Research Article

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Fractional calculus containing certain bivariate Mittag-Leffler kernel with respect to function

https://doi.org/10.1515/dema-2025-0156
received December 14, 2024; accepted May 26, 2025

Abstract: In the present study, we introduce a general integral operator containing bivariate Mittag-Leffler (M-L) kernel with respect to a function $\tau(z)$. This general family includes the usual, Hadamard, Katugampola, Erdélyi-Kober, and Tempered versions for specific choices of the function $\tau(z)$. We investigate the main properties of the general family by using series representation and conjugation relation.

Keywords: bivariate M-L function, bivariate fractional calculus, fractional calculus with respect to functions, integral transforms

MSC 2020: 33E12, 26A33, 35A22

1 Introduction

Mittag-Leffler (M-L) functions play an important role in fractional calculus. However, the univariate M-L functions do not provide the addition property

$$E_1(x)E_1(y) \neq E_1(x+y).$$

Due to the lack of this property, bivariate M-L functions are needed in application areas. Over the past decade, some important bivariate M-L functions were defined [1–3]. In this study, we consider the following bivariate M-L function [3]:

$$E_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta}}(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\overline{\delta})_{k+l}}{\Gamma(\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon})}\frac{x^k y^l}{k!l!}, \quad \mathrm{Re}(\overline{\alpha})>0, \mathrm{Re}(\overline{\beta})>0.$$

The integral operator involving the above function in the kernel is given as follows [3]:

$$ {}_c\mathcal{E}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta};\,\omega_1,\omega_2}f(z) := \int_c^z (z-t)^{\varepsilon-1}E_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta}}(\omega_1(z-t)^{\overline{\alpha}},\omega_2(z-t)^{\overline{\beta}})f(t)\mathrm{d}t, \tag{1}$$

where $\overline{\alpha},\overline{\beta},\overline{\varepsilon},\overline{\delta},\omega_1,\omega_2$ are complex numbers with $\mathrm{Re}(\overline{\alpha}),\mathrm{Re}(\overline{\beta}),\mathrm{Re}(\overline{\varepsilon})>0$.

In this work, in order to obtain new findings and viewpoints, we will generate this integral operator involving the bivariate M-L function in the kernel with respect to the function $\tau(z)$. With the special choices of the function $\tau(z)$ other than the usual one, the general operator includes the Hadamard, Katugampola, and

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Erdélyi-Kober, and Tempered versions. We now proceed by recalling the necessary background to be used throughout the study.

Definition 1.1. [4] Suppose $a \in \mathbb{R}$ is a constant and $f \in L^1[a,b]$ is a function. The Riemann-Liouville fractional integral of order ν is defined by

$$ {}_{a}^{RL}I^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_{a}^{z} (z-\xi)^{\nu-1} f(\xi) \mathrm{d}\xi, \quad \mathrm{Re}(\nu) > 0, z \in [a,b] \subset \mathbb{R}. $$

Definition 1.2. [4] Let $a \in \mathbb{R}$ be a constant, $f \in C^n[a,b]$ a function. The Riemann-Liouville fractional derivative of order ν is defined by

$$ {}_{a}^{RL}D^{\nu} f(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m} \left({}_{a}^{RL}I^{m-\nu} f(z) \right), \quad m := \lfloor \mathrm{Re}(\nu) \rfloor + 1, \mathrm{Re}(\nu) \geq 0. $$

The Riemann-Liouville fractional integral operator exhibits semigroup behavior with respect to the parameters μ and ν

$$ {}_{a}^{RL}I^{\mu} \circ {}_{a}^{RL}I^{\nu} f(z) = {}_{a}^{RL}I^{\mu+\nu} f(z), $$

where $\circ$ indicates the composition of operators.

Definition 1.3. Let $f \in L^1[a,b]$, $\nu \in \mathbb{C}$ such that $\nu > 0$, ϕ a monotonic $C^1[a,b]$ function, and $a \in \mathbb{R}$ is a constant. Then, with respect to the function ϕ, the Riemann-Liouville fractional integral of the function f of order ν is defined as follows:

$$ {}_{a}^{RL}I_{\phi(z)}^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_{a}^{z} \phi'(\xi) (\phi(z) - \phi(\xi))^{\nu-1} f(\xi) \mathrm{d}\xi. \tag{2} $$

Remark 1.1. Setting $\phi(z) = z$, Definition 1.3 recovers the Riemann-Liouville fractional integral operator of order ν.

Remark 1.2. Setting $\phi(z) = \log z$, Definition 1.3 recovers the Hadamard fractional integral operator [5]

$$ {}_{a}^{H}I_{\log z}^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_{a}^{z} \frac{1}{\xi} \log\left(\frac{z}{\xi} \right)^{\nu-1} f(\xi) \mathrm{d}\xi. $$

Remark 1.3. Setting $\phi(z) = z^{\rho+1}$ and multiplying the resultant operator by $(\rho+1)^{-\nu}$, we obtain the Katugampola fractional integral operator [6]

$$ (\rho+1)^{-\nu} \left[{}_{K}^{a}I_{z^{\rho+1}}^{\nu} f(z) \right] = \frac{(\rho+1)^{1-\nu}}{\Gamma(\nu)} \int_{a}^{z} z^{\rho} (z^{\rho+1} - \xi^{\rho+1})^{\nu-1} f(\xi) \mathrm{d}\xi. $$

Remark 1.4. Setting $\phi(z) = z^{\sigma}$, replacing $f(z)$ by $z^{\sigma\eta} f(z)$, and multiplying by $z^{-\sigma(\nu+\eta)}$, we obtain the Erdélyi-Kober fractional integral operator [7]

$$ z^{-\sigma(\nu+\eta)} \left[\frac{1}{\Gamma(\nu)} {}_{a}I_{z^{\sigma}}^{\nu} f(z) \right] = \frac{\sigma z^{-\sigma(\nu+\eta)}}{\Gamma(\nu)} \int_{a}^{z} \xi^{\sigma\eta+\sigma-1} (z^{\sigma} - \xi^{\sigma})^{\nu-1} f(\xi) \mathrm{d}\xi. $$

Remark 1.5. Setting $\phi(z) = z$, and replacing $f(z)$ by $e^{-\lambda(z-\xi)} f(z)$, we obtain the Tempered fractional integral operator [8,9]

$$ {}_{a}^{T}I_{z}^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_{a}^{z} (z-\xi)^{\nu-1} e^{-\lambda(z-\xi)} f(\xi) \mathrm{d}\xi. $$

It should be noted that the operators in (2) satisfy the semigroup property

$$ {}^{RL}_{a}I^{\mu}_{\phi(z)} \circ {}^{RL}_{a}I^{\nu}_{\phi(z)} f(z) = {}^{RL}_{a}I^{\nu}_{\phi(z)} \circ {}^{RL}_{a}I^{\mu}_{\phi(z)} f(z) = {}^{RL}_{a}I^{\mu+\nu}_{\phi(z)} f(z). $$

Definition 1.4. Suppose $f, \phi \in C^n[a,b]$ and $\phi'(z) > 0$. For every $z \in [a,b]$ and $\nu \in \mathbb{C}$, with $\mathrm{Re}(\nu) \geq 0$, the Riemann-Liouville fractional derivative of order ν with respect to the function ϕ is defined as follows:

$$ {}^{RL}_{a}D^{\nu}_{\phi(z)} f(z) = \left(\frac{1}{\phi'(z)} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^{n} {}^{RL}_{a}I^{n-\nu}_{\phi(z)} f(z), $$

with $n - 1 \leq \mathrm{Re}(\nu) < n \in \mathbb{Z}^{+}$.

According to the group theory, the fractional operators with regard to functions can be written as conjugations of the original fractional operators with a few compositional operators. As a result, it is simple to illustrate different results about fractional operators in relation to functions.

Now, define Q_ϕ by $Q_\phi f = f \circ \phi$ such that

$$ (Q_\phi f)(z) = f(\phi(z)). $$

Now, define the inverse operator as $Q_\phi^{-1} = Q_{\phi^{-1}}$.

It is not hard to see that

$$ \begin{aligned} {}^{RL}_{a}I^{\nu}_{\phi(z)} &= Q_\phi \circ {}^{RL}_{\phi}I^{\nu}_{z} \circ Q_\phi^{-1}, \\ {}^{RL}_{a}D^{\nu}_{\phi(z)} &= Q_\phi \circ {}^{RL}_{\phi}D^{\nu}_{z} \circ Q_\phi^{-1}. \end{aligned} $$

The structure of this work is as follows: In Section 2, we recall the fractional calculus properties of the integral operator (1) and also obtain new properties via series approach. In Section 3, we generalize the operators in (1) by taking these operators with respect to the function $\tau(z)$. Section 4 is devoted to the concluding remarks.

2 Analysis of the operators containing bivariate M-L function in the kernel

In this section, we recall the main characteristics of the operators in (1). We also obtain new properties of these operators such as product rule and chain rule.

Theorem 2.1. [3, Theorem 7] *Suppose that* $\overline{\alpha}$, $\overline{\beta}$, $\overline{\varepsilon}$, ω_1, ω_2 *are parameters in* $\mathbb{C}$ *with* $\mathrm{Re}(\overline{\alpha}) > 0$, $\mathrm{Re}(\overline{\beta}) > 0$, $\mathrm{Re}(\overline{\varepsilon}) > 0$. *Then, we obtain*

$$ {}_{a}\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\overline{\delta})_{k+l}\omega_1^k \omega_2^l}{k!l!} {}^{RL}_{a}I^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}} f(z), \tag{3} $$

where $z \in [a,b] \subset \mathbb{R}$ *and* $f \in L^1[a,b]$.

This series formula MGMLO:series:eqn helps to give many properties of the operators ${}_{a}\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}$, which are inherited from the Riemann-Liouville integral operator. For instance, the following composition property holds true [3, Corollary 4]:

$$ {}^{RL}_{a}I^{\mu}\left({}_{a}\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} f(z)\right) = {}_{a}E^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}+\mu} f(z), \tag{4} $$

where $f \in L^1[a,b]$ and $\overline{\alpha}, \overline{\beta}, \overline{\varepsilon}, \overline{\delta}, \omega_1, \omega_2$ are complex numbers with $\mathrm{Re}(\overline{\alpha}) > 0$, $\mathrm{Re}(\overline{\beta}) > 0$, $\mathrm{Re}(\overline{\varepsilon}) > 0$.

Moreover, representation (3) helps to prove the following semigroup property:

$$ {}_a\mathcal{E}^{\overline{\delta}_1;\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}_1}\left({}_a\mathcal{E}^{\overline{\delta}_2;\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}_2} f(z)\right) = {}_a\mathcal{E}^{\overline{\delta}_1+\overline{\delta}_2;\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}_1+\overline{\varepsilon}_2} f(z), $$

where $f \in L^1[a,b]$ and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2\omega_1$, ω_2 be parameters in $\mathbb{C}$ with $\mathrm{Re}(\overline{\alpha}) > 0$, $\mathrm{Re}(\overline{\beta}) > 0$, $\mathrm{Re}(\overline{\varepsilon}_1) > 0$, $\mathrm{Re}(\overline{\varepsilon}_2) > 0$.

We continue to use the series formula (1) to obtain some useful results.

Proposition 2.1. The image of the power function under the action of ${}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}$ is given as

$$ {}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(z-a)^{\mu} = \Gamma(\mu+1)(z-a)^{\mu+\overline{\varepsilon}} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}+\mu+1}(\omega_1(z-a)^{\overline{\alpha}}, \omega_2(z-a)^{\overline{\beta}}), $$

where $\overline{\alpha}$, $\overline{\beta}$, $\overline{\varepsilon}$, $\overline{\delta}$, ω_1, ω_2 are complex parameters with $\mathrm{Re}(\overline{\alpha}) > 0$ and $\mathrm{Re}(\overline{\beta}) > 0$, are real parameters.

Proof. Using the following formula [10,11]:

$$ {}^{RL}_{a}I^{\nu}(z-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} z^{\mu+\nu}, \quad \mathrm{Re}(\mu) > -1, \mathrm{Re}(\nu) > 0, $$

and (1) we obtain

$$ \begin{aligned} {}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(z-a)^{\mu} &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!\,l!} {}^{RL}_{a}I^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}}(z-a)^{\mu} \\ &= (z-a)^{\mu+\overline{\varepsilon}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l\Gamma(\mu+1)}{\Gamma(\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}+\mu+1)k!\,l!}(z-a)^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}+\mu} \\ &= \Gamma(\mu+1)(z-a)^{\mu+\overline{\varepsilon}} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}+\mu+1}(\omega_1(z-a)^{\overline{\alpha}}, \omega_2(z-a)^{\overline{\beta}}), \end{aligned} $$

which is the desired result. □

Example 2.1. By selecting specific parameter values and plotting ${}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(z^{-\frac{1}{2}})$, we can examine how the resulting functions behave and use this to illustrate the conclusion of Proposition 2.1.

We use the function $z^{-1/2}$ with $a = 0$ and $\mu = -\frac{1}{2}$, and select $\overline{\delta} = 1$ and $\omega_1 = \omega_2 = 1$, then we examine the following scenarios:

- For $\overline{\alpha} = \frac{1}{2}$, $\overline{\beta} = \frac{1}{2}$, $\overline{\varepsilon} = \frac{1}{2}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,1/2,1/2}(z^{-1/2}) = \Gamma\left(\frac{1}{2}\right) E^{1}_{1/2,1/2,1}(z^{1/2}, z^{1/2}) = \sqrt{\pi}\sum_{k,l}\frac{(1)_{k+l}\, z^{\frac{k+l}{2}}}{\Gamma(1+\frac{k}{2}+\frac{l}{2})k!\,l!}, $$

  whose red (upper) curve is shown in Figure 1.
- For $\overline{\alpha} = \frac{1}{2}$, $\overline{\beta} = 1$, $\overline{\varepsilon} = \frac{5}{2}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,1,5/2}(z^{-1/2}) = \Gamma\left(\frac{1}{2}\right) z^2 E^{1}_{1/2,1,3}(z^{1/2}, z) = \sqrt{\pi}z^2\sum_{k,l}\frac{(1)_{k+l}\, z^{\frac{2k+l}{2}}}{\Gamma(3+k+\frac{l}{2})k!\,l!}, $$

  whose blue (middle) curve is shown in Figure 1.
- For $\overline{\alpha} = 1$, $\overline{\beta} = \frac{1}{2}$, $\overline{\varepsilon} = \frac{7}{2}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1,1/2,7/2}(z^{-1/2}) = \Gamma\left(\frac{1}{2}\right) z^3 E^{1}_{1,1/2,4}(z^{1}, z^{1/2}) = \sqrt{\pi}z^3\sum_{k,l}\frac{(1)_{k+l}\, z^{\frac{k+2l}{2}}}{\Gamma(4+\frac{k}{2}+l)k!\,l!}, $$

  whose green (lower) curve is shown in Figure 1.

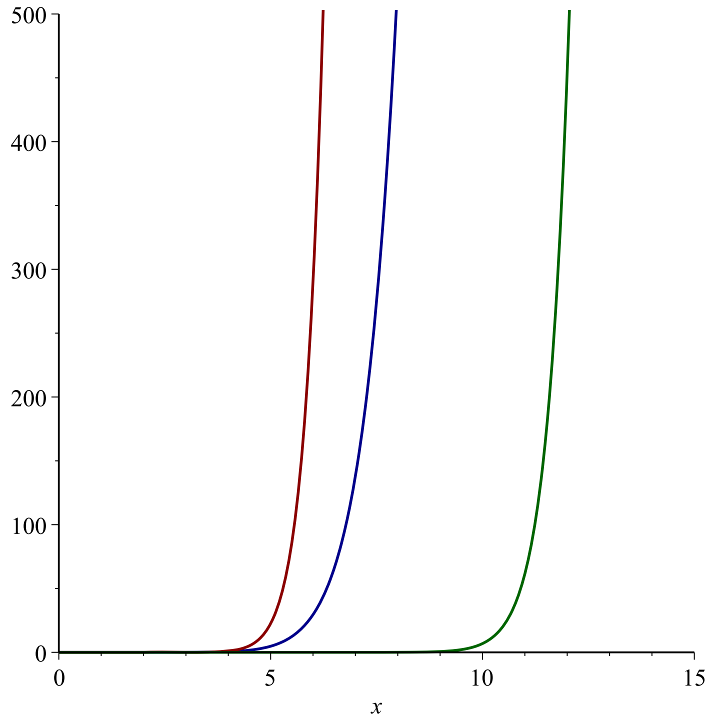

Figure 1: Graph for Example 2.1.

The graphs in Figure 1 are the result of plotting all three of these functions collectively. Figure 1 was created with Maple18 by considering truncated sums $0 \le k, n \le 20$. From the graph, the following conclusions can be drawn.

As z increases, the red function ($\overline{\alpha} = \frac{1}{2}, \overline{\beta} = \frac{1}{2}, \overline{\varepsilon} = \frac{1}{2}$) grows faster than the other two. At least for $0 \le z \le 5$, the magnitudes of the other two functions are more similar; nonetheless, the blue function ($\overline{\alpha} = \frac{1}{2}, \overline{\beta} = 1, \overline{\varepsilon} = \frac{5}{2}$) is significantly greater than the green one ($\overline{\alpha} = 1, \overline{\beta} = \frac{1}{2}, \overline{\varepsilon} = \frac{7}{2}$).

This is understandable as raising the values of $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\varepsilon}$ will increase the value of the gamma function, which reduces the value of the function. At least when z is large, increasing its exponents will typically have the opposite effect. However, since gamma functions are known to increase at a rate faster than power functions, the main determinant of size comparisons between these functions will be the variations in the gamma function's argument.

Proposition 2.2. *The image of the function* $e^{\mu z}$ *under the action of* ${}_{-\infty}\mathcal{E}^{\overline{\delta},\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}$ *is given as*

$$ {}_{-\infty}\mathcal{E}^{\overline{\delta},\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} e^{\mu z} = \mu^{-\overline{\delta}}\left(1 - \frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}} - \omega_2\mu^{-\overline{\beta}} + \frac{\omega_1\omega_2\mu^{-(\overline{\alpha}+\overline{\beta})}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}} e^{\mu z}, $$

where $z \in [a, b]$ *and* $\overline{\alpha}, \overline{\beta}, \overline{\varepsilon}, \omega_1, \omega_2, \mu$ *are complex parameters with* $\mathrm{Re}(\overline{\varepsilon}) > 0$, $\mathrm{Re}(\overline{\alpha})$, $\mathrm{Re}(\overline{\beta}) > 0$ *and* $\mathrm{Re}(\mu) > 0$.

Proof. Using the known formula [7]

$$ {}^{RL}_{\infty}I^{\nu} e^{\mu z} = \mu^{-\nu} e^{\mu z}, \quad \mathrm{Re}(\mu) > 0, \mathrm{Re}(\nu) > 0, $$

and (3), we obtain

$$ {}_{-\infty}\mathcal{E}^{\overline{\delta},\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} e^{\mu z} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!l!} {}^{RL}_{-\infty}I^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}} e^{\mu z} $$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!l!}\mu^{-(\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon})}e^{\mu z}\\
&= \mu^{-\overline{\varepsilon}}e^{\mu z}\sum_{k=0}^{\infty}\frac{(\overline{\delta})_k\omega_1^k}{k!}\mu^{-\overline{\alpha}k}\sum_{l=0}^{\infty}\frac{(\overline{\delta}+k)_l(\omega_2\mu^{-\overline{\beta}})^l}{l!}\\
&= \mu^{-\overline{\varepsilon}}e^{\mu z}(1-\omega_2\mu^{-\overline{\beta}})^{-\overline{\delta}}\sum_{k=0}^{\infty}\frac{(\overline{\delta})_k}{k!}\left(\frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^k\\
&= \mu^{-\overline{\varepsilon}}e^{\mu z}(1-\omega_2\mu^{-\overline{\beta}})^{-\overline{\delta}}\left(1-\frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}}\\
&= \mu^{-\overline{\delta}}\left(1-\frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}}-\omega_2\mu^{-\overline{\beta}}+\frac{\omega_1\omega_2\mu^{-(\overline{\alpha}+\overline{\beta})}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}}e^{\mu z}.
\end{aligned}
$$

Thus, the result is proved. □

Proposition 2.3. *The image of the three parameters M-L function under the action of* ${}_a\mathcal{E}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta};\,\omega_1,\omega_2}$ *is given by*

$$
\begin{aligned}
&{}_a\mathcal{E}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta};\,\omega_1,\omega_2}((z-a)^{\mu-1}E_{\eta,\mu}^{\sigma}(\omega(z-a)^{\eta}))\\
&\quad= (z-a)^{\overline{\varepsilon}+\mu-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\overline{\delta})_{k+l}(\sigma)_i\omega_1^k\omega_2^l\omega^i}{\Gamma(\overline{\alpha}k+\overline{\beta}l+\eta i+\overline{\varepsilon}+\mu)k!l!i!}(z-a)^{\overline{\alpha}k+\overline{\beta}l+\eta i},
\end{aligned}
$$

where $z\in[a,b]$ *and* $\overline{\alpha},\overline{\beta},\overline{\varepsilon},\overline{\delta},\ \omega_1,\omega_2,\ \eta,\mu,\sigma$ *are complex parameters with* $\mathrm{Re}(\overline{\varepsilon})>0$ *and* $\mathrm{Re}(\overline{\alpha})$, $\mathrm{Re}(\overline{\beta})$, $\mathrm{Re}(\eta),\mathrm{Re}(\mu)>0$.

Proof. We know [12,13] that

$$
\begin{aligned}
{}_a^{RL}I^{\nu}((z-a)^{\mu-1}E_{\overline{\beta},\mu}^{\rho}(\omega(z-a)^{\overline{\beta}})) &= (z-a)^{\mu+\nu-1}E_{\overline{\beta},\mu+\nu}^{\rho}(\omega(z-a)^{\overline{\beta}})\\
&= \sum_{i=0}^{\infty}\frac{\omega^i(z-a)^{\overline{\beta}i+\mu+\nu-1}(\rho)_i}{i!\Gamma(\overline{\beta}i+\mu+\nu)}.
\end{aligned}
$$

Therefore, using (3), we obtain

$$
\begin{aligned}
&{}_a\mathcal{E}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}^{\overline{\delta};\,\omega_1,\omega_2}((z-a)^{\mu-1}E_{\eta,\mu}^{\sigma}(\omega(z-a)^{\eta}))\\
&\quad= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!l!}{}_{-\infty}^{RL}I^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}}((z-a)^{\mu-1}E_{\eta,\mu}^{\sigma}(\omega(z-a)^{\eta}))\\
&\quad= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!l!}(z-a)^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}+\mu-1}E_{\eta,\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}+\mu}^{\sigma}(\omega(z-a)^{\eta})\\
&\quad= (z-a)^{\overline{\varepsilon}+\mu-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\overline{\delta})_{k+l}(\sigma)_i\omega_1^k\omega_2^l\omega^i}{\Gamma(\overline{\alpha}k+\overline{\beta}l+\eta i+\overline{\varepsilon}+\mu)k!l!i!}(z-a)^{\overline{\alpha}k+\overline{\beta}l+\eta i},
\end{aligned}
$$

which is the desired result. □

Example 2.2. By selecting specific parameter values and plotting ${}_0\mathcal{E}_{\frac{1}{2},\frac{1}{2},\overline{\varepsilon}}^{1;\,1,1}\left(z^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}^{1}\left(z^{\frac{1}{2}}\right)\right)$, we can examine how the resulting functions behave on a graph and use this to illustrate the conclusion of Proposition 2.3.

Setting $\overline{\delta}=1$, $\omega_1=\omega_2=1$, $\overline{\alpha}=\overline{\beta}=\eta=\mu=\frac{1}{2}$, we consider the following cases:

- For $\overline{\varepsilon}=\frac{8}{5}$, we have

$$
{}_0\mathcal{E}_{1/2,1/2,8/5}^{1;\,1,1}(z^{-1/2}E_{1/2,1/2}^{1}(z^{1/2})) = z^{11/10}\sum_{k,l,i}\frac{(1)_{k+l}\,z^{\frac{k+l+i}{2}}}{\Gamma(\frac{21}{10}+\frac{k}{2}+\frac{l}{2}+\frac{i}{2})k!l!},
$$

  whose blue (lower) curve is shown in Figure 2.

- For $\bar{\varepsilon} = \frac{6}{5}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,1/2,6/5}(z^{-1/2}E^{1}_{1/2,1/2}(z^{1/2})) = z^{7/10}\sum_{k,l,i}\frac{(1)_{k+l}\, z^{\frac{k+l+i}{2}}}{\Gamma(\frac{17}{10}+\frac{k}{2}+\frac{l}{2}+\frac{i}{2})k!l!}, $$

  whose red (middle) curve is shown in Figure 2.
- For $\bar{\varepsilon} = \frac{4}{5}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,1/2,4/5}(z^{-1/2}E^{1}_{1/2,1/2}(z^{1/2})) = z^{3/10}\sum_{k,n,m}\frac{(1)_{k+l}z^{\frac{k+l+i}{2}}}{\Gamma\left(\frac{13}{10}+\frac{k}{2}+\frac{l}{2}+\frac{i}{2}\right)k!l!}, $$

  whose green (upper) curve is shown in Figure 2.

Figure 2 illustrates the abovementioned three cases. Figure 2 was created by considering truncated sums $0 \leq k, n \leq 20$.

As z increases, the fastest growing function is the green function ($\bar{\varepsilon} = \frac{8}{5}$), followed by the red function ($\bar{\varepsilon} = \frac{6}{5}$), and finally, the blue function ($\bar{\varepsilon} = \frac{4}{5}$).

This is understandable as raising the values of $\bar{\varepsilon}$ will increase the value of the gamma function, which reduces the value of the function. At least when z is large, increasing its exponents will typically have the opposite effect. However, since gamma functions are known to increase at a rate faster than power functions, the main determinant of size comparisons between these functions will be the variations in the gamma function's argument.

Now, we continue to prove the product value and chain rule by using representation (3).

Theorem 2.2. *Suppose that f and g are functions such that f is continuous and g is differentiable. Then, the following product rule holds true*:

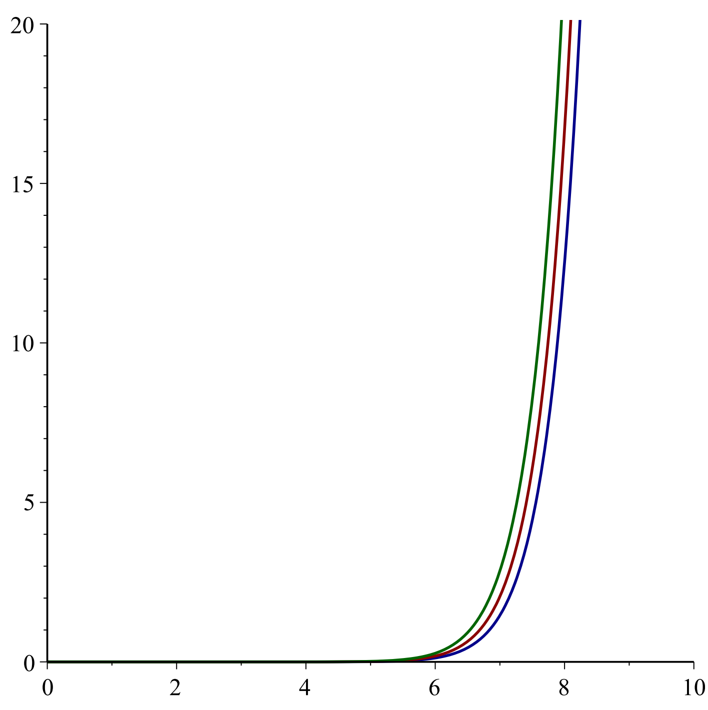

Figure 2: Graph for Example 2.2.

$$ {}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}(f(z)g(z)) = \sum_{m=0}^{\infty} \frac{\mathrm{d}^m g(z)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\, {}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m} f(z)\right]. $$

Proof. Using the series formula and the product rule for the Riemann-Liouville integral operator [10,20],

$$ \begin{aligned} {}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}(f(z)g(z)) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\varepsilon(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\, {}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}}(f(z)g(z)) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\left[\sum_{m=0}^{\infty}\binom{-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}}{m} {}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m} f(z)\frac{\mathrm{d}^m g(z)}{\mathrm{d}z^m}\right] \\ &= \sum_{m=0}^{\infty} \frac{\mathrm{d}^m g(z)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\binom{-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}}{m} {}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m} f(z)\right]. \end{aligned} $$

Since $\binom{-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}}{m} = \frac{\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})}{m!\,\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)}$, we obtain the result. □

Theorem 2.3. *Suppose that f and g are functions such that f is continuous and g is differentiable. Then, the following chain rule holds true*:

$$ \begin{aligned} {}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}[f(g(z))] = (z-c)^{\bar{\varepsilon}} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!} \sum_{m=0}^{\infty} \frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)} \\ \times\left[\sum_{r=1}^{m}\frac{\mathrm{d}^r f(g(z))}{dg(z)^r}\sum_{P_1,\ldots,P_m}\left[\prod_{j=1}^{m}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{\mathrm{d}^j g(z)}{\mathrm{d}z^j}\right)^{P_j}\right]\right]. \end{aligned} $$

Proof. Applying Theorem 2.2, we obtain

$$ \begin{aligned} &{}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}[f(g(z))] \\ &= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\, {}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}(1)\right] \\ &= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\frac{(z-c)^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}}{\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m+1)}\right] \\ &= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\pi\frac{(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}{\sin(\pi(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m))k!l!}}\frac{(z-c)^{\bar{\varepsilon}+m}}{m!}\right] \\ &= (z-c)^{\bar{\varepsilon}}\sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\pi\frac{(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}{\sin(\pi(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}))k!l!}}\frac{(c-z)^{m}}{m!}\right] \\ &= (z-c)^{\bar{\varepsilon}}\sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\frac{(c-z)^{m}}{m!}\right] \\ &= (z-c)^{\bar{\varepsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\sum_{m=0}^{\infty}\frac{\mathrm{d}^m(f\circ g)}{\mathrm{d}z^m}\frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)} \\ &= (z-c)^{\bar{\varepsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\sum_{m=0}^{\infty}\frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)} \\ &\quad\times\left[\sum_{r=1}^{m}\frac{\mathrm{d}^r f(g(z))}{dg(z)^r}\sum_{P_1,\ldots,P_m}\left[\prod_{j=1}^{m}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{\mathrm{d}^j g(z)}{\mathrm{d}z^j}\right)^{P_j}\right]\right], \end{aligned} $$

which is the desired result. □

3 Operators with respect to functions

Erdélyi introduced the idea of taking fractional integrals with regard to a power function in 1964 [14]. In a 1970 publication [15], Osler presented the entire extension of this concept to Riemann-Liouville fractional calculus. This construction and its characteristics are covered in greater detail in textbooks [4,7,11]. The popularity of "Ψ-fractional calculus" has increased in recent years, referring to the study of fractional operators with respect to a function $\Psi(z)$, since Almeida introduced a Caputo version in 2017 [16]. Other fractional operators, such as tempered [17], Hilfer [18], and operators with general analytic kernels [19], have also been taken with regard to functions, in addition to the classical Riemann-Liouville and Caputo.

The operators with bivariate M-L kernel, which were summarized and studied in Section 2, will be generalized in this section by taking them with respect to a monotonic function of C^1. It is then easy to apply many of the findings from Section 2 to the generalized operators by using conjugation relation.

Definition 3.1. Let $f \in L^1[a,b]$ and $g \in C^1[a,b]$ be two functions with g positive and monotonically increasing. Let $\overline{\alpha}, \overline{\beta}, \overline{\varepsilon}, \overline{\delta}, \omega_1, \omega_2$ be parameters in $\mathbb{C}$ with $\mathrm{Re}(\overline{\alpha}) > 0$ and $\mathrm{Re}(\overline{\beta}) > 0$. The fractional integral operator containing bivariate M-L kernel of the function f with respect to the function τ is defined by

$$ {}_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)} f(z) \coloneqq \int\limits_a^z (\tau(z) - \tau(t))^{\overline{\varepsilon}-1} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(\tau(z) - \tau(t))^{\overline{\alpha}}, \omega_2(\tau(z) - \tau(t))^{\overline{\beta}}) f(t)\tau'(t)\mathrm{d}t. \tag{5} $$

Remark 3.1. Setting $\tau(z) = z$, (5) recovers the integral operator involving $E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(z,y)$ in the kernel, as given in (1).

Remark 3.2. Setting $\tau(z) = \log(z)$, (5) reduces to the Hadamard-type version of bivariate M-L integral operator:

$$ {}^H_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\log(z)} f(z) = \frac{1}{\Gamma(\overline{\varepsilon})}\int\limits_a^z \frac{1}{t}\left(\log\frac{z}{t}\right)^{\overline{\varepsilon}-1} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}\left(\omega_1\left(\log\frac{z}{t}\right)^{\overline{\alpha}}, \omega_2\left(\log\frac{z}{t}\right)^{\overline{\beta}}\right) f(t)\mathrm{d}t. $$

Remark 3.3. Setting $\tau(z) = z^{\rho+1}$ and multiplying the resultant operator by $\frac{(\rho+1)^{-\overline{\varepsilon}}}{\Gamma(\overline{\varepsilon})}$, we obtain the Katugampola-type version of bivariate M-L integral operator:

$$ \frac{(\rho+1)^{-\overline{\varepsilon}}}{\Gamma(\overline{\varepsilon})}\left[{}^K_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};z^{\rho+1}} f(z)\right] = \frac{(\rho+1)^{1-\overline{\varepsilon}}}{\Gamma(\varepsilon)}\int\limits_a^z z^{\rho}(z^{\rho+1} - t^{\rho+1})^{\overline{\varepsilon}-1} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(z^{\rho+1} - t^{\rho+1})^{\overline{\alpha}}, \omega_2(z^{\rho+1} - t^{\rho+1})^{\overline{\beta}}) f(t)\mathrm{d}t. $$

Remark 3.4. Setting $\tau(z) = z^{\sigma}$, replacing $f(z)$ by $z^{\sigma\eta}f(z)$, and multiplying by $\frac{z^{-\sigma(\nu+\eta)}}{\Gamma(\overline{\varepsilon})}$, we obtain the Erdélyi-Kober-type version of bivariate M-L integral operator:

$$ \frac{z^{-\sigma(\nu+\eta)}}{\Gamma(\overline{\varepsilon})}\left[\frac{1}{\Gamma(\overline{\varepsilon})}{}_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};z^{\sigma}} f(z)\right] = \frac{\sigma z^{-\sigma(\nu+\eta)}}{\Gamma(\overline{\varepsilon})}\int\limits_a^z z^{\sigma\nu+\sigma-1}(z^{\sigma} - t^{\sigma})^{\overline{\varepsilon}-1} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(z^{\sigma} - t^{\sigma})^{\overline{\alpha}}, \omega_2(z^{\rho+1} - t^{\rho+1})^{\overline{\beta}}) f(t)\mathrm{d}t. $$

Remark 3.5. Setting $\tau(z) = z$, and replacing $f(z)$ by $\frac{e^{-\lambda(z-\xi)}}{\Gamma(\overline{\varepsilon})}f(z)$, we obtain the Tempered-type version of bivariate M-L integral operator:

$$ {}^T_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};z} f(z) = \frac{1}{\Gamma(\overline{\varepsilon})}\int\limits_a^z (z-t)^{\overline{\varepsilon}-1} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(z-t)^{\overline{\alpha}}, \omega_2(z-t)^{\overline{\beta}}) e^{-\lambda(z-\xi)} f(t)\mathrm{d}t. $$

Definition 3.2. We consider $\tau \in C^1[a,b]$ to be a positive function that increases monotonically, and $[a,b]$ to be a fixed real interval. Suppose that $\overline{\alpha}, \overline{\beta}, \overline{\varepsilon}, \overline{\delta}, \omega_1, \omega_2 \in \mathbb{C}$ have $\mathrm{Re}(\overline{\alpha}) \geq 0$ and $\mathrm{Re}(\overline{\beta}) > 0$. Consequently, $N \in \mathbb{N}$ is the natural number such that $N - 1 \leq \mathrm{Re}(\overline{\alpha}) < N$. For a function $f \in C^N[c,d]$, the fractional derivative with bivariate M-L kernel with regard to the function g is defined by

$$ {}_a\mathcal{D}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)}f(z) = \left(\frac{1}{\tau'(z)}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^N \left({}_a\mathcal{E}^{-\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},N-\overline{\varepsilon};\tau(z)}f(z)\right). $$

Theorem 3.1. *The following representation holds true*:

$$ {}_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)} = Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}f(z)\circ Q_\tau^{-1}, \tag{6} $$

$$ {}_a\mathcal{D}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)} = Q_\tau \circ {}_{\tau(a)}\mathcal{D}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}f(z)\circ Q_\tau^{-1}, \tag{7} $$

where the inner composition of τ *is defined as the operator* Q_τ, *working as*

$$ Q_\tau(f) = f\circ\tau, \quad \text{i.e. } (Q_\tau f)(z) = f(\tau(z)). $$

Proof. For functions, the proof can be comparable to that of the classical Riemann-Liouville fractional calculus [4,7]. As we already know, the relevant operational identity is satisfied by the simple first-order derivative with respect to a function

$$ {}^{RL}D^1_{\tau(z)} = \frac{\mathrm{d}}{\mathrm{d}\tau(z)} = \frac{1}{\tau'(z)}\cdot\frac{\mathrm{d}}{\mathrm{d}z} = Q_\tau \circ {}^{RL}D^1 \circ Q_\tau^{-1}. \tag{8} $$

We set a function $f \in L^1[a,b]$ for fractional integrals and then take the following steps:

$$ \begin{aligned} f &: z \mapsto f(z);\\ Q_\tau^{-1}f &: z \mapsto f(\tau^{-1}(z));\\ {}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} \circ Q_\tau^{-1}f &: z \mapsto {}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(f\circ\tau^{-1})(z),\\ Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} \circ Q_\tau^{-1}f &: z \mapsto ({}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(f\circ\tau^{-1}))(\tau(z)). \end{aligned} $$

According to the definition (1), we have

$$ \begin{aligned} &{}_{\tau(a)}\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(f\circ\tau^{-1})(z)\\ &= \int_{\tau(a)}^{z} (z-t)^{\varepsilon-1}E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(z-t)^{\overline{\alpha}},\omega_2(z-t)^{\overline{\beta}})f(\tau^{-1}(t))\mathrm{d}t\\ &= \int_{a}^{\tau^{-1}(z)} (z-\tau(u))^{\overline{\varepsilon}-1}E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(z-\tau(u))^{\overline{\alpha}},\omega_2(z-\tau(u))^{\overline{\beta}})(\omega_1(z-\tau(u))^{\overline{\beta}})f(u)\tau'(u)\mathrm{d}u, \end{aligned} $$

where $u = \tau^{-1}(z)$. Finally, we substitute $\tau(z)$ for z and we obtain

$$ \begin{aligned} &Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\overline{\delta},\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)} \circ Q_\tau^{-1}f(z)\\ &= \int_{a}^{z}(\tau(z)-\tau(u))^{\overline{\alpha}-1}E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(\omega_1(\tau(z)-\tau(u))^{\overline{\alpha}},\omega_2(\tau(z)-\tau(u))^{\overline{\beta}})f(u)\tau'(u)\mathrm{d}u\\ &= {}_{\tau(a)}\mathcal{E}^{\overline{\delta},\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)}f(z), \end{aligned} $$

which proves (6). N times repeatedly use of (6) and (8), we obtain the desired expression (7) for fractional derivatives. □

Theorem 3.2. *The following series formula holds true*:

$$ \left({}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\tau(z)}f\right)(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\overline{\delta})_{k+l}\omega_1^k\omega_2^l}{k!\,l!}\left({}^{RL}_{\ a}I^{\overline{\alpha}k+\overline{\beta}l+\overline{\varepsilon}}_{\tau(z)}f\right)(z). $$

Proof. According to (5), we obtain

$$\begin{aligned}
({}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}f)(z) &= \int_a^z (\tau(z)-\tau(t))^{\bar\varepsilon-1}E^{\bar\delta}_{\bar\alpha,\bar\beta,\bar\varepsilon}(\omega_1(\tau(z)-\tau(t))^{\bar\alpha},\omega_2(\tau(z)-\tau(t))^{\bar\beta})f(t)\tau'(t)\mathrm{d}t\\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\bar\delta)_{k+l}\omega_1^k\omega_2^l}{\Gamma(\bar\alpha k+\bar\beta l+\bar\varepsilon)k!l!}\int_a^z(\tau(z)-\tau(t))^{\bar\alpha k+\bar\beta l+\bar\varepsilon-1}f(t)\tau'(t)\mathrm{d}t\\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\bar\delta)_{k+l}\omega_1^k\omega_2^l}{k!l!}({}^{RL}_{a}I^{\bar\alpha k+\bar\beta l+\bar\varepsilon}_{\tau(z)}f)(z).
\end{aligned}$$

□

Proposition 3.1. *The following composition property holds true*:

$${}^{RL}_{a}I^{\sigma}_{\tau(z)}({}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}f(z)) = {}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon+\sigma;\tau(z)}f(z) = {}^{\bar\alpha,\bar\beta,\bar\varepsilon}_{a}\mathcal{E}_{\bar\delta;\,\omega_1,\omega_2};\tau(z)({}^{RL}_{a}I^{\sigma}_{\tau(z)}f(z)),$$

where $f\in L^1[a,b]$, *monotonic function* $\tau\in C^1[a,b]$, *and* $\bar\alpha,(\bar\beta_i),(\rho_i),(\omega_i)$ *are complex parameters with* $\mathrm{Re}(\bar\varepsilon)>0$, $\mathrm{Re}(\bar\alpha)>0$, *and* $\mathrm{Re}(\bar\beta_i)>0$ *for all* i.

Proof. Using (4) and Theorem 3.1, we have

$$\begin{aligned}
{}_aI^{\sigma}_{\tau(z)}({}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}f(z)) &= Q_\tau\circ{}^{RL}_{\tau(a)}I^{\sigma}\circ Q_\tau^{-1}\circ Q_\tau\circ{}_{\tau(a)}\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon}\circ Q_\tau^{-1}\\
&= Q_\tau\circ{}^{RL}_{\tau(a)}I^{\sigma}\circ{}_{\tau(a)}\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon}\circ Q_\tau^{-1}\\
&= Q_\tau\circ{}_{\tau(a)}\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon+\sigma}\circ Q_\tau^{-1}\\
&= {}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon+\sigma;\tau(z)}f(z)
\end{aligned}$$

and

$$\begin{aligned}
{}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}({}_aI^{\sigma}_{\tau(z)}f(z)) &= Q_\tau\circ{}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}\circ Q_\tau^{-1}\circ Q_\tau\circ{}^{RL}_{\tau(a)}I^{\sigma}\circ Q_\tau^{-1}\\
&= Q_\tau\circ{}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}\circ{}^{RL}_{\tau(a)}I^{\sigma}\circ Q_\tau^{-1}\\
&= Q_\tau\circ{}_{\tau(a)}\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon+\sigma}\circ Q_\tau^{-1}\\
&= {}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon+\sigma;\tau(z)}f(z).
\end{aligned}$$

□

Proposition 3.2. *The following result*

$${}^{RL}_{a}D^{\sigma}_{\tau(z)}({}_a\mathcal{E}^{\bar\delta;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon;\tau(z)}f(z)) = {}_a\mathcal{E}^{\bar\alpha-\sigma,\bar\beta,\bar\varepsilon;\tau(z)}_{\bar\delta;\,\omega_1,\omega_2}f(z) \tag{9}$$

holds true for any function $f\in C^k[a,b]$, $k\coloneqq\lceil\bar\varepsilon\rceil$, *any monotonic function* $\tau\in C^1[a,b]$, *and any complex parameters* $\bar\alpha,\bar\beta,\bar\varepsilon,\omega_1,\omega_2$ *with* $\mathrm{Re}(\bar\alpha)>0$ *and* $\mathrm{Re}(\bar\beta)>0$.

Proof. The proof of (9) comes from Theorem 5.1 in [3], once more utilizing the classical relation and Theorem 3.1.

$${}^{RL}_{a}D^{\sigma}_{\tau(z)} = Q_\tau\circ{}^{RL}_{\tau(a)}D^{\sigma}\circ Q_\tau^{-1}.$$

□

Theorem 3.3. *The following semigroup property holds true*:

$${}_a\mathcal{E}^{\bar\delta_1;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon_1;\tau(z)}({}_a\mathcal{E}^{\bar\delta_2;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon_2;\tau(z)}f(z)) = {}_a\mathcal{E}^{\bar\delta_1+\bar\delta_2;\,\omega_1,\omega_2}_{\bar\alpha,\bar\beta,\bar\varepsilon_1+\bar\varepsilon_2;\tau(z)}f(z),$$

where $f\in L^1[a,b]$, $\tau\in C^1[a,b]$ *is a monotonic function, and* $\bar\alpha,\bar\beta,\bar\varepsilon_1,\bar\varepsilon_2,\bar\delta_1,\bar\delta_2,\omega_1,\omega_2$ *are complex parameters with* $\mathrm{Re}(\bar\alpha)>0$, $\mathrm{Re}(\bar\varepsilon)>0$ *and* $\mathrm{Re}(\bar\beta_i)>0$ *for all* i.

Proof. Since

$$ {}_a\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)} = Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}} \circ Q_\tau^{-1}, $$

we have

$$ \begin{aligned} {}_a\mathcal{E}^{\bar{\delta}_1;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_1;\tau(z)}\big({}_a\mathcal{E}^{\bar{\delta}_2;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_2;\tau(z)}f(z)\big) &= Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\bar{\delta}_1;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_1} \circ Q_\tau^{-1} \circ Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\bar{\delta}_2;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_2} \circ Q_\tau^{-1} \\ &= Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\bar{\delta}_1+\bar{\delta}_2;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_1+\bar{\varepsilon}_2} \circ Q_\tau^{-1} \\ &= {}_a\mathcal{E}^{\bar{\delta}_1+\bar{\delta}_2;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}_1+\bar{\varepsilon}_2;\tau(z)}. \end{aligned} $$

□

Theorem 3.4. *The action of the operator ${}_a\mathcal{D}^{\bar{\delta};\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)}$ on ${}_a\mathcal{E}^{\sigma;\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)}$ is given by*

$$ {}_a\mathcal{D}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)}\big({}_a\mathcal{E}^{\sigma;\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\mu;\tau(z)}f(z)\big) = {}_a\mathcal{E}^{(\sigma-\bar{\delta};\,\omega_1,\omega_2)}_{\bar{\alpha},\bar{\beta},\mu-\bar{\varepsilon};\tau(z)}f(z), $$

where $f \in L^1[a,b]$, $\tau \in C^1[a,b]$ is a monotonic function, and $\bar{\alpha}, \bar{\beta}, \bar{\varepsilon}, \mu, \omega_1, \omega_2$ are complex parameters with $\mathrm{Re}(\bar{\alpha}) \geq 0$ *and* $\mathrm{Re}(\bar{\varepsilon}) > 0$.

Proof. The semigroup property for fractional integrals in this context is given by Theorem 3.3, Definition 3.2 for the fractional derivative with bivariate M-L kernel with respect to a function, and Proposition 3.2 for composition with a standard fractional derivative with respect to a function. □

Example 3.1. The following method can be used to apply the bivariate M-L integral operator with regard to a power function to another power function:

$$ {}_a\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};(z-a)^\sigma}(z-a)^\mu = \Gamma\left(\frac{\mu}{\sigma}+1\right)(z-a)^{\mu+\sigma\bar{\varepsilon}}E^{\bar{\delta}}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}+\frac{\mu}{\sigma}+1}(\omega_1(z-a)^{\sigma\bar{\alpha}}, \omega_2(z-a)^{\sigma\bar{\beta}}), $$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\varepsilon}$, $\bar{\delta}$, ω_1, ω_2 are complex parameters with $\mathrm{Re}(\bar{\alpha}) > 0$ and $\mathrm{Re}(\bar{\beta}) > 0$ and $\sigma > 0$, $\mu > 0$ are real parameters.

Remark 3.6. Setting $\tau(z) = z^\sigma$, Example 3.1 contains a generalized version of Erdelyi-type version of the bivariate M-L integral operator.

Proof. By using Definition 3.1, we suppose $\tau(z) = (z-a)^\sigma$ and $f(z) = (z-a)^\mu$. Then, with the help of the Example 3.1, such that

$$ {}_a\mathcal{E}^{\bar{\delta};\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)} = Q_\tau \circ {}_{\tau(a)}\mathcal{E}^{\bar{\delta};\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)} \circ Q_\tau^{-1}, $$

we have

$$ {}_a\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};(z-a)^\sigma}(z-a)^\mu = Q_\tau \circ {}_a\mathcal{E}^{\bar{\delta};\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}} \circ Q_\tau^{-1}(z-a)^\mu. $$

Clearly,

$$ Q_\tau^{-1}(z-a)^\mu = (z-a)^\mu \circ \tau^{-1} = (z-a)^\mu \circ (\sqrt[\sigma]{z}+a) = (\sqrt[\sigma]{z})^\mu = z^{\frac{\mu}{\sigma}}. $$

Then,

$$ \begin{aligned} {}_a\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}z^{\frac{\mu}{\sigma}} &= \int_a^z (z-t)^{\bar{\varepsilon}-1}E^{\bar{\delta}}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}(\omega_1(z-t)^{\bar{\alpha}}, \omega_2(z-t)^{\bar{\beta}})t^{\frac{\mu}{\sigma}}\mathrm{d}t \\ &= \sum_{k=0}^\infty\sum_{l=0}^\infty \frac{(\bar{\delta})_{k+l}\omega_1^k\omega_2^l}{\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\int_a^z (z-t)^{\bar{\varepsilon}k+\bar{\beta}l+\bar{\varepsilon}-1}t^{\frac{\mu}{\sigma}}\mathrm{d}t \\ &= \Gamma\left(\frac{\mu}{\sigma}+1\right)(z)^{\frac{\mu}{\sigma}+\bar{\varepsilon}}\sum_{k=0}^\infty\sum_{l=0}^\infty \frac{(\bar{\delta})_{k+l}\omega_1^k\omega_2^l z^{\bar{\alpha}k+\bar{\beta}l}}{\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+\frac{\mu}{\sigma}+1)k!l!} \\ &= \Gamma\left(\frac{\mu}{\sigma}+1\right)(z)^{\frac{\mu}{\sigma}+\bar{\varepsilon}}E^{\bar{\delta}}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}+\frac{\mu}{\sigma}+1}(\omega_1 z^{\bar{\alpha}}, \omega_2 z^{\bar{\beta}}) \end{aligned} $$

and

$$Q_\tau \circ {}_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} \circ Q_\tau^{-1} = {}_a\mathcal{E}^{\overline{\delta};\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};(z-a)^\sigma}(z-a)^\mu$$
$$= \Gamma\left(\frac{\mu}{\sigma}+1\right)(z-a)^{\mu+\sigma\overline{\varepsilon}} E^{\overline{\delta}}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}+\frac{\mu}{\sigma}+1}(\omega_1(z-a)^{\sigma\overline{\alpha}}, \omega_2(z-a)^{\sigma\overline{\beta}}). \qquad \square$$

Example 3.2. By selecting specific parameter values and plotting ${}_a\mathcal{E}^{\overline{\delta};\ \omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}}(z^{\frac{1}{2}})$, we can examine how the resulting functions behave and use this to illustrate the conclusion of Example 3.1.

Setting $\overline{\delta} = 1$, $\omega_1 = \omega_2 = 1$, $\overline{\alpha} = 1$, $\overline{\beta} = \frac{1}{2}$, and using the function $z^{1/2}$ with $a = 0$ and $\mu = \frac{1}{2}$, we consider the following cases:

- For $\sigma = \frac{1}{2}$, we have

$$ {}_0\mathcal{E}^{1,1/2,1;z^{1/2}}_{1;\ 1,1}(z^{1/2}) = zE^1_{1,1/2,3}(z^{1/2}, z^{1/4}) = z\sum_{k,l}\frac{(1)_{k+l}\, z^{\frac{2k+n}{4}}}{\Gamma(3+k+\frac{n}{2})k!l!},$$

  whose blue (lower) curve is shown in Figure 3.
- For $\sigma = \frac{2}{3}$, we have

$$ {}_0\mathcal{E}^{1;\ 1,1}_{1,1/2,1;z^{2/3}}(z^{1/2}) = \Gamma\left(\frac{7}{4}\right)z^{7/6}E^1_{1,1/2,\frac{11}{4}}(z^{2/3}, z^{1/3}) = \Gamma\left(\frac{7}{4}\right)z^{7/6}\sum_{k,n}\frac{(1)_{k+l}\, z^{\frac{2k+n}{3}}}{\Gamma(\frac{11}{4}+k+\frac{n}{2})k!l!},$$

  whose red (middle) curve is shown in Figure 3.
- For $\sigma = \frac{3}{4}$, we have

$$ {}_0\mathcal{E}^{1;\ 1,1}_{1,1/2,1;z^{3/4}}(z^{1/2}) = \Gamma\left(\frac{5}{3}\right)z^{5/4}E^1_{1,1/2,\frac{8}{3}}(z^{3/4}, z^{3/8}) = \Gamma\left(\frac{5}{3}\right)z^{5/4}\sum_{k,n}\frac{(1)_{k+l} z^{\frac{6k+3n}{8}}}{\Gamma(\frac{8}{3}+k+\frac{n}{2})k!l!},$$

  whose green (upper) curve is shown in Figure 3.

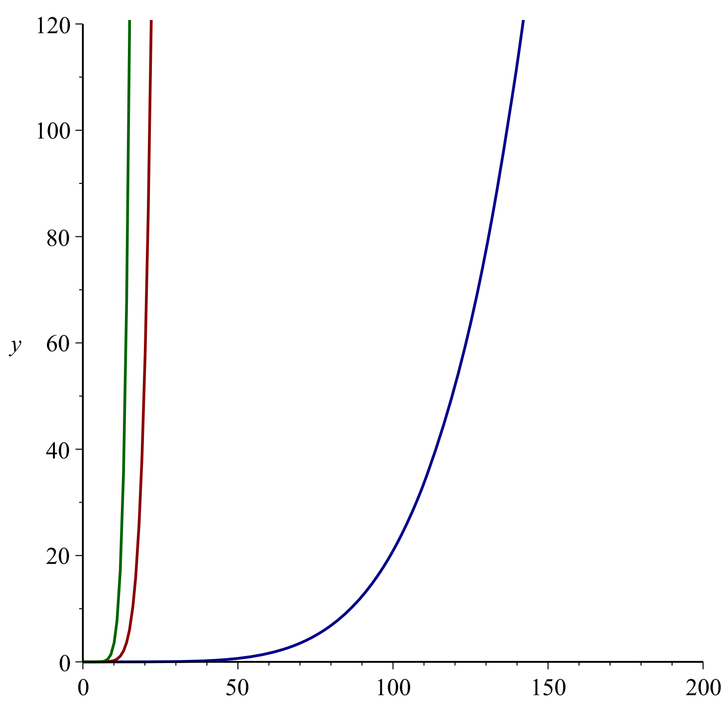

Figure 3: Graph for Example 3.2.

Figure 3 illustrate the abovementioned three cases. Figure 3 was created with Maple18 by considering truncated sums.

The fastest growing function is the green function ($\sigma = \frac{3}{4}$), followed by the red function ($\sigma = \frac{2}{3}$), and finally the blue function ($\sigma = \frac{1}{2}$).

This is understandable as raising the value of σ has two effects: it decreases the argument of the gamma function in the denominator and increases the exponent of x in the numerator.

Example 3.3. The following is an application of the bivariate M-L integral operator with regard to a logarithm function to a power function:

$$ {}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};\log(z-a)}(z-a)^{\mu} = \mu^{-\overline{\delta}}\left(1 - \frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}} - \omega_2\mu^{-\overline{\beta}} + \frac{\omega_1\omega_2\mu^{-(\overline{\alpha}+\overline{\beta})}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}}(z-a)^{\mu}, $$

where $\overline{\alpha}$, $(\overline{\beta}_1, \ldots, \overline{\beta}_n)$, $(\rho_1, \ldots, \rho_n)$, $(\omega_1, \ldots, \omega_n)$ are parameters in $\mathbb{C}$ with $\operatorname{Re}(\overline{\alpha}) > 0$ and $\operatorname{Re}(\overline{\beta}_j) > 0$ for all j and $\mu > 0$ is a real parameter.

Remark 3.7. Setting $\tau(z) = \log(z)$, Example 3.3 contains a generalized version of Hadamard-type version of the bivariate M-L integral operator.

Proof. An example of a monotonically growing function on any interval $(a, b]$ is $\tau(z) = \log(z - a)$, where $\tau(z) \to -\infty$ is $z \to a^+$. Additionally, $(z-a)^{\mu} = e^{\mu\tau(z)}$ is the function to which we are applying the fractional integral operator. Thus, applying Theorem 2.2's finding, we obtain

$$ \begin{aligned} f : z &\mapsto (z-a)^{\mu}, \\ Q_{\tau}^{-1} f : z &\mapsto e^{\mu z}, \\ {}_{-\infty}\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} \circ Q_{\tau}^{-1} f : z &\mapsto {}_{-\infty}\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} e^{\mu z} \\ &= \mu^{-\overline{\delta}}\left(1 - \frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}} - \omega_2\mu^{-\overline{\beta}} + \frac{\omega_1\omega_2\mu^{-(\overline{\alpha}+\overline{\beta})}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}} e^{\mu z}, \\ Q_{\tau} \circ {}_0\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon}} \circ Q_{\tau}^{-1} f : z &\mapsto {}_a\mathcal{E}^{\overline{\delta};\,\omega_1,\omega_2}_{\overline{\alpha},\overline{\beta},\overline{\varepsilon};g(z)}(z-a)^{\mu} \\ &= \mu^{-\overline{\delta}}\left(1 - \frac{\omega_1\mu^{-\overline{\alpha}}}{1-\omega_2\mu^{-\overline{\beta}}} - \omega_2\mu^{-\overline{\beta}} + \frac{\omega_1\omega_2\mu^{-(\overline{\alpha}+\overline{\beta})}}{1-\omega_2\mu^{-\overline{\beta}}}\right)^{-\overline{\delta}}(z-a)^{\mu}, \end{aligned} $$

which is the desired result. □

Example 3.4. By selecting specific parameter values, we can examine the resulting functions and use this to illustrate the conclusion of Example 3.3.

Setting $\overline{\delta} = 1$, $\omega_1 = \omega_2 = 1$, $\overline{\alpha} = \frac{1}{2}$, $\overline{\beta} = \frac{2}{3}$, $\overline{\varepsilon} = \frac{3}{2}$ with $a = 0$, we consider the following cases:

- For $\mu = \frac{1}{3}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,2/3,3/2;\log(z)}(z^{1/3}) = \left(\frac{1}{3}\right)^{-3/2}\left(1 - \frac{(1/3)^{-1/2}}{1-(1/3)^{-2/3}} - (1/3)^{-2/3} + \frac{(1/3)^{-(1/2+2/3)}}{1-(1/3)^{-2/3}}\right)^{-3/2}(z)^{1/3}, $$

  whose blue (lower) curve is shown in Figure 4.

- For $\mu = \frac{2}{5}$, we have

$$ {}_0\mathcal{E}^{1;\,1,1}_{1/2,2/3,3/2;\log(z)}(z^{2/5}) = \left(\frac{2}{5}\right)^{-3/2}\left(1 - \frac{(2/5)^{-1/2}}{1-(2/5)^{-2/3}} - (2/5)^{-2/3} + \frac{(2/5)^{-(1/2+2/3)}}{1-(2/5)^{-2/3}}\right)^{-3/2}(z)^{2/5}, $$

  whose red (middle) curve is shown in Figure 4.

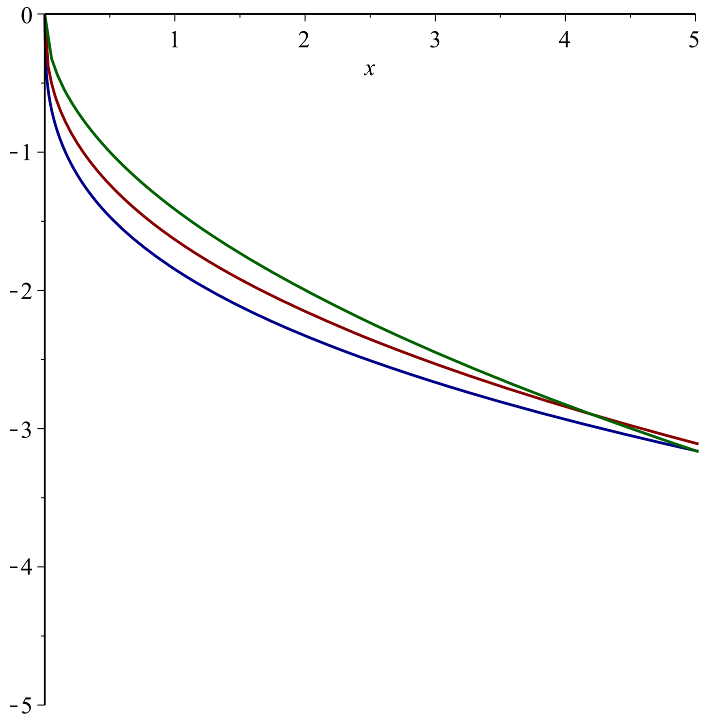

Figure 4: Graph for Example 3.4.

- For $\mu = \frac{1}{2}$, we have

$$_{0}\mathcal{E}^{1;\,1,1}_{1/2,2/3,3/2;\log(z)}(z^{1/2}) = \left(\frac{1}{2}\right)^{-3/2}\left(1 - \frac{(1/2)^{-1/2}}{1-(1/2)^{-2/3}} - (1/2)^{-2/3} + \frac{(1/2)^{-(1/2+2/3)}}{1-(1/2)^{-2/3}}\right)^{-3/2}(z)^{1/2},$$

whose green (upper) curve is shown in Figure 4.

When these three functions are plotted together using Maple18 by considering truncated sums, the graphs are obtained in Figure 4.

As the value of μ increases, the green function ($\mu = \frac{1}{2}$) decreases the slowest, followed by the red function ($\mu = \frac{2}{5}$), while the blue function ($\mu = \frac{1}{3}$) decreases the fastest.

Theorem 3.5. *Consider the complex functions f and h. Then, the integral operator* (5) *satisfies the following product rule*:

$$_{c}\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon};\tau(z)}(f(z)h(z)) = \sum_{m=0}^{\infty}\frac{\mathrm{d}^m h(z)}{z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\,{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}_{\tau(z)}f(z)\right].$$

Proof. Using the series formula (3.2) and known results [10,20], we obtain

$$\begin{aligned}
{}_{c}\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}(f(z)h(z)) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\,{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}}_{g(z)}(f(z)h(z))\\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\left[\sum_{m=0}^{\infty}\binom{-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}}{m}{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}_{g(z)}f(z)\frac{\mathrm{d}^m h(z)}{z^m}\right]\\
&= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m h(z)}{z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})k!l!}\binom{-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}}{m}{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}_{g(z)}f(z)\right]\\
&= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m h(z)}{z^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\,{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\varepsilon+m}_{g(z)}f(z)\right].
\end{aligned}$$

□

Theorem 3.6. *Consider the complex functions f and h. Then, the integral operator* (5) *satisfies the following chain rule*:

$$
{}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}[f(h(z))] = (z-c)^{\bar{\varepsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\sum_{m=0}^{\infty}\frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}
\times\left[\sum_{r=1}^{m}\frac{d^r f(h(z))}{dh(z)^r}\sum_{P_1,\dots,P_m}\left[\prod_{j=1}^{m}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{d^j h(z)}{dz^j}\right)^{P_j}\right]\right].
$$

Proof. It yields that

$$
\begin{aligned}
&{}_c\mathcal{E}^{\bar{\delta};\,\omega_1,\omega_2}_{\bar{\alpha},\bar{\beta},\bar{\varepsilon}}[f(h(z))]\\
&= \sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\,{}^{RL}_{c}I^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}_{g(z)}(1)\right]\\
&= \sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})\omega_1^k\omega_2^l}{\Gamma(\bar{\delta})\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon}-m)k!l!m!}\frac{(z-c)^{\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m}}{\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m+1)}\right]\\
&= \sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\varepsilon(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\pi\frac{(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}{\sin(\pi(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m))k!l!}}\frac{(z-c)^{\bar{\varepsilon}+m}}{m!}\right]\\
&= (z-c)^{\bar{\varepsilon}}\sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)\Gamma(1-\bar{\alpha}k-\bar{\beta}l-\bar{\varepsilon})[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\pi\frac{(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}{\sin(\pi(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}))k!l!}}\frac{(c-z)^m}{m!}\right]\\
&= (z-c)^{\bar{\varepsilon}}\sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\frac{(c-z)^m}{m!}\right]\\
&= (z-c)^{\bar{\varepsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\varepsilon(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\sum_{m=0}^{\infty}\frac{d^m(f\circ h)}{dz^m}\frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}\\
&= (z-c)^{\bar{\varepsilon}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\bar{\delta}+k+l)[\omega_1(z-c)^{\bar{\alpha}}]^k[\omega_2(z-c)^{\bar{\beta}}]^l}{\Gamma(\bar{\delta})\Gamma(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon})k!l!}\sum_{m=0}^{\infty}\frac{(c-z)^m}{m!(\bar{\alpha}k+\bar{\beta}l+\bar{\varepsilon}+m)}\\
&\quad\times\left[\sum_{r=1}^{m}\frac{d^r f(h(z))}{dh(z)^r}\sum_{P_1,\dots,P_m}\left[\prod_{j=1}^{m}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{d^j h(z)}{dz^j}\right)^{P_j}\right]\right],
\end{aligned}
$$

which is the desired result. □

4 Conclusion

The generalized fractional integral operators in this study are related to the bivariate M-L function in the kernel. In the process, we have examined a recently defined bivariate M-L and its corresponding integral operator. Numerous features and consequences of the integral operator are recalled by the series formula.

Then, we have defined the generalized version of integral operator with bivariate M-L function in the kernel with respect to function $\tau(z)$. In the special choices of $\tau(z)$, this generalized integral operator contains the usual, Hadamard, Erdélyi-Kober, Katugampola, and Tempered-type version of the bivariate M-L integral operator. We have examined the fundamental properties of the general integral operator using series representations and conjugation relations.

In the future study, we will define and investigate the Hilfer derivative and integral versions of this general family.

Funding information: The authors state no funding involved.

Author contributions: Both authors contributed equally to this work.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Not applicable.

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