Research Article

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The description of entire solutions of complex PDEs and PDDEs

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Abstract: By utilizing the Hadamard factorization theory and Nevanlinna theory of meromorphic functions in $\mathbb{C}^n$, we mainly give some description of the solutions of complex partial differential equation (PDE) $(au_{z_1}+bu_{z_2})(cu_{z_1}+du_{z_2})=e^g$, and complex partial differential-difference equation (PDDE) $[\alpha u(z+\eta)-\beta u(z)](au_{z_1}+bu_{z_2})=e^g$, where $u_x=\frac{\partial u}{\partial x}$, $\eta=(\eta_1,\eta_2)$, α,β,a,b,c,d are constants and g is a polynomial with two variables. Some results are obtained which are the description on the existence and the forms of the finite order transcendental entire solutions of the above equations. Meantime, some examples indicate that our results of this article are accurate to a certain extent.

Keywords: entire function; complex partial differential equations; Hadamard factorization theory; Nevanlinna theory

MSC 2020: 32W50; 35M10; 39B32

1 Introduction

The article is devoted to provide some descriptions of the solutions of several complex PDEs and PDDEs by using the Hadamard factorization theory and the Nevanlinna theory and its difference analogous theory in $\mathbb{C}^2$. We all know that the study of PDEs is very important and has always attracted much attention [1, 2]. However, there were few references focusing on the solutions of PDEs and PDDEs in the complex domain. In 1995, D. Khavinson [3] investigate the entire solution of eikonal equations and obtained

Theorem A (see [3]). *Let u be an entire solution of*

$$\left(u_{z_1}\right)^2+\left(u_{z_2}\right)^2=1 \tag{1.1}$$

in $\mathbb{C}^2$. Then u is a linear function, that is, $u=c_1z_1+c_2z_2+c_0$, where $c_1,c_2,c_0\in\mathbb{C}$, and $c_1^2+c_2^2=1$.

After that, the problem on the solutions of complex PDEs has attracted consideration attention of many mathematical scholars. For example, Khavinson, Lundberg, Chang, Chen investigated the entire solution of eikonal equations and their deformation equations including $\left(u^lu_z\right)^m\left(u^lu_w\right)^n=p(z,w)$, $\left(u^lu_z\right)^m\left(u^lu_w\right)^n=$

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$p(z,w)e^{q(z,w)}$, etc., the existence theorems and the forms of solutions of some equations were listed in Refs. [4–8]; Hemmati, Hu, Li, discussed the entire and meromorphic solution of some first order PDEs, and gave the forms of these equations (see [9–13]); Saleeby, Li, Lü studied the entire solutions of PDEs in $\mathbb{C}^n (n \geq 2)$, and obtained a large number of results (see [14–20]); Mandal, Biswas, Cao, Liu, Xu investigated the solutions of complex partial differential and difference equations and systems, and obtained a series of interesting results (see [21–26]) by using the Nevanlinna theory of meromorphic function with several complex variables (see [27, 28]). In 2005, Li [13] further obtained the following results by replacing the number 1 in equation (1.1) with a more general function e^g.

Theorem B (see [13]). *Let g be a polynomial in $\mathbb{C}^2$. Then u is an entire solution of*

$$\left(u_{z_1}\right)^2 + \left(u_{z_2}\right)^2 = e^g \tag{1.2}$$

in $\mathbb{C}^2$ if and only if
(i) $u = f(c_1 z_1 + c_2 z_2)$; *or*
(ii) $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$,

where f is an entire function in $\mathbb{C}$ such that

$$f'(c_1 z_1 + c_2 z_2) = \pm e^{g/2},$$

c_1 and c_2 are two constants satisfying $c_1^2 + c_2^2 = 1$, and ϕ_1 and ϕ_2 are entire functions in $\mathbb{C}$ and

$$\phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2) = \frac{1}{4}e^g.$$

Khavinson [3] and Li [29] pointed out that (1.2) can be transformed to

$$4U_x U_y = P, \tag{1.3}$$

where $P(x,y) = e^g$ and $U(x,y) = u(z_1,z_2)$, by taking a suitable linear transformation. Differentiating equation (1.3) with respect to x, y, respectively, we have that $4U_{xy}U_y = P_x - 4U_x U_{yx}$ and $U_{xy}U_y = P_y - 4U_x U_{yy}$. this leads to

$$16U_x U_y U_{xx} U_{yy} = P_x P_y - 4U_x P_y U_{xy} - 4P_x U_y U_{xy} + 16U_x U_y U_{xy}^2.$$

Applying a simple calculation for the above equation, one can deduce that

$$A\left(U_{xx}U_{yy} - U_{xy}^2\right) + BU_{xy} + C = 0, \tag{1.4}$$

where $A = U_x U_y$, $B = \frac{1}{4}(U_x P_y + U_y P_x)$ and $C = -\frac{1}{16}P_x P_y$. Equation (1.4) may be said as a Monge-Ampère equation. As it is known to all, Monge-Ampère equation usual appeared in variational methods, differential geometry, optimization problems, etc.

In 2018 and 2020, Xu-Cao [21, 22] discussed the forms and the existence of solutions of Fermat type PDEs by using the Nevanlinna theory and difference analogue results, and obtained

Theorem C (see [21]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any finite order transcendental entire solution of the Fermat type PDDE*

$$\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

is of the form $f(z_1,z_2) = \sin(Az_1 + Bz_2 + H(z_2))$, where A, B are constants on $\mathbb{C}$ satisfying $A^2 = 1$ and $Ae^{i(Ac_1 + Bc_2)} = 1$, and $H(z_2)$ is a polynomial in z_2 such that $H(z_2) = H(z_2 + c_2)$. Especially, whenever $c_1 \neq 0$, we have $f(z_1,z_2) = \sin(Az_1 + Bz_2 + \text{Const.})$.

Inspired by the above results and the notes of Khavinson [3] and Li [29], two following questions will be suggested

Question 1.1. What had happened on the solutions of the equation when the equation in Theorem C is transformed into product type partial differential equation?

Question 1.2. How to characterise the forms of the entire solutions of these equations when the partial derivative u_{z_1}, u_{z_2} of the equation in Theorems A and B is replaced by the linear combination of u_{z_1}, u_{z_2}?

2 Our main results

Motivated by the above problems, we aim to describe the existence and the forms of solutions of the product type nonlinear PDEs and PDDEs. If there is no special explanation, let $z = (z_1, z_2)$, $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$, $\alpha, \beta, \eta_1, \eta_2$ are constants in $\mathbb{C}$, $s := z_2 - \frac{b}{a}z_1$, $s_0 := \eta_2 - \frac{b}{a}\eta_1$, $s_1 := \eta_2 z_1 - \eta_1 z_2$, here and below.

Theorem 2.1. *Let $D = ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$. Let u be an entire solution of*

$$(au_{z_1} + bu_{z_2})(cu_{z_1} + du_{z_2}) = 1 \tag{2.1}$$

in $\mathbb{C}^2$, then u only is the form of

$$u = \frac{d\xi - b\xi^{-1}}{D}z_1 + \frac{a\xi^{-1} - c\xi}{D}z_2 + \xi_0,$$

where $\xi_0, \xi \in \mathbb{C}$, $\xi \neq 0$.

Remark 2.1. Taking $a = c = 1, b = -d = i$ in Theorem 2.1, then equation (2.1) become $u_{z_1}^2 + u_{z_2}^2 = 1$. Thus, one can get that $u = \xi_1 z_1 + \xi_2 z_2 + \xi_0$, where $\xi_1^2 + \xi_2^2 = 1$.

Theorem 2.2. *Let $D \neq 0$ and g be a nonconstant polynomial in $\mathbb{C}^2$, $a, b, c, d \in \mathbb{C}\backslash\{0\}$. Suppose that u is a transcendental entire solution with finite order of*

$$(au_{z_1} + bu_{z_2})(cu_{z_1} + du_{z_2}) = e^{g}. \tag{2.2}$$

(i) *If $g = g(z_1)$, then we have $u(z_1, z_2) = \frac{1}{q}F_1(z_1)$, where $F_1'(t) = e^{\frac{g(t)+\xi}{2}}$, $a = ce^{\xi}$;*
(ii) *If $g = g(z_2)$, then we have $u(z_1, z_2) = \frac{1}{b}F_2(z_2)$, where $F_2'(t) = e^{\frac{g(t)+\xi}{2}}$, $b = de^{\xi}$;*
(iii) *If $g = g\left(z_2 - \frac{b-de^{\xi}}{a-ce^{\xi}}z_1\right)$, then we have*

$$u(z_1, z_2) = \frac{ae^{-\xi} - c}{D}F_3\left(z_2 - \frac{b - de^{\xi}}{a - ce^{\xi}}z_1\right),$$

where $F_3'(t) = e^{\frac{g(t)+\xi}{2}}$, $\xi \in \mathbb{C}$;
(iv) *If $g = q(z_2 - \frac{b}{a}z_1) + p(z_2 - \frac{d}{c}z_1)$, then*

$$u(z_1, z_2) = \frac{1}{D}\left[aF_4\left(z_2 - \frac{b}{a}z_1\right) - cF_5\left(z_2 - \frac{d}{c}z_1\right)\right],$$

where $F_4'(t_1) = e^{q(t_1)}, F_5'(t_2) = e^{p(t_2)}$, p, q are polynomials, $t_1 = z_2 - \frac{b}{a}z_1$, $t_2 = z_2 - \frac{d}{c}z_1$.

Remark 2.2. Obviously, Eq. (1.2) is a special case of Eq. (2.2). In fact, we give the description of the solutions of the product type PDE (2.2) in Theorem 2.2 when $a_1 = a_2 = 0$, g is an arbitrary polynomial.

Examples show the existence of solutions of (2.2) in every case of Theorem 2.2.

Example 2.1. Let

$$u = \frac{1}{e}\int_0^{z_1} e^{\frac{t^2+1}{2}}\,\mathrm{d}t.$$

Obviously, u is transcendental and $\rho(u) = 2$. Moreover, u is a solution of (2.2) for the case $g = z_1^2$, $a = e$, $c = 1$, $b, d \in \mathbb{C}$.

Example 2.2. Let

$$u = \frac{1}{2}\int_0^{z_2} e^{\frac{t^3+t^2+\pi i}{2}}\,\mathrm{d}t.$$

Obviously, u is transcendental and $\rho(u) = 3$. Moreover, u is a solution of equation (2.2), where $g = z_2^3 + z_2^2$, $b = 2$, $d = -2$, $a, c \in \mathbb{C}$.

Example 2.3. Let

$$u = \frac{2-e}{e}\int_0^{z_2} e^{2\left(t-\frac{1-e}{2-e}z_1\right)^2+\left(t-\frac{1-e}{2-e}z_1\right)}\,\mathrm{d}t.$$

Obviously, u is transcendental and $\rho(u) = 2$. Moreover, u is a solution of equation (2.2), where $g = 4(z_2 - \frac{1-e}{2-e}z_1)^2 + 2(z_2 - \frac{1-e}{2-e}z_1)$, $a = 2$, $b = c = d = 1$.

Example 2.4. Let

$$u = -\int_0^{z_2} e^{t-2z_1}\,\mathrm{d}t + \int_0^{z_2} e^{(t-z_1)^2}\,\mathrm{d}t.$$

Obviously, u is transcendental and $\rho(u) = 2$. Moreover, u is a solution of equation (2.2), where $g = (z_2 - 2z_1) + (z_2 - z_1)^2$, $a = 1$, $b = 2$, $c = d = 1$.

Theorem 2.3. *Let $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$, $\eta \neq \{0, 0\}$ and α, β, a, b be nonzero constants. Let u be a finite order transcendental entire solution of equation*

$$[\alpha u(z+\eta) - \beta u(z)][au_{z_1} + bu_{z_2}] = 1, \tag{2.3}$$

then u must be one of the following forms

(i)

$$u(z_1, z_2) = \frac{z_1}{a}e^{-A\left(z_2-\frac{b}{a}z_1\right)-A_0} + \psi(s),$$

where $e^{As_0} = \frac{\alpha}{\beta}$, $A_0 \in \mathbb{C}$, $\psi(s)$ is a finite order transcendental entire function such that

$$\alpha\psi(s+s_0) - \beta\psi(s) = e^{As+A_0} - \frac{\beta\eta_1}{a}e^{-As-A_0};$$

(ii)

$$u(z_1, z_2) = \frac{1}{a\xi}z_1 + \frac{a\xi^2 - \eta_1\alpha}{\alpha a s_0 \xi}\left(z_2 - \frac{b}{a}z_1\right) + G(s),$$

and $\alpha = \beta$, where $\xi \in \mathbb{C}\backslash\{0\}$, $G(s)$ is a periodic entire function with s_0.

Example 2.5. Let

$$u = \frac{z_1}{3}e^{-\left(z_2-\frac{2}{3}z_1\right)\log 2} + \frac{1}{3}e^{\left(z_2-\frac{2}{3}z_1\right)\log 2} - \left(z_2 - \frac{2}{3}z_1\right)e^{-\left(z_2-\frac{2}{3}z_1\right)\log 2}.$$

Obviously, u is transcendental and $\rho(u) = 1$. Moreover, u is a solution of (2.3) for the case $a = 3$, $b = 2$, $\alpha = 2, \beta = 1$ and $\eta_1 = \eta_2 = 3$ and

$$\psi(s) = \frac{1}{3}e^{s\,\log 2} - se^{-s\,\log 2}.$$

Example 2.6. Let

$$u = z_1 + \frac{2}{3}(z_2 - 2z_1) + e^{\frac{2}{3}\pi i(z_2-2z_1)}.$$

Obviously, u is transcendental and $\rho(u) = 1$. Moreover, u is a solution of (2.3) for the case $a = 1$, $b = 2$, $\alpha = \beta = 1$ and $\eta_1 = -1, \eta_2 = 1, \xi = 1$.

Remark 2.3. Examples 2.5 and 2.6 show that the form in case (i) and (ii) of Theorem 2.3 are accurate, respectively.

By using the same argument as in Theorem 2.2, one can obtain the following corollary

Corollary 2.1. *Let $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2\backslash\{0, 0\}$, $\alpha(\neq 0), a, b \in \mathbb{C}$ such that $ab \neq 0$, then the equation*

$$\alpha u(z + \eta)[au_{z_1} + bu_{z_2}] = 1 \tag{2.4}$$

has no any entire solution.

Let us sketch the proof of the above corollary as follows.

Suppose that u is a nonconstant entire solution of equation (2.4), there exists an entire function p such that

$$\alpha u(z + \eta) = e^p, \quad au_{z_1} + bu_{z_2} = e^{-p}.$$

It follows that

$$ap_{z_1} + bp_{z_2} = \alpha e^{-p(z+\eta)}.$$

By using the basic Nevanlinna results (see e.g. [30, p. 99], [31]) or [32, Lemma 3.2], one can get that p is a constant. This leads to that u is a constant. This is a contradiction.

Suppose that u is a constant, it is easy to get a contradiction by a simple calculation.

In view of Theorem 2.1, Eq. (2.4) has no any entire solution. However, the following theorem shows that there exists entire solution when the number 1 in (2.4) is replaced by the function e^g, g is a nonconstant polynomial.

Theorem 2.4. *Let g be a nonconstant polynomial in $\mathbb{C}^2$ and $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2\backslash\{0, 0\}$, $a, b \in \mathbb{C}$ such that $ab \neq 0$. If u is a finite order transcendental entire solution of PDDE*

$$u(z + \eta)[au_{z_1} + bu_{z_2}] = e^g, \tag{2.5}$$

then u must be the form of

$$u(z_1, z_2) = e^{g - \frac{e^\xi}{a}z_1 - \varphi(s) - \xi},$$

where $\xi \in \mathbb{C}$, $\varphi(s)$ is a polynomial satisfying

$$\varphi(s) + \varphi(s - s_0) = g(z_1, z_2) - 2a^{-1}e^\xi z_1 - \xi + \eta_1 a^{-1}e^\xi.$$

Examples show that the conclusions on the forms in Theorem 2.4 are accurate.

Example 2.7. Let

$$u = e^{\varphi(z_2-2z_1)+z_1},$$

where $\varphi(z_2 - 2z_1)$ is a polynomial in $z_2 - 2z_1$. Obviously, u is transcendental and $\rho(u) < \infty$. Moreover, u is a solution of equation (2.5), where $a = 1$, $b = 2$, $\eta_1 = 2\pi i, \eta_2 = 4\pi i, \xi = 2\pi i, g = 2\varphi(z_2 - 2z_1) + 2z_1$.

Theorem 2.5. *Let g be a nonconstant polynomial in $\mathbb{C}^2$ and $a\eta_2 - b\eta_1 \neq 0$, where $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$ and $\alpha, \beta, a, b \in \mathbb{C}\backslash\{0\}$. If u is a finite order transcendental entire solution of PDDE*

$$[\alpha u(z+\eta) - \beta u(z)][au_{z_1} + bu_{z_2}] = e^g, \tag{2.6}$$

then u must be one of the following forms

(i)

$$u(z_1, z_2) = \chi(z)e^{A_1z_1+A_2z_2+A_0} + \psi(s),$$

where $\chi(z) = \frac{1}{aA_1+bA_2}$, if $aA_1 + bA_2 \neq 0$; $\chi(z) = \frac{z_1}{a}$ if $aA_1 + bA_2 = 0$; and $A_1z_1 + A_2z_2 = \frac{g-A_0-B_0}{2}$, $A_1, A_2, A_0, B_0 \in \mathbb{C}$ satisfy

$$\alpha e^{A_1\eta_1+A_2\eta_2} - \beta = (aA_1 + bA_2)e^{B_0-A_0}, \tag{2.7}$$

$\psi(s)$ a finite order transcendental entire solution in s. Further, $\psi(s)$ is a periodic function with s_0 if $\alpha = \beta$, and $\psi(s) = e^{\frac{\log\beta-\log\alpha}{s_0}s}$ if $\alpha \neq \beta$;

(ii)

$$u(z_1, z_2) = F\left(\frac{z_1}{a}, z_2 - \frac{b}{a}z_1\right) + \phi\left(z_2 - \frac{b}{a}z_1\right),$$

where $F(t,s) = \int_0^t e^{Q(t,s)}\mathrm{d}t$, $Q(t,s) = q(at, bt+s)$, $q(z_1,z_2) = A_1z_1 + A_2z_2 + \varphi(\eta_2z_1 - \eta_1z_2)$, $A_1, A_2 \in \mathbb{C}$, $\varphi(s_1)$ is a polynomial in s_1 satisfying $e^{A_1\eta_1+A_2\eta_2} = \frac{\beta}{\alpha}$ and

$$\alpha\phi(s+s_0) - \beta\phi(s) = e^{p(s)} + \beta F(t,s) - \alpha F(t, s+s_0), \quad p(z_1,z_2) + q(z_1,z_2) = g(z_1,z_2). \tag{2.8}$$

We provide some examples to show that the solutions of (2.7) for every cases in Theorem 2.5 can occur.

Example 2.8. Let

$$u = \frac{1}{3}e^{2z_1+z_2} + e^{\frac{\pi i}{2\log 2}(z_2-z_1)}.$$

Obviously, u is a transcendental entire solution and $\rho(u) = 1$. Moreover, u is a solution of (2.6) for the case $a = 1$, $b = 1$, $\eta_1 = 2\log 2$, $\eta_2 = -2\log 2$, $\alpha = \beta = 1$, $g = 4z_1 + 2z_2$.

Remark 2.4. This example corresponds to the subcase of $aA_1 + bA_2 \neq 0$ and $\alpha = \beta$ in the case (i) of Theorem 2.5.

Example 2.9. Let

$$u = \frac{z_1}{2}e^{z_1+2z_2} + e^{z_1+2z_2}.$$

Obviously, u is transcendental and $\rho(u) = 1$. Moreover, u is a solution of (2.6) for the case $a = 2$, $b = -1$, $\eta_1 = 3\log 2$, $\eta_2 = -\log 2$, $\alpha = 1$, $\beta = 2$, $g = 2z_1 + 4z_2 + \log(3\log 2)$.

Remark 2.5. This example corresponds to the subcase of $aA_1 + bA_2 = 0$ and $\alpha \neq \beta$ in Theorem 2.5.

Example 2.10. Let

$$u = \frac{1}{\log 3 - 1}e^{3z_1+2z_2-(z_1+z_2)\log 3} + e^{2(z_2+2z_1)}.$$

Obviously, u is transcendental and $\rho(u) = 1$. Moreover, u is a solution of (2.6) for the case $a = 1$, $b = -2$, $\eta_1 = \log 3$, $\eta_2 = -\log 3$, $\alpha = 1$, $\beta = 3$, $g = 7z_1 + 4z_2 - (z_1 + z_2)\log 3$.

Remark 2.6 This example corresponds to the subcase of $q = 3z_1 + 2z_2 - (z_1 + z_2)\log 3$ and $p = 2(z_2 + 2z_1)$ in the case (ii) of Theorem 2.5.

Remark 2.7. From Examples 2.5–2.10 and the conclusions of Theorems 2.3–2.5, it can be seen that the solutions of our theorems are very different from Theorem C, this shows that the solutions of the product type PDDEs are different from the solution of Fermat type PDDEs.

Remark 2.8. By comparing with the main results from our article and the recent articles given by Xu, Liu and Li [26], Biswas and Manal [24], we find that there significant differences. The right side of Eqs. (2.2), (2.5) and (2.6) in our main theorems are e^g, while all the right side of systems in Refs. [24, 26] are the constant 1. Obviously, this makes our results fundamentally different from Refs. [24, 26]. Moreover, the equations in our article only include the product type of the differences and first-order partial derivatives, while the article [26] discussed the Fermat type partial differential difference equations. The article [24] investigated the entire solution of the quadratic trinomials partial differential difference equation (systems) in $\mathbb{C}^n$. All above cases can illustrate that our results exist fundamentally different from Refs. [24, 26].

3 Proofs of theorems

For convenience, we assume uniformly that u is a finite order transcendental entire solution of the equation in every proof of Theorems.

3.1 The proof of Theorem 2.1

Noting the form of equation (2.1), one can deduce that $au_{z_1} + bu_{z_2}$ and $cu_{z_1} + du_{z_2}$ do not have any zeros and any poles. So, in view of the results in Refs. [31,33,34], we obtain that there exists a polynomial h in $\mathbb{C}^2$ satisfying

$$au_{z_1} + bu_{z_2} = e^h, \quad cu_{z_1} + du_{z_2} = e^{-h}. \tag{3.1}$$

Noting that $D := ad - bc \neq 0$, we obtain

$$u_{z_1} = \frac{1}{D}\left(de^h - be^{-h}\right), \quad u_{z_2} = \frac{1}{D}\left(ae^{-h} - ce^h\right). \tag{3.2}$$

Combining with the fact $u_{z_1z_2} = u_{z_2z_1}$, we have from (3.2) that

$$-(ah_{z_1} + bh_{z_2}) = (ch_{z_1} + dh_{z_2})e^{2h}. \tag{3.3}$$

Case 1. Assume that h is a nonconstant polynomial. If $ch_{z_1} + dh_{z_2} \neq 0$, in view of (3.3), we can get

$$e^{2h} = -\frac{ah_{z_1} + bh_{z_2}}{ch_{z_1} + dh_{z_2}}. \tag{3.4}$$

Combining with [31] and (3.4), it yields

$$T(r, e^{2h}) = O\{T(r, h) + \log r\} \tag{3.5}$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

In addition, combining with the results in Ref. [35], one can obtain that

$$\lim_{r\to+\infty} \frac{T(r, e^{2h})}{T(r, h) + \log r} = +\infty. \tag{3.6}$$

The above equation holds for both the transcendental entire function and nonconstant polynomial. Thus, a contradiction between (3.5) and (3.6).

If $ch_{z_1} + dh_{z_2} = 0$, by (3.3), we know that $ah_{z_1} + bh_{z_2} = 0$. We see from $D = ad - bc \neq 0$ that $h_{z_1} = h_{z_2} = 0$, which follows a contradiction.

Case 2. Assume that h is a constant. Denote $e^h = \xi(\neq 0)$. From (3.1), it is easy to get that

$$u_{z_1} = \frac{d\xi - b\xi^{-1}}{D}, \quad u_{z_2} = \frac{a\xi^{-1} - c\xi}{D}.$$

Thus,

$$u(z_1,z_2)=\frac{d\xi-b\xi^{-1}}{D}z_1+\frac{a\xi^{-1}-c\xi}{D}z_2+\xi_0,\qquad \xi_0\in\mathbb{C}.$$

Therefore, we obtain the conclusions of Theorem 2.1.

3.2 The proof of Theorem 2.2

In view of the results in Refs. [31,33,34], there exist two polynomials p and q in $\mathbb{C}^2$ satisfying $p+q=g$ and

$$au_{z_1}+bu_{z_2}=e^p,\quad cu_{z_1}+du_{z_2}=e^q. \tag{3.7}$$

Noting that $D=ad-bc\neq 0$, we have from (3.7) that

$$u_{z_1}=\frac{1}{D}(de^p-be^q),\quad u_{z_2}=\frac{1}{D}(ae^q-ce^p). \tag{3.8}$$

Noting that the fact $u_{z_1z_2}=u_{z_2z_1}$, we have from (3.8) that

$$(cp_{z_1}+dp_{z_2})e^{p-q}=aq_{z_1}+bq_{z_2}. \tag{3.9}$$

Hence, two following cases are discussed below.

Case 1. Assume that $p-q=\xi$(Const.). This yields that $p_{z_1}=q_{z_1}$, $p_{z_2}=q_{z_2}$. And (3.9) can be expressed as

$$(a-ce^\xi)p_{z_1}+(b-de^\xi)p_{z_2}=0. \tag{3.10}$$

Subcase 1.1. If $a-ce^\xi=0$, $b-de^\xi\neq 0$, then $p_{z_2}=0$, that is, $p=p(z_1)$. Thus, $q=p-\xi=p(z_1)-\xi$, $p(z_1)=\frac{g+\xi}{2}$. Substituting these into u_{z_1}, u_{z_2}, we have

$$u_{z_1}=\frac{1}{D}(d-be^{-\xi})e^p,\quad u_{z_2}=\frac{1}{D}(ae^{-\xi}-c)e^p,$$

and by the assumptions in subcase 1.1, we obtain $u_{z_2}=0$. Therefore,

$$u=\frac{1}{D}(d-be^{-\xi})F_1(z_1)=\frac{1}{a}F_1(z_1), \tag{3.11}$$

where $F_1'(z_1)=e^{p(z_1)}=e^{\frac{g+\xi}{2}}$. Hence, we obtain the conclusion of Theorem 2.2(i).

Subcase 1.2. If $a-ce^\xi\neq 0$, $b-de^\xi=0$, then we have $p_{z_1}=0$, that is, $p=p(z_2)$. Thus, $q=p-\xi=p(z_2)-\xi$, $p(z_2)=\frac{g+\xi}{2}$. Similarly, we obtain

$$u=\frac{1}{b}F_2(z_2), \tag{3.12}$$

where $F_2'(z_2)=e^{p(z_2)}=e^{\frac{g+\xi}{2}}$. Hence, we get the conclusion of Theorem 2.2(ii).

Subcase 1.3. If $a-ce^\xi\neq 0$, $b-de^\xi\neq 0$, then (3.10) can be expressed as

$$p_{z_1}+\frac{b-de^\xi}{a-ce^\xi}p_{z_2}=0.$$

The above equation contains

$$p=p\left(z_2-\frac{b-de^\xi}{a-ce^\xi}z_1\right). \tag{3.13}$$

Therefore, in view of $p-q=\xi$, we have

$$q=p\left(z_2-\frac{b-de^\xi}{a-ce^\xi}z_1\right)-\xi. \tag{3.14}$$

Combining with $p+q=g$, we obtain $p=\frac{g+\xi}{2}$. Furthermore, substituting (3.13) and (3.14) into (3.8), we have

$$u_{z_1}=\frac{1}{D}(d-be^{-\xi})e^{p},\quad u_{z_2}=\frac{1}{D}(ae^{-\xi}-c)e^{p}. \tag{3.15}$$

By solving the second equation of (3.15), we have

$$u=\frac{1}{D}(ae^{-\xi}-c)F_3\left(z_2-\frac{b-de^{\xi}}{a-ce^{\xi}}z_1\right)+\psi(z_1),$$

where $F_3'(t)=e^{p(t)}=e^{\frac{g(t)+\xi}{2}}$. Substituting the above equation into the first equation of (3.15), we see $\psi'(z_1)\equiv 0$, that is, $\psi(z_1)$ is a constant. Thus, it follows

$$u=\frac{1}{D}(ae^{-\xi}-c)F_3\left(z_2-\frac{b-de^{\xi}}{a-ce^{\xi}}z_1\right), \tag{3.16}$$

where $F_3'(t)=e^{p(t)}=e^{\frac{g(t)+\xi}{2}}$. Hence, we get the conclusion of Theorem 2.2(iii).

Case 2. Assume that $p-q$ is a nonconstant. Thus, $cp_{z_1}+dp_{z_2}\neq 0$ does not hold. Otherwise, it follows from (3.9) that

$$e^{p-q}=\frac{aq_{z_1}+bq_{z_2}}{cp_{z_1}+dp_{z_2}}. \tag{3.17}$$

Noting that p,q are two polynomials, similar to the discussion of Theorem 2.1, a contradiction can be obtained.

If $cp_{z_1}+dp_{z_2}\equiv 0$, in view of (3.9), we have $aq_{z_1}+bq_{z_2}\equiv 0$. Thus, it yields that $p=p(z_2-\frac{d}{c}z_1)$, $q=q(z_2-\frac{b}{a}z_1)$, $p(z_2-\frac{d}{c}z_1)+q(z_2-\frac{b}{a}z_1)=g$. Substituting p and q into (3.8), we deduce

$$u=\frac{1}{D}[aF_4(t_1)-cF_5(t_2)]+\psi(z_1), \tag{3.18}$$

where $F_4'(t_1)=e^{q(t_1)}, F_5'(t_2)=e^{p(t_2)}$, $t_1=z_2-\frac{b}{a}z_1$, $t_2=z_2-\frac{d}{c}z_1$, $\psi(z_1)$ is a finite order entire function in z_1. Substituting (3.18) into (3.8), we have $\psi'(z_1)\equiv 0$, that is, $\psi(z_1)$ is a constant. Thus,

$$u=\frac{1}{D}\left[aF_4\left(z_2-\frac{b}{a}z_1\right)-cF_5\left(z_2-\frac{d}{c}z_1\right)\right], \tag{3.19}$$

where $F_4'(t_1)=e^{q(t_1)}, F_5'(t_2)=e^{p(t_2)}$, and $p(z_2-\frac{b}{a}z_1)+q(z_2-\frac{b}{a}z_1)=g$. Thus, this proves the conclusion of Theorem 2.2(iv).

Therefore, the conclusions of Theorem 2.2 are proved completely.

3.3 The proof of Theorem 2.3

In view of the results in Refs. [31,33,34], there exists a polynomial h in $\mathbb{C}^2$ such that

$$\alpha u(z+\eta)-\beta u(z)=e^{h},\quad au_{z_1}+bu_{z_2}=e^{-h}. \tag{3.20}$$

It follows that

$$\alpha e^{-h(z+\eta)}-\beta e^{-h}=(ah_{z_1}+bh_{z_2})e^{h},$$

that is,

$$\alpha e^{h(z)-h(z+\eta)}-\beta=(ah_{z_1}+bh_{z_2})e^{2h}. \tag{3.21}$$

Hence, two cases are discussed as follows.

Case 1. Assume that h is a nonconstant. Thus, we have $ah_{z_1} + bh_{z_2} \not\equiv 0$. It yields that $h(z) - h(z+\eta)$ is a nonconstant. Hence, $e^{h(z)-h(z+\eta)}$ is transcendental. Noting that (3.21) and $\alpha\beta \neq 0$, and applying the Nevanlinna second fundamental theorem for $e^{h(z)-h(z+\eta)}$, we can deduce that

$$T(r, e^H) < N(r, e^H) + N\left(r, \frac{1}{e^H}\right) + N\left(r, \frac{1}{e^H - \frac{\beta}{\alpha}}\right) + S(r, e^H)$$
$$< N\left(r, \frac{1}{(ah_{z_1} + bh_{z_2})e^{2h}}\right) + S(r, e^H)$$
$$< O(\log r) + S(r, e^H) = S(r, e^H),$$

where $H = h(z) - h(z+\eta)$. Obviously, this is impossible.

If $ah_{z_1} + bh_{z_2} \equiv 0$, it leads to $h = h(z_2 - \frac{b}{a}z_1)$. This shows that $h(s)$ is a polynomial in $s = z_2 - \frac{b}{a}z_1$. Combining with (3.21), we have $e^{h(z)-h(z+\eta)} = \frac{\beta}{\alpha}$, this shows that $h(z) - h(z+\eta)$ is a constant. So we can deduce that $h = As + A_0$, where $A, A_0 \in \mathbb{C}$ satisfy $e^{As_0} = \frac{\alpha}{\beta}$. By using the characteristic equation of the second equation in (3.20), we have

$$\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = \frac{b}{a}, \quad \frac{du}{dt} = \frac{1}{a}e^{-h}.$$

Using the initial condition $z_1 = 0, z_2 = s$, and $u = \psi(s)$, we obtain that $z_1 = t, z_2 = \frac{b}{a}t + s$, and

$$u = \frac{1}{a}\int_0^t e^{-h}dt + \psi(s) = \frac{t}{a}e^{-h(s)} + \psi(s),$$

that is,

$$u(z_1, z_2) = \frac{z_1}{a}e^{-A\left(z_2 - \frac{b}{a}z_1\right) - A_0} + \psi\left(z_2 - \frac{b}{a}z_1\right), \tag{3.22}$$

where $\psi(s)$ is a transcendental entire function with finite order.

Substituting (3.22) into (3.20), we can get

$$\alpha u(z+\eta) - \beta u(z) = \alpha\left[\frac{z_1 + \eta_1}{a}e^{-A\left(z_2 - \frac{b}{a}z_1\right) - A_0 - A\left(\eta_2 - \frac{b}{a}\eta_1\right)} + \psi\left(z_2 - \frac{b}{a}z_1 + \eta_2 - \frac{b}{a}\eta_1\right)\right]$$
$$- \beta\left[\frac{z_1}{a}e^{-A\left(z_2 - \frac{b}{a}z_1\right) - A_0} + \psi\left(z_2 - \frac{b}{a}z_1\right)\right] = e^{A\left(z_2 - \frac{b}{a}z_1\right) + A_0}.$$

By combining with $e^{h(z)-h(z+\eta)} = \frac{\beta}{\alpha}$, we can get that $\psi(s)$ satisfies

$$\alpha\psi(s + s_0) - \beta\psi(s) = e^{As + A_0} - \frac{\beta\eta_1}{a}e^{-As - A_0}.$$

Case 2. Assume that h is a constant. Let $e^h = \xi$ where $\xi \in \mathbb{C}$. Thus, we have from (3.21) that $\alpha = \beta$. On the other hand, solving the second equation of (3.20), we obtain

$$u = \frac{1}{a\xi}z_1 + \varphi\left(z_2 - \frac{b}{a}z_1\right), \tag{3.23}$$

where $\varphi(s)$ is a transcendental entire function with finite order. Substituting (3.23) into (3.20), we have

$$\varphi(s + s_0) - \varphi(s) = \frac{\xi}{\alpha} - \frac{\eta_1}{a\xi}. \tag{3.24}$$

Thus, the above equation means that $\varphi(s) = Bs + G(s)$, where $B = (\frac{\xi}{\alpha} - \frac{\eta_1}{a\xi})/(s_0)$ and $G(s)$ is a period transcendental entire function of finite order with period $s_0 = \eta_2 - \frac{b}{a}\eta_1$. Thus, we obtain that

$$u = \frac{1}{a\xi}z_1 + B\left(z_2 - \frac{b}{a}z_1\right) + G(s). \tag{3.25}$$

Therefore, the conclusions of Theorem 2.3 are proved completely.

3.4 The proof of Theorem 2.4

In view of the results in Refs. [31,33,34], there exist two polynomials p and q in $\mathbb{C}^2$ such that

$$u(z+\eta)=e^{p},\quad au_{z_1}+bu_{z_2}=e^{q}, \tag{3.26}$$

where $p+q=g$. Thus, p is a nonconstant polynomial. Otherwise, noting that u is a transcendental entire function with finite order, this is a contradiction. In view of (3.26), we have

$$e^{q(z+\eta)}=(ap_{z_1}+bp_{z_2})e^{p},$$

that is

$$e^{q(z+\eta)-p}=ap_{z_1}+bp_{z_2}. \tag{3.27}$$

Therefore, $q(z+\eta)-p$ is a constant. Otherwise, noting that p is a polynomial, a contradiction can be obtained based on the types of functions on both sides of the equation (3.27). Let $q(z+\eta)-p=\xi$, where $\xi\in\mathbb{C}-\{0\}$. Thus, it leads to $ap_{z_1}+bp_{z_2}=e^{\xi}$, its characteristic equation is $\frac{\mathrm{d}z_1}{\mathrm{d}t}=a$, $\frac{\mathrm{d}z_2}{\mathrm{d}t}=b$, $\frac{\mathrm{d}p}{\mathrm{d}t}=e^{\xi}$. By using the initial condition $z_1=0, z_2=s$, and $p=\varphi(s)$, we obtain $z_1=at$, $z_2=bt+s$, $p=te^{\xi}+\varphi(s)$, that is, $p=z_1\frac{e^{\xi}}{a}+\varphi(z_2-\frac{b}{a}z_1)$, where $\varphi(s)$ is a polynomial in $s=z_2-\frac{b}{a}z_1$. Combining with $q(z+\eta)-p=\xi$, we have

$$q=\xi+p(z-\eta)=\frac{e^{\xi}}{a}z_1+\varphi\left(z_2-\frac{b}{a}z_1-\eta_2+\frac{b}{a}\eta_1\right)+\xi-\frac{e^{\xi}}{a}\eta_1.$$

Noting that $p+q=g$, we have that

$$\varphi(s-s_0)+\varphi(s)=g-2\frac{e^{\xi}}{a}z_1+\frac{e^{\xi}}{a}\eta_1-\xi, \tag{3.28}$$

where $s=z_2-\frac{b}{a}z_1$, $s_0=\eta_2-\frac{b}{a}\eta_1$. Thus, in view of (3.26), we have

$$u=e^{p(z-\eta)}=e^{g-\frac{e^{\xi}}{a}z_1-\varphi(s)-\xi},$$

where $\varphi(s)$ satisfies (3.28).

Therefore, the conclusions of Theorem 2.4 are proved completely.

3.5 The proof of Theorem 2.5

In view of the results in Refs. [31,33,34], there exist two polynomials p and q in $\mathbb{C}^2$ such that

$$\alpha u(z+\eta)-\beta u(z)=e^{p},\quad au_{z_1}+bu_{z_2}=e^{q}, \tag{3.29}$$

where $p+q=g$. From the above equations, it can be obtained

$$\alpha e^{q(z+\eta)}-\beta e^{q}=(ap_{z_1}+bp_{z_2})e^{p},$$

that is,

$$\alpha e^{q(z+\eta)-q}=\beta+(ap_{z_1}+bp_{z_2})e^{p-q}. \tag{3.30}$$

Hence, two following cases are discussed below.

Case 1. Assume that $p-q=\xi_1$, where $\xi_1\in\mathbb{C}$. So, we obtain that $q(z+\eta)-q$ is a constant. Let $q(z+\eta)-q=\xi_2$(Const.). Otherwise, a contradiction can be obtained based on the types of functions on both sides of the equation (3.30). Noting that $p+q=g$ and $p-q=\xi_1$, we can obtain that $q=\frac{g-\xi_1}{2}$. Meanwhile, in view of $q(z+\eta)-q=\xi_2$, we have

$$q=A_1z_1+A_2z_2+\varphi(\eta_2z_1-\eta_1z_2), \tag{3.31}$$

where $\varphi(x)$ is a polynomial in $s_1 = \eta_2 z_1 - \eta_1 z_2$ satisfying

$$\alpha e^{\xi_{s_2}} = \beta + \left[aA_1 + bA_2 + (a\eta_2 - b\eta_1)\varphi'\right] e^{\xi_{s_1}}. \tag{3.32}$$

This shows that $(a\eta_2 - b\eta_1)\varphi'$ is a constant. Combining with $a\eta_2 - b\eta_1 \neq 0$, it can be seen that φ' is a constant, that is, $\deg_{s_1}\varphi \leq 1$. Thus, q is of a linear form about z_1, z_2, and $q = A_1 z_1 + A_2 z_2 + A_0$ is still recorded, where A_1, A_2, A_0 are constants. Therefore, we deduce $p = A_1 z_1 + A_2 z_2 + B_0$, where B_0 is a constant. Combining with (3.32), we have that A_1, A_2, A_0, B_0 satisfy

$$\alpha e^{A_1\eta_1 + A_2\eta_2} = \beta + (aA_1 + bA_2)e^{B_0 - A_0}. \tag{3.33}$$

By $a \neq 0$, and noting that the characteristic equation of the second equation (3.29) is $\frac{dz_1}{dt} = a$, $\frac{dz_2}{dt} = b$, $\frac{du}{dt} = e^q$. Using the initial condition $z_1 = 0, z_2 = s$, and $u = \psi(s)$, we have $z_1 = at$, $z_2 = bt + s$, and

$$u(t,s) = \int_0^t e^{A_1 at + A_2(bt+s) + A_0} dt + \psi(s), \tag{3.34}$$

where $\psi(s)$ is a polynomial in $s = z_2 - \frac{b}{a}z_1$. If $aA_1 + bA_2 \neq 0$, then we have

$$u(z_1, z_2) = \frac{1}{aA_1 + bA_2} e^{A_1 z_1 + A_2 z_2 + A_0} + \psi\left(z_2 - \frac{b}{a}z_1\right). \tag{3.35}$$

Substituting (3.35) into (3.29), we obtian

$$\begin{aligned}\alpha u(z+\eta) - \beta u(z) = \alpha &\left[\frac{1}{aA_1 + bA_2} e^{A_1 z_1 + A_2 z_2 + A_0 + A_1\eta_1 + A_2\eta_2} + \psi(s + s_0)\right] \\ &- \beta\left[\frac{1}{aA_1 + bA_2} e^{A_1 z_1 + A_2 z_2 + A_0} + \psi(s)\right] = e^{A_1 z_1 + A_2 z_2 + B_0},\end{aligned}$$

where $s_0 = \eta_2 - \frac{b}{a}\eta_1$. Combining with (3.33), we have $\alpha\psi(s + s_0) = \beta\psi(s)$, it means that $\psi(s)$ is a finite order period entire function with period s_0 for the case $\alpha = \beta$, and

$$\psi(s) = e^{\frac{\log\beta - \log\alpha}{s_0}s}. \tag{3.36}$$

for the case $\alpha \neq \beta$.

If $aA_1 + bA_2 = 0$, then it follows

$$u(z_1, z_2) = \frac{z_1}{a} e^{A_1 z_1 + A_2 z_2 + A_0} + \psi(s). \tag{3.37}$$

Substituting (3.37) into (3.29), $\psi(s)$ can be obtained as described above and

$$e^{B_0 - A_0} = \frac{\beta}{a}\eta_1. \tag{3.38}$$

Hence we have

$$u(z_1, z_2) = \chi(z)e^{A_1 z_1 + A_2 z_2 + A_0} + \psi(s), \tag{3.39}$$

where $\chi(z) = \frac{1}{aA_1 + bA_2}$ if $aA_1 + bA_2 \neq 0$; $\chi(z) = \frac{z_1}{a}$ if $aA_1 + bA_2 = 0$, and $A_1 z_1 + A_2 z_2 = \frac{g - A_0 - B_0}{2}$, A_1, A_2, A_0, B_0 satisfy

$$\alpha e^{\frac{g(\eta) - A_0 - B_0}{2}} = \beta + (aA_1 + bA_2)e^{B_0 - A_0},$$

and for $\alpha = \beta$, $\psi(s)$ is a period entire function with period s_0 and is also of finite order, for $\alpha \neq \beta$, $\psi(s)$ satisfies (3.36).

Thus, the conclusions of Theorem 2.5(i) are proved completely.

Case 2. Assume that $p - q$ is a nonconstant. Thus, (3.30) can be converted into

$$\alpha e^{q(z+\eta)-p} = \beta e^{q-p} + a p_{z_1} + b p_{z_2}. \tag{3.40}$$

So, we obtain that $q(z+\eta) - p$ is also a nonconstant. Otherwise, a contradiction can be obtained based on the types of functions on both sides of the equation (3.40). If $a p_{z_1} + b p_{z_2} \neq 0$, in view of $\alpha\beta \neq 0$, and the Nevanlinna second fundamental theorem for e^{q-p}, we obtain

$$\begin{aligned} T(r, \beta e^P) &< N(r, \beta e^P) + N\left(r, \frac{1}{\beta e^P}\right) + N\left(r, \frac{1}{\beta e^P + a p_{z_1} + b p_{z_2}}\right) + S(r, e^P) \\ &< N\left(r, \frac{1}{\alpha e^{q(z+\eta)-p}}\right) + S(r, e^P) \\ &< O(\log r) + S(r, e^P) = S(r, e^P), \end{aligned}$$

where $P = q - p$. Obviously, this is a contradiction.

If $a p_{z_1} + b p_{z_2} = 0$, then it yields $p = p(s) = p(z_2 - \frac{b}{a} z_1)$, where $p(s)$ is a polynomial in $s = z_2 - \frac{b}{a} z_1$. By (3.30), it leads to $\alpha e^{q(z+\eta)-q} = \beta$, which means that $q(z+\eta) - q$ is a constant. Thus, we have

$$q(z_1, z_2) = A_1 z_1 + A_2 z_2 + \varphi(\eta_2 z_1 - \eta_1 z_2), \tag{3.41}$$

where $\varphi(s_1)$ is a polynomial in $s_1 = \eta_2 z_1 - \eta_1 z_2$, and $e^{A_1\eta_1 + A_2\eta_2} = \frac{\beta}{\alpha}, A_1, A_2 \in \mathbb{C}$. The characteristic equation of the second equation of (3.29) is $\frac{\mathrm{d}z_1}{\mathrm{d}t} = a$, $\frac{\mathrm{d}z_2}{\mathrm{d}t} = b$, $\frac{\mathrm{d}u}{\mathrm{d}t} = e^q$, Using the initial condition $z_1 = 0, z_2 = s$, and $u = \psi(s)$, we can get that $z_1 = at, z_2 = bt + s$ and

$$u(t, s) = \int_0^t e^{Q(t,s)} \mathrm{d}t + \phi(s) = F(t, s) + \phi(s), \tag{3.42}$$

where $\phi(s)$ is a polynomial in $s = z_2 - \frac{b}{a} z_1$, $F(t, s) = \int_0^t e^{Q(t,s)} \mathrm{d}t$, and

$$Q(t, s) = q(at, bt + s) = (A_1 a + A_2 b)t + A_2 s + \varphi[(\eta_2 a - \eta_1 b)t - \eta_1 s].$$

Substituting (3.42) into (3.29), and noting with $p + q = g$, we can deduce

$$u(z_1, z_2) = F\left(\frac{z_1}{a}, z_2 - \frac{b}{a} z_1\right) + \phi\left(z_2 - \frac{b}{a} z_1\right),$$

where

$$\alpha\phi(s + s_0) - \beta\phi(s) = e^{p(s)} + \beta F(t, s) - \alpha F(t, s + s_0), \quad p(z_1, z_2) + q(z_1, z_2) = g(z_1, z_2).$$

Thus, the conclusions of Theorem 2.5(ii) are proved completely.

Therefore, the proof of Theorem 2.5 is completed.

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