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Research Article

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Generalization of Sheffer-λ polynomials

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Abstract: The aim of this study is to define a new generalization of Sheffer-λ polynomials with the help of Sheffer polynomials and λ-polynomials. For this family, explicit form, summation formulas, quasi-monomiality properties, differential equation and determinant representation are obtained. Subfamilies of these polynomials are introduced and similar properties for subfamilies are found. In addition, 3D graphs and the distribution of real roots are plotted for these subfamilies. Similarly, twice iterated Sheffer-λ polynomials are defined and basic properties for this family and its subfamilies are obtained, and 3D graphs and the distribution of real roots are also investigated for the subfamilies.

Keywords: Sheffer-λ polynomials, monomiality principle, differential equations, determinant form

MSC 2020: 33C65, 11B83, 11B68

1 Introduction

The Sheffer sequence class, which is one of the polynomial sequences used in many different areas of science, has an important place in both pure and applied mathematics [1–3]. There are also studies in number theory, approximation theory, combinatorics, hypothetical physics, and some other scientific fields. Different concepts have been adopted to define Sheffer polynomials and many other well-known polynomial families and to examine their properties. One of these concepts is the monomiality principle and operational techniques and these polynomial families have been investigated from different perspectives [4–10]. The idea of monomiality was first introduced by Steffensen [11] and later redefined by Dattoli [4]. Recently, this principle has been intensively investigated in relation to hybrid polynomials [12–15]. A polynomial set $\{a_j(x)\}_{j\in\mathbb{N}}$ is said to be "quasi-monomial" in accordance with the monomiality principle if two operators, Λ^+ and Λ^-, which are regarded as multiplicative and derivative operators [4], are as follows:

$$\Lambda^+\{a_j(x)\} = a_{j+1}(x), \tag{1.1}$$

$$\Lambda^-\{a_j(x)\} = j\ a_{j-1}(x). \tag{1.2}$$

The commutation identity is satisfied by the operators Λ^+ and Λ^-

$$(\Lambda^-, \Lambda^+) = I,$$

where I denotes the identity operator [4]. Using equations (1.1) and (1.2), $a_j(x)$ satisfies from [4]

$$\Lambda^+\Lambda^-\{a_j(x)\} = ja_j(x). \tag{1.3}$$

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On the other hand, the λ-polynomials were defined by Dattoli et al. [16]. These polynomials have the following generating function [16,17]:

$$e^{yt}\cos(\sqrt{xt}) = \sum_{j=0}^{\infty} \lambda_j(x,y)\frac{t^j}{j!} \tag{1.4}$$

and has the following series representation:

$$\lambda_j(x,y) = \sum_{k=0}^{j} \frac{(-1)^k x^k y^{j-k} j!}{(2k)!(j-k)!}.$$

On the other hand, the Sheffer polynomials were defined by the following generating function [1–3]:

$$A(t)\exp(xQ(t)) = \sum_{j=0}^{\infty} s_j(x)\frac{t^j}{j!}, \tag{1.5}$$

where

$$A(t) = \sum_{j=0}^{\infty} a_j \frac{t^j}{j!}, \quad a_0 \neq 0, \quad Q(t) = \sum_{j=1}^{\infty} q_j \frac{t^j}{j!}, \quad q_1 \neq 0 \tag{1.6}$$

and

$$\exp(xQ(t)) = \sum_{j=0}^{\infty} p_j(x)\frac{t^j}{j!}. \tag{1.7}$$

The Sheffer-λ polynomials, which are the combination of Sheffer polynomials and λ-polynomials, were defined by [17] and have the following generating function from [17]:

$$A(t)e^{yQ(t)}\cos(\sqrt{xt}) = \sum_{j=0}^{\infty} {}_s\lambda_j(x,y)\frac{t^j}{j!} \tag{1.8}$$

and has the following series representation from [17]:

$${}_s\lambda_j(x,y) = \sum_{k=0}^{j} \frac{(-1)^k x^k s_{j-k}(y) j!}{(2k)!(j-k)!}.$$

The new family of mixed polynomials formed by the combination of two different polynomials can be called hybrid polynomials. On the one hand, a different generalization of Appell polynomials, the twice iterated Appell polynomials, has been studied with interest recently [18–20]. This polynomial family is defined in [18] and has the following generating function from [18]:

$$A(t)B(t)e^{xt} = \sum_{j=0}^{\infty} A_j^{[2]}(x)\frac{t^j}{j!}, \tag{1.9}$$

where

$$A(t) = \sum_{j=0}^{\infty} \alpha_j \frac{t^j}{j!}, \quad \alpha_0 \neq 0, \quad B(t) = \sum_{j=0}^{\infty} \beta_j \frac{t^j}{j!}, \quad \beta_0 \neq 0.$$

We introduce a new generalization of Sheffer-λ polynomials, a hybrid polynomial family inspired by variations of Sheffer polynomials and λ-polynomials and study their properties. This study is organized as follows: In Section 2, the generalization of Sheffer-λ polynomials is defined with the help of generating function and series representation, summation formulas, quasi-monomiality properties, differential equation and determinant representation are derived for this family. In Section 3, the subfamilies of generalized Sheffer-λ polynomials are introduced and various properties are obtained for these subfamilies. Also, their 3D surface plots and the graphs of the distributions of the real roots of the polynomials are given. In Section 4, the twice-iterated Sheffer-λ polynomials are introduced and their corresponding similar properties are obtained.

In Section 5, the subfamilies of twice-iterated Sheffer-λ polynomials are introduced and their corresponding similar properties are obtained.

2 Some properties of the generalization of Sheffer-λ polynomials

In this section, we define a new class of Sheffer-λ polynomials and then derive their corresponding properties.

Definition 2.1. The generalization of Sheffer-λ polynomials ${}_s\lambda_j(x,y,z)$ is defined by the following generating function:

$$A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)=\sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^j}{j!},\tag{2.1}$$

where

$$A(t)=\sum_{j=0}^{\infty}a_j\frac{t^j}{j!},\quad a_0\neq 0,\quad Q(t)=\sum_{j=1}^{\infty}q_j\frac{t^j}{j!},\quad q_1\neq 0\tag{2.2}$$

and

$$\phi(z,t)=\sum_{j=0}^{\infty}f_j(z)\frac{t^j}{j!},\quad f_0(z)\neq 0.\tag{2.3}$$

It should be noted here that by adding a new function $\phi(z,t)$ to the polynomial family in (1.8), a large family is obtained that reduces the polynomial to the existing ones' families in the literature. Thus, with the help of the properties of this family, it is easier to obtain properties of different polynomial families.

Remark 2.2. When we take $A(t)=\phi(z,t)=1$ and $Q(t)=t$ in equation (2.1), the λ-polynomials given in (1.4) are found.

Remark 2.3. In equation (2.1), when $\phi(z,t)=1$, we obtain a generating function for Sheffer-λ polynomials ${}_s\lambda_j(x,y)$ as

$$A(t)e^{yQ(t)}\cos(\sqrt{xt})=\sum_{j=0}^{\infty}{}_s\lambda_j(x,y)\frac{t^j}{j!}.\tag{2.4}$$

Theorem 2.1. *The generalization of Sheffer-λ polynomials ${}_s\lambda_j(x,y,z)$ satisfies the following series representation*:

$${}_s\lambda_j(x,y,z)=\sum_{l=0}^{j}\sum_{k=0}^{j-l}\frac{(-1)^k x^k s_{j-k-l}(y)f_l(z)j!}{(2k)!(j-k-l)!l!}.\tag{2.5}$$

Proof. Using equations (1.5), (2.3) and the expansion of the cosine function, applying the Cauchy's product rule and comparing the coefficients of $\frac{t^j}{j!}$, respectively, (2.5) is obtained. □

Theorem 2.2. *The generalization of Sheffer-λ polynomials ${}_s\lambda_j(x,y,z)$ satisfies the following summation formulas*:

$${}_s\lambda_j(x,y+v,z)=\sum_{k=0}^{j}\binom{j}{k}{}_s\lambda_{j-k}(x,y,z)p_k(v)\tag{2.6}$$

or

$$ {}_s\lambda_j(x, y+v, z) = \sum_{k=0}^{j} \binom{j}{k} {}_s\lambda_{j-k}(x, v, z) p_k(y), \tag{2.7}$$

where $p_k(v)$ *is as in* (1.7).

Proof. If we take $y+v$ instead of y in the generating function in (2.1) and use equation (1.7) and applying Cauchy's product rule, we obtain the following equation:

$$\begin{aligned}\sum_{j=0}^{\infty} {}_s\lambda_j(x, y+v, z)\frac{t^j}{j!} &= A(t)e^{(y+v)Q(t)}\cos(\sqrt{xt})\phi(z,t)\\ &= A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)e^{vQ(t)}\\ &= \left(\sum_{j=0}^{\infty} {}_s\lambda_j(x,y,z)\frac{t^j}{j!}\right)\left(\sum_{k=0}^{\infty} p_k(v)\frac{t^k}{k!}\right)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k} {}_s\lambda_{j-k}(x,y,z)p_k(v)\frac{t^j}{j!}.\end{aligned}$$

In the last equation, it is obtained from the polynomial equation (2.6) by comparing of the coefficients of $\frac{t^j}{j!}$. Similarly, (2.7) is also easily seen and the proof is completed. □

Theorem 2.3. *The generalization of Sheffer-λ polynomials* ${}_s\lambda_j(x,y,z)$ *satisfies the following quasi-monomiality operators*:

$$\Lambda^{+}_{s^\lambda} = \frac{A'(Q^{-1}(D_y))}{A(Q^{-1}(D_y))} + \frac{\phi'(z, Q^{-1}(D_y))}{\phi(z, Q^{-1}(D_y))} + yQ'(Q^{-1}(D_y)) - \frac{1}{2}\sqrt{\frac{x}{Q^{-1}(D_y)}}\tan\sqrt{xQ^{-1}(D_y)}, \tag{2.8}$$

$$\Lambda^{-}_{s^\lambda} = Q^{-1}(D_y), \tag{2.9}$$

where $D_y := \frac{\partial}{\partial y}$, $\phi'(z,t) = \frac{\partial}{\partial t}(\phi(z,t))$, *and Q is the invertible function.*

Proof. Upon taking the derivative on each side of (2.1) with respect to t, we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} {}_s\lambda_{j+1}(x,y,z)\frac{t^j}{j!} = {}& \frac{A'(t)}{A(t)}A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t) + yQ'(t)e^{yQ(t)}A(t)\cos(\sqrt{xt})\phi(z,t)\\ &- \frac{1}{2}\sqrt{\frac{x}{t}}\sin(\sqrt{xt})A(t)e^{yQ(t)}\phi(z,t) + \frac{\phi'(z,t)}{\phi(z,t)}\phi(z,t)A(t)e^{yQ(t)}\cos(\sqrt{xt}).\end{aligned}$$

When the terms in the last equation are arranged, we obtain the following equation:

$$\sum_{j=0}^{\infty} {}_s\lambda_{j+1}(x,y,z)\frac{t^j}{j!} = A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)\left[\frac{A'(t)}{A(t)} + yQ'(t) - \frac{1}{2}\sqrt{\frac{x}{t}}\tan(\sqrt{xt}) + \frac{\phi'(z,t)}{\phi(z,t)}\right]. \tag{2.10}$$

In addition, considering the following equation for the generalization of Sheffer-λ polynomials:

$$D_y\{A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)\} = Q(t)A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t) \tag{2.11}$$

or equivalently for invertible function $Q(t)$,

$$Q^{-1}(D_y)A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t) = t\, A(t)e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t). \tag{2.12}$$

Equation (2.10) becomes

$$\sum_{j=0}^{\infty} {}_s\lambda_{j+1}(x,y,z)\frac{t^j}{j!} = \left[\frac{A'(Q^{-1}(D_y))}{A(Q^{-1}(D_y))} + yQ'(Q^{-1}(D_y)) - \frac{1}{2}\sqrt{\frac{x}{Q^{-1}(D_y)}}(\tan\sqrt{xQ^{-1}(D_y)})\right.$$

$$+\frac{\phi'(z,Q^{-1}(D_y))}{\phi(z,Q^{-1}(D_y))}\Bigg]\Bigg(\sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^j}{j!}\Bigg).$$

By using equation (2.1) in the last equation and comparing the coefficients of $\frac{t^j}{j!}$, (2.8) is obtained.

Next using (2.1) in (2.12), we obtain

$$\begin{aligned}Q^{-1}(D_y)\sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^j}{j!} &= t\sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^j}{j!}\\ &= \sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^{j+1}}{j!}\\ &= j\sum_{j=1}^{\infty}{}_s\lambda_{j-1}(x,y,z)\frac{t^j}{j!},\quad j\geq 1.\end{aligned}$$

In the last equation, comparing the coefficients of $\frac{t^j}{j!}$, we have

$$Q^{-1}(D_y)_s\lambda_j(x,y,z) = j_s\lambda_{j-1}(x,y,z).$$

Hence, equation (2.9) is obtained and the proof is completed. □

Theorem 2.4. *The differential equation of the generalization of Sheffer-λ polynomials* ${}_s\lambda_j(x,y,z)$ *is as follows*:

$$\begin{aligned}&\Bigg[\Bigg(\frac{A'(Q^{-1}(D_y))}{A(Q^{-1}(D_y))}+yQ'(Q^{-1}(D_y))-\frac{1}{2}\sqrt{\frac{x}{Q^{-1}(D_y)}}\tan\sqrt{xQ^{-1}(D_y)}\Bigg)Q^{-1}(D_y)\\ &+\frac{\phi'(z,Q^{-1}(D_y))}{\phi(z,Q^{-1}(D_y))}Q^{-1}(D_y)-j\Bigg]{}_s\lambda_j(x,y,z)=0,\qquad j=1,2,\ldots.\end{aligned} \tag{2.13}$$

Proof. By substituting equations (2.8) and (2.9) into relation (1.3) and rearranging, equation (2.13) is obtained. □

Theorem 2.5. *The generalization of Sheffer-λ polynomials* ${}_s\lambda_j(x,y,z)$ *satisfies the following determinant representation*:

$${}_s\lambda_j(x,y,z)=\frac{(-1)^j}{(\delta_0)^{j+1}}\begin{vmatrix} {}_p\lambda_0 & {}_p\lambda_1 & \cdots & {}_p\lambda_{j-1} & {}_p\lambda_j\\ \delta_0 & \delta_1 & \cdots & \delta_{j-1} & \delta_j\\ 0 & \delta_0 & \cdots & \binom{j-1}{1}\delta_{j-2} & \binom{j}{1}\delta_{j-1}\\ 0 & 0 & \cdots & \binom{j-1}{2}\delta_{j-3} & \binom{j}{2}\delta_{j-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \delta_0 & \binom{j}{j-1}\delta_1\end{vmatrix}, \tag{2.14}$$

where $\sum_{j=0}^{\infty}{}_p\lambda_j\frac{t^j}{j!}=e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)$, $\frac{1}{A(t)}=\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}$, *and* ${}_p\lambda_j \coloneqq {}_p\lambda_j(x,y,z)$.

Proof. Using

$$[A(t)]^{-1}=\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}$$

and the generating function (2.1), we obtain

$$e^{yQ(t)}\cos(\sqrt{xt})\phi(z,t)=\Bigg(\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}\Bigg)\Bigg(\sum_{j=0}^{\infty}{}_s\lambda_j(x,y,z)\frac{t^j}{j!}\Bigg).$$

Hence, it follows that

$$\sum_{j=0}^{\infty} {}_p\lambda_j(x,y,z)\frac{t^j}{j!} = \left(\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty} {}_s\lambda_j(x,y,z)\frac{t^j}{j!}\right).$$

Applying the Cauchy's product rule, we have

$$\sum_{j=0}^{\infty} {}_p\lambda_j(x,y,z)\frac{t^j}{j!} = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}\delta_k\, {}_s\lambda_{j-k}(x,y,z)\frac{t^j}{j!}.$$

In the last equation, comparing the coefficients of $\frac{t^j}{j!}$, we obtain

$${}_p\lambda_j(x,y,z) = \sum_{k=0}^{j}\binom{j}{k}\delta_k\, {}_s\lambda_{j-k}(x,y,z).$$

Thus, we establish the system of equations given below

$$\begin{gathered}
{}_p\lambda_0(x,y,z) = \delta_0\, {}_s\lambda_0(x,y,z),\\
{}_p\lambda_1(x,y,z) = \delta_0\, {}_s\lambda_1(x,y,z) + \delta_1\, {}_s\lambda_0(x,y,z),\\
{}_p\lambda_2(x,y,z) = \delta_0\, {}_s\lambda_2(x,y,z) + \binom{2}{1}\delta_1\, {}_s\lambda_1(x,y,z) + \delta_2\, {}_s\lambda_0(x,y,z),\\
\vdots\\
{}_p\lambda_{j-1}(x,y,z) = \delta_0\, {}_s\lambda_{j-1}(x,y,z) + \ldots + \delta_{j-1}\, {}_s\lambda_0(x,y,z),\\
{}_p\lambda_j(x,y,z) = \delta_0\, {}_s\lambda_j(x,y,z) + \binom{j}{1}\delta_1\, {}_s\lambda_{j-1}(x,y,z) + \ldots + \delta_j\, {}_s\lambda_0(x,y,z).
\end{gathered}$$

Using Cramer's rule, we obtain

$${}_s\lambda_j(x,y,z) = \frac{\begin{vmatrix}
\delta_0 & 0 & \cdots & 0 & {}_p\lambda_0\\
\delta_1 & \delta_0 & \cdots & 0 & {}_p\lambda_1\\
\delta_2 & \binom{2}{1}\delta_1 & \cdots & 0 & {}_p\lambda_2\\
\delta_3 & \binom{3}{2}\delta_2 & \cdots & 0 & {}_p\lambda_3\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\delta_{j-1} & \binom{j-1}{1}\delta_{j-2} & \cdots & \delta_0 & {}_p\lambda_{j-1}\\
\delta_j & \binom{j}{1}\delta_{j-1} & \cdots & \binom{j}{j-1}\delta_1 & {}_p\lambda_j
\end{vmatrix}}{\begin{vmatrix}
\delta_0 & 0 & \cdots & 0 & 0\\
\delta_1 & \delta_0 & \cdots & 0 & 0\\
\delta_2 & \binom{2}{1}\delta_1 & \cdots & 0 & 0\\
\delta_3 & \binom{3}{2}\delta_2 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\delta_{j-1} & \binom{j-1}{1}\delta_{j-2} & \cdots & \delta_0 & 0\\
\delta_j & \binom{j}{1}\delta_{j-1} & \cdots & \binom{j}{j-1}\delta_1 & \delta_0
\end{vmatrix}}.$$

Taking the above equation and transposing, it gives us

$$
{}_s\lambda_j(x,y,z)=\frac{1}{(\delta_0)^{j+1}}\begin{vmatrix}
\delta_0 & \delta_1 & \cdots & \delta_{j-1} & \delta_j\\
0 & \delta_0 & \cdots & \binom{j-1}{1}\delta_{j-2} & \binom{j}{1}\delta_{j-1}\\
0 & 0 & \cdots & \binom{j-1}{2}\delta_{j-3} & \binom{j}{2}\delta_{j-2}\\
0 & 0 & \cdots & \binom{j-1}{3}\delta_{j-4} & \binom{j}{3}\delta_{j-3}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \delta_0 & \binom{j}{j-1}\delta_1\\
{}_p\lambda_0 & {}_p\lambda_1 & \cdots & {}_p\lambda_{j-1} & {}_p\lambda_j
\end{vmatrix}.
$$

Thus, the basic row operations are used to finish the proof. □

One can define several new hybrid subfamilies by some particular choices of $A(t)$, $Q(t)$ and $\phi(z,t)$. Furthermore, various properties of these new subfamily could be derived as have been done here for the main results.

3 Special cases

In this section, we introduce generalization of second kind Bernoulli-λ polynomials and Laguerre-λ polynomials. We also examine the characteristic properties corresponding to these new subpolynomials.

3.1 Generalization of second kind Bernoulli-λ polynomials

The generalization of second kind Bernoulli-λ polynomials is defined as follows:

$$\frac{t}{\ln(1+t)}(1+t)^y\cos(\sqrt{xt})e^{zt}=\sum_{j=0}^{\infty}{}_b\lambda_j(x,y,z)\frac{t^j}{j!},\tag{3.1}$$

where

$$A(t)=\frac{t}{\ln(1+t)},\quad Q(t)=\ln(1+t),\quad \text{and}\quad \phi(z,t)=e^{zt}.\tag{3.2}$$

Corollary 3.1. *The generalization of second kind Bernoulli-λ polynomials satisfies the following operators*:

$$
\begin{aligned}
\Lambda^{+}_{b^\lambda}&=\ln(1+D_y)-\frac{1}{(1+(\ln(1+D_y))^{-1})\ln(1+(\ln(1+D_y))^{-1})}+z+y\frac{\ln(1+D_y)}{\ln(1+D_y)+1}\\
&\quad-\frac{1}{2}\sqrt{x\ln(1+D_y)}\tan\left(\sqrt{\frac{x}{\ln(1+D_y)}}\right),\\
\Lambda^{-}_{b^\lambda}&=(\ln(1+D_y))^{-1}
\end{aligned}\tag{3.3}
$$

and differential equation

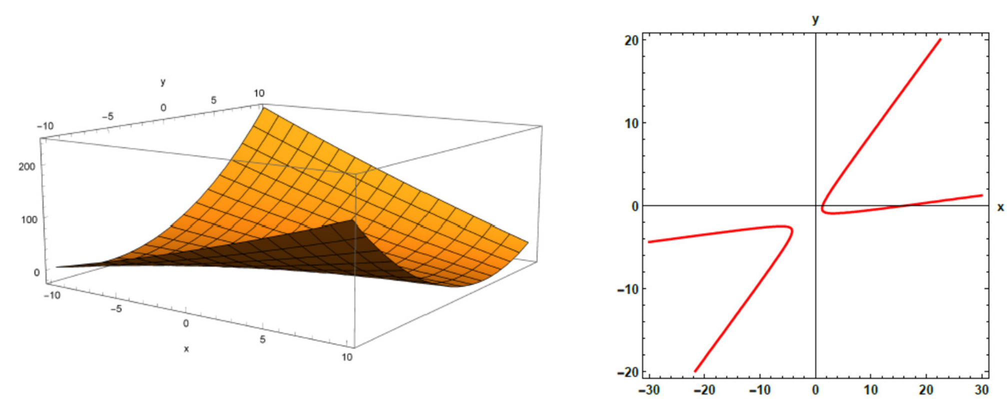
$$
\begin{aligned}
&\Bigg[\Bigg\{\ln(1+D_y)-\frac{1}{(1+(\ln(1+D_y))^{-1})\ln(1+(\ln(1+D_y))^{-1})}+z+y\frac{\ln(1+D_y)}{\ln(1+D_y)+1}\\
&\quad-\frac{1}{2}\sqrt{x\ln(1+D_y)}\tan\left(\sqrt{\frac{x}{\ln(1+D_y)}}\right)\Bigg\}(\ln(1+D_y))^{-1}-j\Bigg]\,{}_b\lambda_j(x,y,z)=0.
\end{aligned}\tag{3.4}
$$

Corollary 3.2. *The generalization of second kind Bernoulli-λ polynomials satisfies the following determinant representation*:

$$
{}_b\lambda_j(x,y,z) = (-1)^j \begin{vmatrix}
{}_p\lambda_0 & {}_p\lambda_1 & {}_p\lambda_2 & \cdots & {}_p\lambda_{j-1} & {}_p\lambda_j \\
1 & \frac{-1}{2} & \frac{2}{3} & \cdots & (-1)^{j-1}\frac{(j-1)!}{j} & (-1)^j\frac{j!}{j+1} \\
0 & 1 & -1 & \cdots & (-1)^{j-2}(j-2)! & (-1)^{j-1}(j-1)! \\
0 & 0 & 1 & \cdots & \frac{(-1)^{j-3}}{2!}\frac{(j-1)!}{j-2} & \frac{(-1)^{j-2}}{2!}\frac{j!}{j-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -\frac{j}{2}
\end{vmatrix}. \tag{3.5}
$$

${}_p\lambda_j$'s here are the special case where the functions in (3.2) are used in generating function of ${}_p\lambda_j$ in Theorem 2.5.

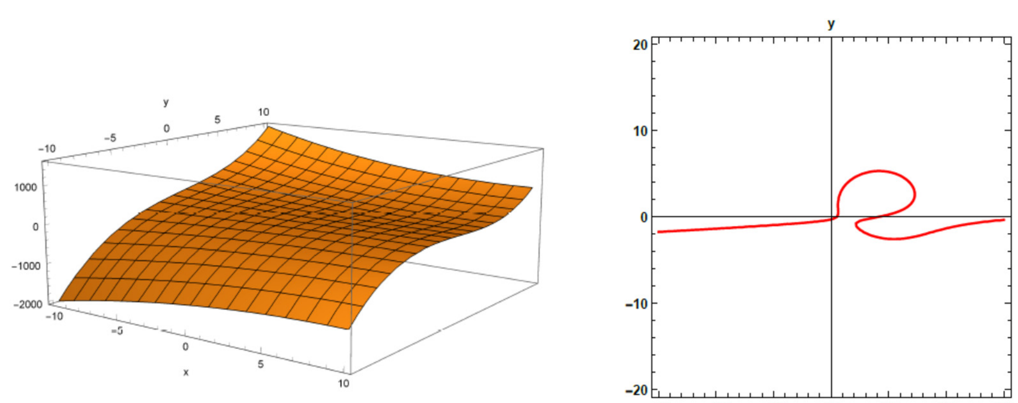
The 3D surface plots of the generalization of second kind Bernoulli-λ polynomials ${}_b\lambda_2(x,y,1)$ and the graph of the distribution of real roots for the generalization of second kind Bernoulli-λ polynomials ${}_b\lambda_2(x,y,1)$ are shown in Figure 1.



(a) *Surface plot of* ${}_b\lambda_2(x,y,1)$ (b) *Distribution of real roots of* ${}_b\lambda_2(x,y,1)$

Figure 1: Figures related to ${}_b\lambda_2(x,y,1)$.

The 3D surface plots of the generalization of second kind Bernoulli-λ polynomials ${}_b\lambda_3(x,y,1)$ and the graph of the distribution of real roots for the generalization of second kind Bernoulli-λ polynomials ${}_b\lambda_3(x,y,1)$ are shown in Figure 2.



(a) *Surface plot of* ${}_b\lambda_3(x,y,1)$ (b) *Distribution of real roots of* ${}_b\lambda_3(x,y,1)$

Figure 2: Figures related to ${}_b\lambda_3(x,y,1)$.

3.2 Generalization of Laguerre-λ polynomials

The generalization of Laguerre-λ polynomials is defined as follows:

$$\frac{1}{1-t}e^{\frac{yt}{t-1}}\cos(\sqrt{xt})e^{zt} = \sum_{j=0}^{\infty} {}_{\mathcal{L}}\lambda_j(x,y,z)\frac{t^j}{j!}, \tag{3.6}$$

where

$$A(t) = \frac{1}{1-t}, \quad Q(t) = \frac{t}{t-1}, \quad \text{and} \quad \phi(z,t) = e^{zt}. \tag{3.7}$$

Corollary 3.3. *The generalization of Laguerre-λ polynomials satisfies the following operators*:

$$\Lambda^{+}_{\mathcal{L}^{\lambda}} = D_y + z - yD_y^2 - \frac{1}{2}\sqrt{\frac{x}{1-D_y^{-1}}}\tan\left(\sqrt{x(1-D_y^{-1})}\right), \tag{3.8}$$

$$\Lambda^{-}_{\mathcal{L}^{\lambda}} = 1 - D_y^{-1} \tag{3.9}$$

and differential equation

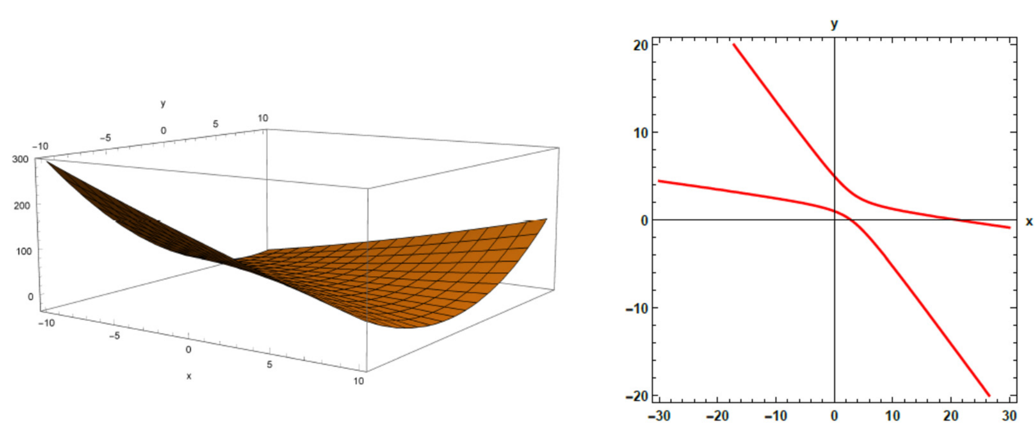
$$\left[\left[D_y + z - yD_y^2 - \frac{1}{2}\sqrt{\frac{x}{1-D_y^{-1}}}\tan\left(\sqrt{x(1-D_y^{-1})}\right)\right](1-D_y^{-1}) - j\right] {}_{\mathcal{L}}\lambda_j(x,y,z) = 0. \tag{3.10}$$

Corollary 3.4. *The generalization of Laguerre-λ polynomials satisfies the following determinant representation*:

$${}_{\mathcal{L}}\lambda_j(x,y,z) = (-1)^j \begin{vmatrix} {}_p\lambda_0 & {}_p\lambda_1 & {}_p\lambda_2 & \cdots & {}_p\lambda_{j-1} & {}_p\lambda_j \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -j \end{vmatrix}. \tag{3.11}$$

${}_p\lambda_j$'s here are the special case where the functions in (3.7) are used in generating function of ${}_p\lambda_j$ in Theorem 2.5.

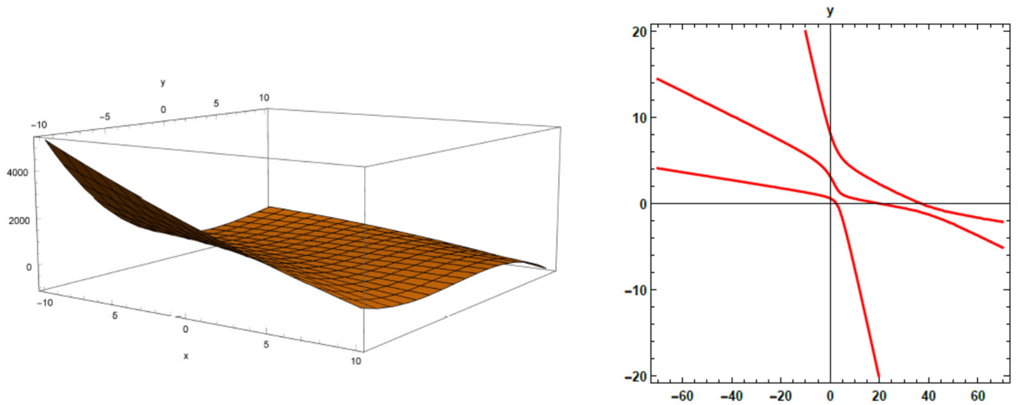
The 3D surface plots of the generalization of Laguerre-λ polynomials ${}_{\mathcal{L}}\lambda_2(x,y,1)$ and the graph of the distribution of real roots for these polynomials ${}_{\mathcal{L}}\lambda_2(x,y,1)$ are shown in Figure 3.



(a) *Surface plot of* ${}_{\mathcal{L}}\lambda_2(x,y,1)$ (b) *Distribution of real roots of* ${}_{\mathcal{L}}\lambda_2(x,y,1)$

Figure 3: Figures related to ${}_{\mathcal{L}}\lambda_2(x,y,1)$.

The 3D surface plots of the generalization of Laguerre-λ polynomials ${}_{\mathcal{L}}\lambda_3(x, y, 1)$ and the graph of the distribution of real roots for the generalization of Laguerre-λ polynomials ${}_{\mathcal{L}}\lambda_3(x, y, 1)$ are shown in Figure 4.



(a) *Surface plot of* ${}_{\mathcal{L}}\lambda_3(x, y, 1)$ (b) *Distribution of real roots of* ${}_{\mathcal{L}}\lambda_3(x, y, 1)$

Figure 4: Figures related to ${}_{\mathcal{L}}\lambda_3(x, y, 1)$.

4 Twice-iterated Sheffer-λ polynomials

In this section, we replace $\phi(z, t)$ in equation (2.1), by $B(t)$ which lead us to a new family which is called twice-iterated Sheffer-λ polynomials. Choosing $\phi(z, t) = B(t)$ in Theorems 2.1–2.5, we have the following results (Corollaries 4.1–4.5), respectively.

Definition 4.1. The twice-iterated Sheffer-λ polynomials ${}_s\lambda_j^{[2]}(x, y)$ are defined by the following generating function:

$$A(t)B(t)e^{yQ(t)}\cos(\sqrt{xt}) = \sum_{j=0}^{\infty} {}_s\lambda_j^{[2]}(x, y)\frac{t^j}{j!}, \tag{4.1}$$

where

$$A(t) = \sum_{j=0}^{\infty} a_j \frac{t^j}{j!}, \quad a_0 \neq 0, \quad B(t) = \sum_{j=0}^{\infty} b_j \frac{t^j}{j!}, \quad b_0 \neq 0$$

and

$$Q(t) = \sum_{j=1}^{\infty} q_j \frac{t^j}{j!}, \quad q_1 \neq 0. \tag{4.2}$$

Corollary 4.1. *The twice-iterated Sheffer-λ polynomials ${}_s\lambda_j^{[2]}(x, y)$ have the following series representation:*

$${}_s\lambda_j^{[2]}(x, y) = \sum_{l=0}^{j}\sum_{k=0}^{j-l} \frac{(-1)^k x^k s_{j-k-l}(y) b_l\, j!}{(2k)!(j-k-l)!\,l!}. \tag{4.3}$$

Corollary 4.2. *The twice-iterated Sheffer-λ polynomials ${}_s\lambda_j^{[2]}(x, y)$ satisfies properties*

$${}_s\lambda_j^{[2]}(x, y + v) = \sum_{k=0}^{j}\binom{j}{k} {}_s\lambda_{j-k}^{[2]}(x, y)p_k(v) \tag{4.4}$$

or

$${}_s\lambda_j^{[2]}(x, y + v) = \sum_{k=0}^{j}\binom{j}{k} {}_s\lambda_{j-k}^{[2]}(x, v)p_k(y). \tag{4.5}$$

Corollary 4.3. *The twice-iterated Sheffer-λ polynomials* ${}_s\lambda_j^{[2]}(x,y)$ *satisfies the following quasi-monomiality operators*:

$$(\Lambda_{s^\lambda}^{[2]})^{+} = \frac{A'(Q^{-1}(D_y))}{A(Q^{-1}(D_y))} + \frac{B'(Q^{-1}(D_y))}{B(Q^{-1}(D_y))} + yQ'(Q^{-1}(D_y)) - \frac{1}{2}\sqrt{\frac{x}{Q^{-1}(D_y)}}(\tan\sqrt{xQ^{-1}(D_y)}), \tag{4.6}$$

$$(\Lambda_{s^\lambda}^{[2]})^{-} = Q^{-1}(D_y). \tag{4.7}$$

Corollary 4.4. *The differential equation of the twice-iterated Sheffer-λ polynomials* ${}_s\lambda_j^{[2]}(x,y)$ *is as follows*:

$$\begin{aligned}\Bigg[\bigg(\frac{A'(Q^{-1}(D_y))}{A(Q^{-1}(D_y))} + \frac{B'(Q^{-1}(D_y))}{B(Q^{-1}(D_y))} + yQ'(Q^{-1}(D_y))\bigg)Q^{-1}(D_y)\\ - \frac{1}{2}\sqrt{\frac{x}{Q^{-1}(D_y)}}(\tan\sqrt{xQ^{-1}(D_y)})Q^{-1}(D_y) - j\Bigg]\, {}_s\lambda_j^{[2]}(x,y) = 0.\end{aligned} \tag{4.8}$$

Corollary 4.5. *The twice-iterated Sheffer-λ polynomials* ${}_s\lambda_j^{[2]}(x,y)$ *satisfies the following determinant representation*:

$${}_s\lambda_j^{[2]}(x,y) = \frac{(-1)^j}{(\gamma_0)^{j+1}}\begin{vmatrix} {}_s\lambda_0 & {}_s\lambda_1 & \cdots & {}_s\lambda_{j-1} & {}_s\lambda_j \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{j-1} & \gamma_j \\ 0 & \gamma_0 & \cdots & \binom{j-1}{1}\gamma_{j-2} & \binom{j}{1}\gamma_{j-1} \\ 0 & 0 & \cdots & \binom{j-1}{2}\gamma_{j-3} & \binom{j}{2}\gamma_{j-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_0 & \binom{j}{j-1}\gamma_1 \end{vmatrix}, \tag{4.9}$$

where ${}_s\lambda_j \coloneqq {}_s\lambda_j(x,y)$ *are defined in equation* (1.8) *and* $\frac{1}{B(t)} = \sum_{k=0}^{\infty}\gamma_k\frac{t^k}{k!}$.

5 Examples

In this part, we introduce the twice-iterated second kind Bernoulli-λ polynomials and the twice-iterated Laguerre-λ polynomials with some particular selections of the functions $A(t)$, $B(t)$ and $Q(t)$. We also examine the corresponding characteristic properties of these new subpolynomials.

5.1 Twice-iterated second kind Bernoulli-λ polynomials

The twice-iterated second kind Bernoulli-λ polynomials are defined as follows:

$$\left(\frac{t}{\ln(1+t)}\right)^2(1+t)^y\cos(\sqrt{xt}) = \sum_{j=0}^{\infty}{}_b\lambda_j^{[2]}(x,y)\frac{t^j}{j!}, \tag{5.1}$$

where

$$A(t) = B(t) = \frac{t}{\ln(1+t)} \quad\text{and}\quad Q(t) = \ln(1+t). \tag{5.2}$$

Corollary 5.1. *The twice-iterated second kind Bernoulli-λ polynomials satisfies the following operators*:

$$\begin{aligned}(\Lambda^{[2]}_{b^{\lambda}})^{+} = {} & 2\ln(1+D_y) - \frac{2}{(1+(\ln(1+D_y))^{-1})\ln(1+(\ln(1+D_y))^{-1})} + y\frac{\ln(1+D_y)}{\ln(1+D_y)+1} \\ & - \frac{1}{2}\sqrt{x\ln(1+D_y)}\tan\left(\sqrt{\frac{x}{(1+Dy)}}\right),\end{aligned} \tag{5.3}$$

$$(\Lambda^{[2]}_{b^{\lambda}})^{-} = (\ln(1+D_y))^{-1} \tag{5.4}$$

and differential equation

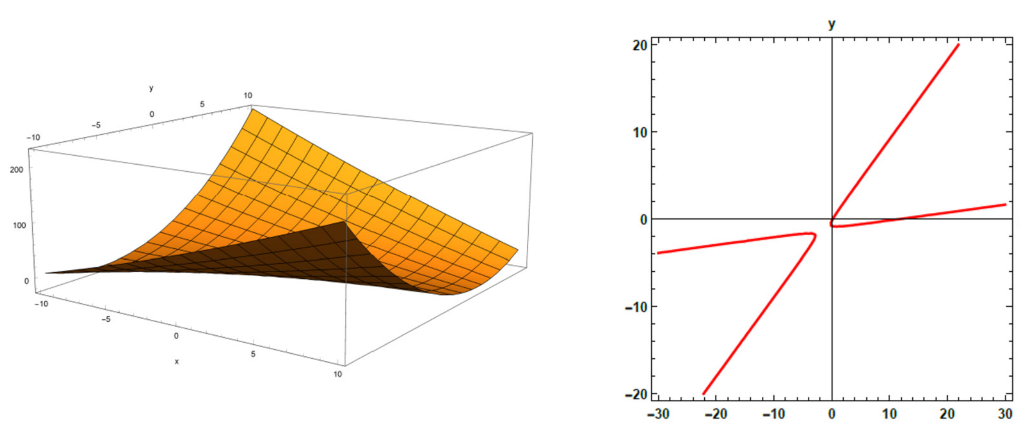
$$\begin{aligned}\Bigg[\Bigg\{ & 2\ln(1+D_y) - \frac{2}{(1+(\ln(1+D_y))^{-1})\ln(1+(\ln(1+D_y))^{-1})} + y\frac{\ln(1+D_y)}{\ln(1+D_y)+1} \\ & - \frac{1}{2}\sqrt{x\ln(1+D_y)}\tan\left(\sqrt{\frac{x}{\ln(1+D_y)}}\right)\Bigg\}(\ln(1+D_y))^{-1} - j\Bigg]\, {}_b\lambda^{[2]}_j(x,y) = 0.\end{aligned} \tag{5.5}$$

Corollary 5.2. *The twice-iterated second kind Bernoulli-λ polynomials satisfies the following determinant representation*:

$${}_b\lambda^{[2]}_j(x,y) = (-1)^j \begin{vmatrix} {}_s\lambda_0 & {}_s\lambda_1 & {}_s\lambda_2 & \cdots & {}_s\lambda_{j-1} & {}_s\lambda_j \\ 1 & \frac{-1}{2} & \frac{2}{3} & \cdots & (-1)^{j-1}\frac{(j-1)!}{j} & (-1)^j\frac{j!}{j+1} \\ 0 & 1 & -1 & \cdots & (-1)^{j-2}(j-2)! & (-1)^{j-1}(j-1)! \\ 0 & 0 & 1 & \cdots & \frac{(-1)^{j-3}}{2!}\frac{(j-1)!}{j-2} & \frac{(-1)^{j-2}}{2!}\frac{j!}{j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{j}{2} \end{vmatrix}. \tag{5.6}$$

${}_s\lambda_j$'s here are the special case where the functions in (5.2) are used in generating function of ${}_s\lambda_j$ in Corollary 4.5.

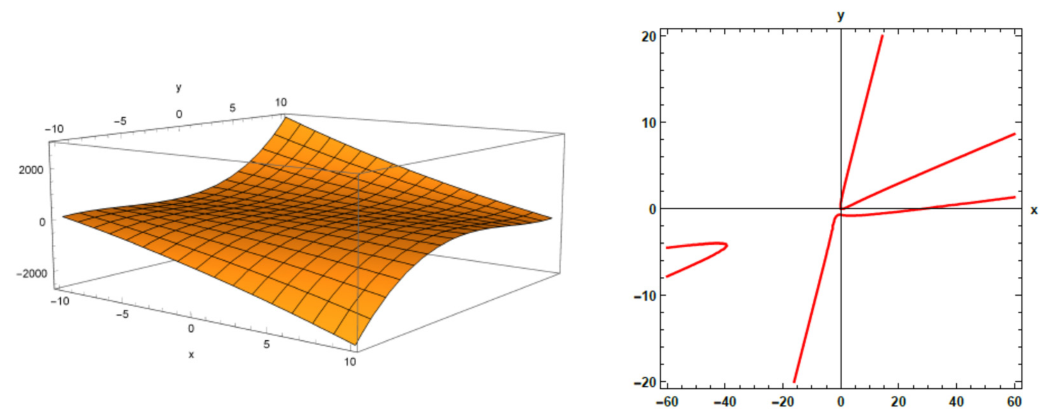
The 3D surface plots of twice-iterated second kind Bernoulli-λ polynomials ${}_b\lambda^{[2]}_2(x,y)$ and the graph of the distribution of real roots for these polynomials ${}_b\lambda^{[2]}_2(x,y)$ are shown in Figure 5.



(a) *Surface plot of* ${}_b\lambda^{[2]}_2(x,y)$ (b) *Distribution of real roots of* ${}_b\lambda^{[2]}_2(x,y)$

Figure 5: Figures related to ${}_b\lambda^{[2]}_2(x,y)$.

The 3D surface plots of twice-iterated second kind Bernoulli-λ polynomials ${}_b\lambda_3^{[2]}(x,y)$ and the graph of the distribution of real roots for these polynomials ${}_b\lambda_3^{[2]}(x,y)$ are shown in Figure 6.



(a) *Surface plot of* ${}_b\lambda_3^{[2]}(x,y)$ (b) *Distribution of real roots of* ${}_b\lambda_3^{[2]}(x,y)$

Figure 6: Figures related to ${}_b\lambda_3^{[2]}(x,y)$.

5.2 Twice-iterated Laguerre-λ polynomials

The twice-iterated Laguerre-λ polynomials are defined as follows:

$$\left(\frac{1}{1-t}\right)^2 e^{\frac{yt}{t-1}}\cos(\sqrt{xt}) = \sum_{j=0}^{\infty} {}_{\mathcal{L}}\lambda_j^{[2]}(x,y)\frac{t^j}{j!}, \tag{5.7}$$

where

$$A(t) = B(t) = \frac{1}{1-t} \quad \text{and} \quad Q(t) = \frac{t}{t-1}. \tag{5.8}$$

Corollary 5.3. *The twice-iterated Laguerre-λ polynomials satisfies the following operators*:

$$(\Lambda_{\mathcal{L}\lambda}^{[2]})^+ = 2D_y - yD_y^2 - \frac{1}{2}\sqrt{\frac{x}{1-D_y^{-1}}}\tan\left(\sqrt{x(1-D_y^{-1})}\right), \tag{5.9}$$

$$(\Lambda_{\mathcal{L}\lambda}^{[2]})^- = 1 - D_y^{-1} \tag{5.10}$$

and differential equation

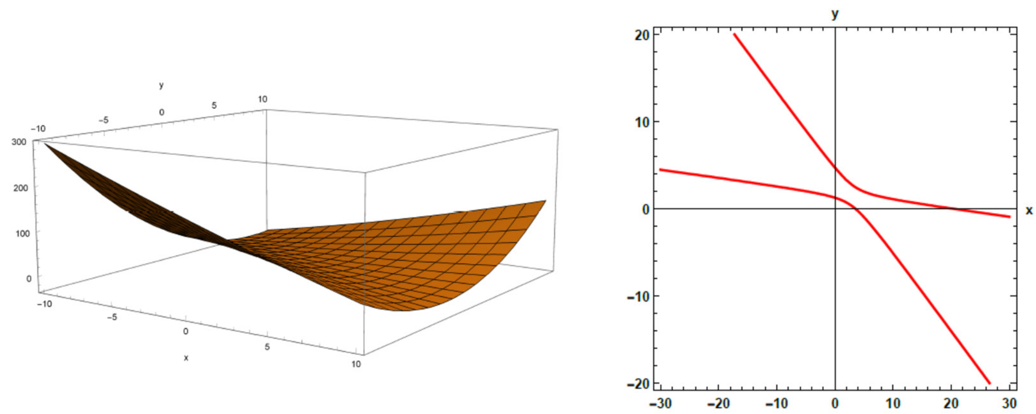
$$\left[\left\{2D_y - yD_y^2 - \frac{1}{2}\sqrt{\frac{x}{1-D_y^{-1}}}\tan\left(\sqrt{x(1-D_y^{-1})}\right)\right\}(1-D_y^{-1}) - j\right]{}_{\mathcal{L}}\lambda_j^{[2]}(x,y) = 0. \tag{5.11}$$

Corollary 5.4. *The twice-iterated Laguerre-λ polynomials satisfies the following determinant representation*:

$${}_{\mathcal{L}}\lambda_j^{[2]}(x,y) = (-1)^j \begin{vmatrix} {}_s\lambda_0 & {}_s\lambda_1 & {}_s\lambda_2 & \cdots & {}_s\lambda_{j-1} & {}_s\lambda_j \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -j \end{vmatrix}. \tag{5.12}$$

${}_s\lambda_j$'s here are the special case where the functions in (5.8) are used in generating function of ${}_s\lambda_j$ in Corollary 4.5.

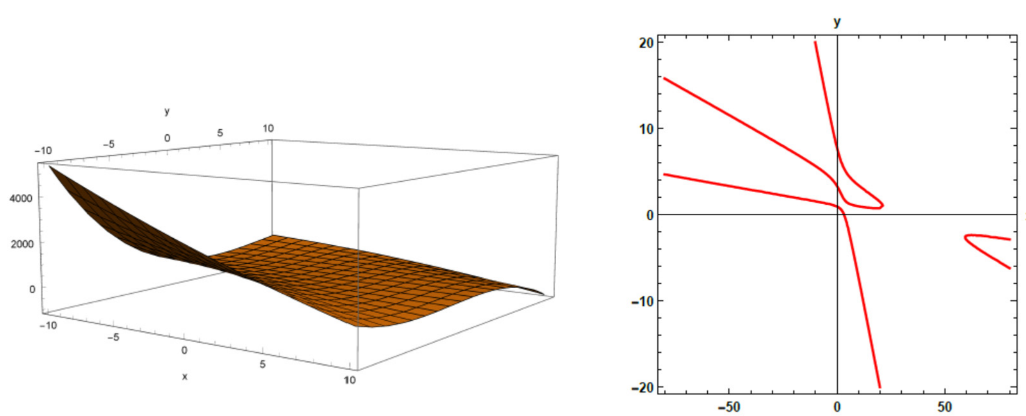
The 3D surface plots of twice-iterated Laguerre-λ polynomials ${}_{\mathcal{L}}\lambda_2^{[2]}(x,y)$ and the graph of the distribution of real roots for these polynomials ${}_{\mathcal{L}}\lambda_2^{[2]}(x,y)$ are shown in Figure 7.



(a) *Surface plot of* ${}_{\mathcal{L}}\lambda_2^{[2]}(x,y)$ (b) *Distribution of real roots of* ${}_{\mathcal{L}}\lambda_2^{[2]}(x,y)$

Figure 7: Figures related to ${}_{\mathcal{L}}\lambda_2^{[2]}(x,y)$.

The 3D surface plots of twice-iterated Laguerre-λ polynomials ${}_{\mathcal{L}}\lambda_3^{[2]}(x,y)$ and the graph of the distribution of real roots for these polynomials ${}_{\mathcal{L}}\lambda_3^{[2]}(x,y)$ are shown in Figure 8.



(a) *Surface plot of* ${}_{\mathcal{L}}\lambda_3^{[2]}(x,y)$ (b) *Distribution of real roots of* ${}_{\mathcal{L}}\lambda_3^{[2]}(x,y)$

Figure 8: Figures related to ${}_{\mathcal{L}}\lambda_3^{[2]}(x,y)$.

6 Conclusion

In this work, we introduce a new generalization of Sheffer-λ polynomials and twice-iterated Sheffer-λ polynomials and obtain some properties corresponding to these polynomials. We obtain subfamilies for these two new polynomial families and show their corresponding properties. In this way, the investigation of different hybrid polynomial families containing Sheffer polynomials and λ-polynomials can be a new field of study. In this context, analyzing these hybrid structures in more detail and investigating their relationships with classical polynomial families can provide significant contributions both theoretically and practically. In addition, the potential uses of these polynomial families in different disciplines can be investigated.

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