |\chi(\rho(v) - k)| \, \mathrm{d}\rho(v) \\ &= \sum_{k \in \mathbb{Z}} \int_a^b |\chi(\rho(v) - k)| \rho'(v) \, \mathrm{d}v \\ &\le \|\rho'\|_{\infty} \sum_{k \in \mathbb{Z}} \int_a^b |\chi(\rho(v) - k)| \, \mathrm{d}v \\ &= \|\rho'\|_{\infty} \int_a^b \sum_{k \in \mathbb{Z}} |\chi(\rho(v) - k)| \, \mathrm{d}v < \infty. \end{aligned}$$

□

4 Convergence of $K_w^{\chi,\rho}$ in $L^\eta(\mathbb{R})$

In this section, we present convergence of $K_w^{\chi,\rho}$ in $L^\eta(\mathbb{R})$. Firstly, we start with the well-definiteness of the operators.

Theorem 1. *Let $f \in L^\eta(\mathbb{R})$ and $\chi \in \psi$ be a ρ-kernel. Then the operator $K_w^{\chi,\rho}$ maps $L^\eta(\mathbb{R})$ in $L^\eta(\mathbb{R})$. Indeed,*

$$I^\eta\left[\lambda K_w^{\chi,\rho} f\right] \leq \frac{1}{M_0^\rho(\chi)} \|\chi\|_1 I^\eta\left[\lambda M_0^\rho(\chi)\left(f \circ \rho^{-1}\right)\right]$$

for some $\lambda > 0$.

Proof. By Lemma 1, we have $f \circ \rho^{-1} \in L^\eta(\mathbb{R})$. This means that, there exist $\widetilde{\lambda} > 0$ such that $I^\eta\left[\widetilde{\lambda}\left(f \circ \rho^{-1}\right)\right] < +\infty$. Choosing $\lambda > 0$ such that $\lambda M_0^\rho(\chi) \leq \widetilde{\lambda}$, then applying Jensen's inequality twice, we get

$$\begin{aligned} I^\eta\left[\lambda K_w^{\chi,\rho} f\right] &= \int_{\mathbb{R}} \eta\left(\lambda\left|\left(K_w^{\chi,\rho} f\right)(x)\right|\right) \mathrm{d}x \\ &\leq \int_{\mathbb{R}} \eta\left(\lambda\left[\sum_{k\in\mathbb{Z}} |\chi(w\rho(x)-k)| w \int_{k/w}^{(k+1)/w} \left|\left(f \circ \rho^{-1}\right)(u)\right| \mathrm{d}u\right]\right) \mathrm{d}x \\ &\leq \frac{1}{M_0^\rho(\chi)} \sum_{k\in\mathbb{Z}} \left\{ \int_{k/w}^{(k+1)/w} \eta\left(\lambda M_0^\rho(\chi)\left|\left(f \circ \rho^{-1}\right)(u)\right|\right) \mathrm{d}u \int_{\mathbb{R}} w|\chi(w\rho(x)-k)| \mathrm{d}x \right\}. \end{aligned}$$

Now, using change of variable $w\rho(x) - (k) = v$ in the last integral and (ρ_3), we get

$$\begin{aligned} I^\eta\left[\lambda K_w^{\chi,\rho} f\right] &\leq \frac{1}{M_0^\rho(\chi)} \sum_{k\in\mathbb{Z}} \left\{ \int_{k/w}^{(k+1)/w} \eta\left(\lambda M_0^\rho(\chi)\left|\left(f \circ \rho^{-1}\right)(u)\right|\right) \mathrm{d}u \right\} \|\chi\|_1 \\ &\leq \frac{1}{M_0^\rho(\chi)} \|\chi\|_1 I^\eta\left[\lambda M_0^\rho(\chi)\left(f \circ \rho^{-1}\right)\right] \\ &< \frac{1}{M_0^\rho(\chi)} \|\chi\|_1 I^\eta\left[\widetilde{\lambda}\left(f \circ \rho^{-1}\right)\right] < +\infty \end{aligned}$$

which is desired. □

Theorem 2. *Let $f \circ \rho^{-1} \in C_c(\mathbb{R})$, $\chi \in \psi$ be a ρ-kernel. Then,*

$$\lim_{w\to+\infty} \left\|K_w^{\chi,\rho} f - f\right\|_\eta = 0. \tag{4.1}$$

Proof. In order to establish the limit in (4.1), it suffices to show that for each fixed $\lambda > 0$

$$\lim_{w\to+\infty} \int_{\mathbb{R}} \eta\left(\lambda\left|K_w^{\chi,\rho} f(x) - f(x)\right|\right) \mathrm{d}x = 0,$$

or, equivalently, that the family of functions

$$\alpha_w(x) = \eta\left(\lambda\left|K_w^{\chi,\rho} f(x) - f(x)\right|\right)$$

converges to zero in $L^1(\mathbb{R})$. To prove this, we invoke the Vitali convergence theorem by verifying the following three properties:

(V_1) $\left(\alpha_w\right)_{w>0}$ converges in measure to zero;

(V_2) For every $\varepsilon > 0$ there exists a measurable set $E_\varepsilon \subset \mathbb{R}$ with $|E_\varepsilon| < \infty$ such that, for any measurable F disjoint from E_ε,

$$\int_F \alpha_w(x)\,\mathrm{d}x < \varepsilon \quad \text{for all sufficiently large } w.$$

(V_3) For each $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $E \subset \mathbb{R}$ is measurable with $|E| < \delta$,

$$\int_E \alpha_w(x)\,\mathrm{d}x < \varepsilon \quad \text{for all sufficiently large } w.$$

Having established (V_1)–(V_3), Vitali's theorem guarantees that $\alpha_w \to 0$ in $L^1(\mathbb{R})$, which completes the proof of (4.1). (V_1) follows directly from Remark 3 (ii) and continuity of η. To verify property (V_2), fix $\varepsilon > 0$ and choose $\tilde{\gamma} > 0$ and $\gamma > \tilde{\gamma} + 1/w$ so that

$$\operatorname{supp} f \subset B(0, \tilde{\gamma}).$$

Observing that $\left[\frac{k}{w}, \frac{k+1}{w}\right] \cap B(0, \tilde{\gamma}) = \emptyset$ whenever $k \notin B(0, w\gamma)$, we can write

$$\int_{k/w}^{(k+1)/w} |f(u)|\,\mathrm{d}u = 0.$$

Next, by Remark 2 (iii) there exists $M > 0$ (which we may take, w.l.o.g, larger than $\tilde{\gamma}$) such that for every sufficiently large w and for all $k \in B(0, w\gamma)$ one has

$$\int_{|\rho(x)|>M} w\left|\chi\left(w\,\rho(x) - k\right)\right|\,\mathrm{d}x < \varepsilon.$$

By Jensen's inequality and Fubini-Tonelli theorem, we can write what follows:

$$\begin{aligned}
&\int_{|\rho(x)|>M} \eta\left(\lambda\left|(K_w^{\chi,\rho})(x)\right|\right)\mathrm{d}x \\
&\quad \le \sum_{|k|\le w\gamma} \frac{\eta\left(\lambda M_0^\rho(\chi)\|f\|_\infty\right)}{M_0^\rho(\chi)w} \int_{|\rho(x)|>M} w|\chi(w\rho(x) - k)|\mathrm{d}x \\
&\quad < \varepsilon \cdot \frac{\eta\left(\lambda M_0^\rho(\chi)\|f\|_\infty\right)}{M_0^\rho(\chi)w} \cdot L,
\end{aligned}$$

where L is the number of integers k such that $|k| \le w\gamma$. For every $w \ge 1$, we have

$$L \le [2(\gamma w + 1)] =: L',$$

where $[\cdot]$ denotes the integer part. Thus,

$$\int_{|\rho(x)|>M} \eta\left(\lambda\left|(K_w^{\chi,\rho})(x)\right|\right)\mathrm{d}x < \varepsilon \cdot \frac{\eta\left(\lambda M_0^\rho(\chi)\|f\|_\infty\right)}{M_0^\rho(\chi)w} \cdot L' =: \varepsilon \cdot C$$

for every $w \ge 1$. Therefore, for $\varepsilon > 0$ we set $\tilde{E}_\varepsilon = B(0, M)$. Then for every measurable set $\tilde{F}$, with $\tilde{F} \cap \tilde{E}_\varepsilon = \emptyset$, we have

$$\int_{\tilde{F}} \eta\Big(\lambda\Big|(K_w^{\chi,\rho})(x) - f(x)\Big|\Big)\mathrm{d}x = \int_{\tilde{F}} \eta\Big(\lambda\Big|(K_w^{\chi,\rho})(x)\Big|\Big)\mathrm{d}x$$
$$\leq \int_{|\rho(x)|>M} \eta\Big(\lambda\Big|(K_w^{\chi,\rho})(x)\Big|\Big)\mathrm{d}x$$
$$< \varepsilon \cdot C.$$

Finally, concerning (V_3), for any fixed $\varepsilon > 0$, there is a measurable set B_ε with

$$|B_\varepsilon| < \frac{\varepsilon}{M_0^\rho(\chi)\eta(\lambda\|f\|_\infty)}$$

for $\|f\|_\infty > 0$. Then, using Remark 3 we have for every $w > 0$

$$\int_{B_\varepsilon} \eta\Big(\lambda\Big|(K_w^{\chi,\rho} f)(x)\Big|\Big)\mathrm{d}x \leq \int_{B_\varepsilon} \eta(\lambda M_0^\rho(\chi)\|f\|_\infty) < \varepsilon.$$

This means that, the integrals

$$\int_{(\cdot)} \eta\Big(\lambda\Big|(K_w^{\chi,\rho} f)(x) - f(x)\Big|\Big)\mathrm{d}x$$

are equi-absolutely continuous. Since $\lambda > 0$ is arbitrary, we have desired. □

As a final and the main result of this section, we mention the modular convergence of the operators $K_w^{\chi,\rho}$.

Theorem 3. *$f \in L^\eta(\mathbb{R})$ and $\chi \in \psi$ be a ρ-kernel. Then, there exist $\lambda > 0$ such that*

$$\lim_{w\to+\infty} I^\eta\big[\lambda(K_w^{\chi,\rho} f - f)\big] = 0.$$

Proof. Using Lemma 1 we know that both f and $f \circ \rho^{-1}$ belong to $L^\eta(\mathbb{R})$. Let $\varepsilon > 0$ be fixed. By density of $C_c(\mathbb{R})$ in $L^\eta(\mathbb{R})$, there exist λ' and function $g \in C_c(\mathbb{R})$ such that

$$I^\eta\big[\lambda'(f - g)\big] < \frac{\varepsilon}{\|\rho'\|_\infty} \tag{4.2}$$

Let $\tilde{g} := g \circ \rho^{-1}$. It is easy to see that $\tilde{g}$ is continuous and has compact support because of properties of ρ. So we can write $f \circ \rho^{-1} - \tilde{g} = (f - g) \circ \rho^{-1}$. Using change of variable and (ρ_4), we have

$$I^\eta\big[\lambda'(f \circ \rho^{-1} - \tilde{g})\big] = I^\eta\big[\lambda'(f - g) \circ \rho^{-1}\big]$$
$$= \int_{\mathbb{R}} \eta\Big(\Big|\lambda'(f - g)(t)\Big|\Big)\rho'(t)\mathrm{d}t$$
$$\leq \|\rho'\|_\infty I^\eta\big[\lambda'(f - g)\big] < \varepsilon.$$

Now, let us set $\lambda > 0$ such that $3\lambda M_0^\rho(\chi) \leq \lambda'$. Considering properties of η, (4.2) and Theorem 1, we get

$$I^\eta\big[\lambda(K_w^{\chi,\rho} f - f)\big]$$
$$\leq I^\eta\big[3\lambda(K_w^{\chi,\rho} f - K_w^{\chi,\rho} g)\big] + I^\eta\big[3\lambda(K_w^{\chi,\rho} g - g)\big] + I^\eta\big[3\lambda(f - g)\big]$$
$$\leq \frac{1}{M_0^\rho(\chi)}\|\chi\|_1 I^\eta\big[\lambda'(f - g) \circ \rho^{-1}\big] + I^\eta\big[3\lambda(K_w^{\chi,\rho} g - g)\big] + I^\eta\big[3\lambda(f - g)\big]$$
$$< \left(\frac{1}{M_0^\rho(\chi)}\|\chi\|_1 + \frac{1}{\|\rho'\|_\infty}\right)\varepsilon + I^\eta\big[3\lambda(K_w^{\chi,\rho} g - g)\big].$$

The assertion follows from Theorem 2 for enough large w. □

5 Examples of ρ-kernels and graphical representations

In this section, we provide concrete examples of ρ-kernels that satisfy the assumptions outlined in Definition 1. We then present graphical comparisons to illustrate the approximation behavior of the modified sampling Kantorovich operators $K_w^{\chi,\rho}$ versus the classical operators K_w^{χ}.

5.1 Examples of ρ-kernels

A crucial step in applying the theory is finding suitable kernel functions. The condition $(\chi 2)$, also known as the partition of unity condition, can be conveniently verified using the Fourier transform and the Poisson summation formula (see [4, p. 123 and Section 5.1.5], [42]). Using Poisson summation formula $(\chi 2)$ is equivalent to

$$\tilde{\chi}(2\pi k) = \begin{cases} 1, & k = 0 \\ 0, & k \in \mathbb{Z}\backslash\{0\} \end{cases}$$

where $\hat{\chi}(v) = \int_{\mathbb{R}} \chi(u) e^{-ivu} \mathrm{d}u$, $v \in \mathbb{R}$.

5.1.1 The central B-spline kernel

A highly effective and widely used family of kernels are the central B-splines of order $n \in \mathbb{N}$, defined by:

$$\sigma_n(t) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{n}{2} + t - j\right)_+^{n-1}, \quad t \in \mathbb{R}$$

where $(t)_+ := \max\{t, 0\}$.

These functions are ideal because they have compact support $[-n/2, n/2]$, which guarantees that the moment condition $(\chi 3)$ holds for any $\beta > 0$. The partition of unity condition $(\chi 2)$ is also satisfied.

Modified sampling Kantorovich operators of $f \in L^\eta(\mathbb{R})$ takes on the form

$$\left(K_w^{\sigma_n,\rho} f\right)(x) = \sum_{k\in\mathbb{Z}} \sigma_n(w\rho(x) - k) w \int_{k/w}^{(k+1)/w} \left(f \circ \rho^{-1}\right)(u) \mathrm{d}u, \ x \in \mathbb{R}, w > 0$$

using B-Spline kernel.

5.1.2 The Fejér kernel

Another important example is the Fejér kernel, which is not compactly supported but decays quickly. It is defined as:

$$F(t) := \frac{1}{2} \operatorname{sinc}^2\left(\frac{t}{2}\right), \quad t \in \mathbb{R}$$

This kernel also satisfies all the necessary assumptions to be a ρ-kernel, see [26].

Modified sampling Kantorovich operators of $f \in L^\eta(\mathbb{R})$ takes on the form

$$\left(K_w^{F,\rho} f\right)(x) = \sum_{k\in\mathbb{Z}} F(w\rho(x) - k) w \int_{k/w}^{(k+1)/w} \left(f \circ \rho^{-1}\right)(u) \mathrm{d}u, \ x \in \mathbb{R}, w > 0$$

using Fejér's kernel.

5.2 Graphical comparisons

This subsection provides a comparative analysis of the modified and classical sampling Kantorovich operators. The comparison is illustrated through graphical representations, which are generated using the central B-spline kernel of order 5. Throughout the examples, we consider the function $\overline{\rho}: \mathbb{R} \to \mathbb{R}$ defined as $\overline{\rho}(t) = 2t + \tanh(t)$. It is easy to see that function $\overline{\rho}$ is continuous on $\mathbb{R}$, $\overline{\rho}(0) = 0$ and $\lim_{t\to\pm\infty} \overline{\rho}(t) = \pm\infty$. The derivative of $\overline{\rho}$, $\overline{\rho}'(t) = 2 + \operatorname{sech}^2(t)$ and its range is (2,3]. So, $\overline{\rho}'(t) > 1$ for all $t \in \mathbb{R}$ and $\overline{\rho}' \in L^\infty(\mathbb{R})$. Hence, the conditions $(\rho_1) - (\rho_4)$ are all satisfied. We also consider functions $f: \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} -1, & \text{if } x < -2 \\ x + 2.5, & \text{if } -2 \le x < -1 \\ 0, & \text{if } -1 \le x < 0 \\ \cos(\pi x), & \text{if } 0 \le x \le 1 \\ 1, & \text{otherwise} \end{cases}$$

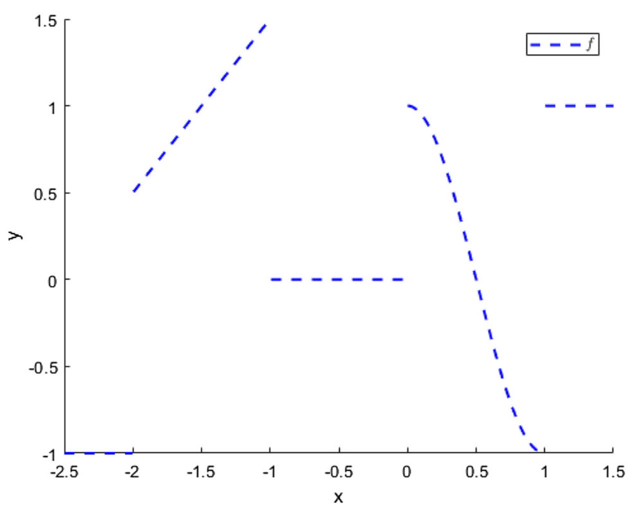

Figure 1: Graph of function f.

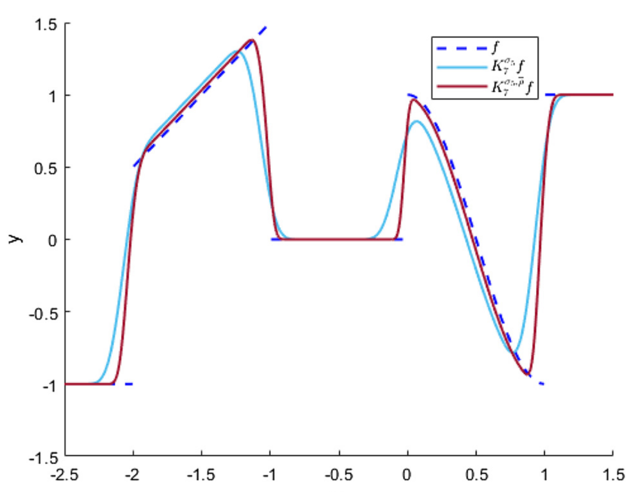

Figure 2: Graph of function f and operators $(K_7^{\sigma_5} f)$, $\left(K_7^{\sigma_5,\overline{\rho}} f\right)$.

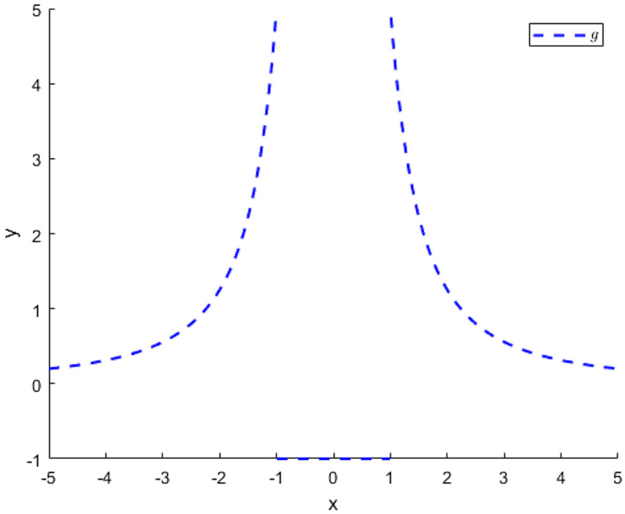

Figure 3: Graph of function g.

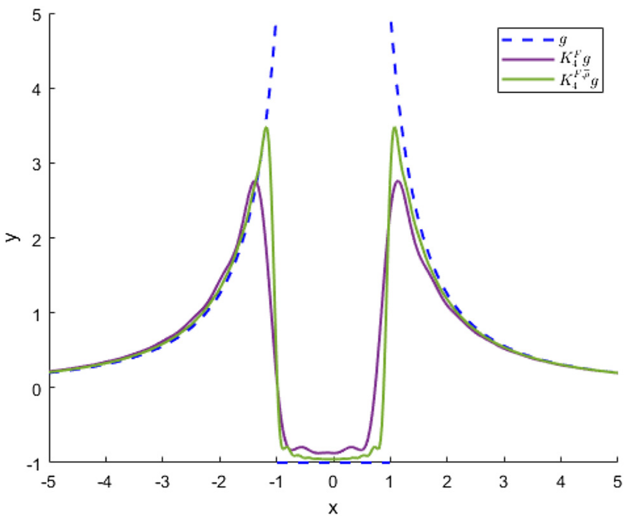

Figure 4: Graph of function f and operators $(K_4^F g)$, $\left(K_4^{F,\bar{\rho}} g\right)$.

and $g : \mathbb{R} \to \mathbb{R}$ as

$$g(t) = \begin{cases} -1, & \text{if } |t| < 1 \\ \dfrac{5}{t^2}, & \text{otherwise} \end{cases}$$

In Figure 1, the graph of function f is presented. In Figure 2, we compare classical sampling Kantorovich operators and modified sampling Kantorovich operators using $w = 7$ and 5th B-Spline kernel.

In Figure 3, the graph of function g is presented. In Figure 4, we compare classical sampling Kantorovich operators and modified sampling Kantorovich operators using $w = 4$ and Fejer's kernel.

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