DE GRUYTER Demonstratio Mathematica 2025; 58: 20250149

Research Article

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Solution of nonlinear Langevin equations involving Hilfer-Hadamard fractional order derivatives and variable coefficients

https://doi.org/10.1515/dema-2025-0149
received September 21, 2024; accepted June 17, 2025

Abstract: This study innovates a novel technique of the nonlinear fractional Langevin equation of Hilfer-Hadamard type, incorporating an initial condition. The research demonstrates that this problem can be reformulated as an integral equation featuring a Mittag-Leffler function within the kernel. Through rigorous analysis, we establish the existence and uniqueness of solutions for this problem without imposing a contractive assumption. Furthermore, the study extends these findings to encompass two specific types of fractional differential equations characterized by two fractional derivatives and a variable coefficient. Theoretical conclusions are supported by the provision of two illustrative examples.

Keywords: fractional Langevin equations, Hilfer-Hadamard fractional derivative, variable coefficients, weighted space

MSC 2020: 26A33, 33E12, 34A08

1 Introduction

Fractional differential equations (FDEs) play a decisive role in addressing practical challenges across various disciplines such as biology, physics, and electrochemistry [1–6]. Recent advancements in FDE research, particularly those involving the Caputo or Riemann-Liouville fractional derivative, have shown notable progress over the past decade. The introduction of the Hadamard derivative [7] by Hadamard in 1892 has provided a new perspective in this field. Detailed information on the Hadamard derivative can be found in [4,8] and its associated references. More recently, the Hilfer and Hilfer-Hadamard fractional derivatives have attracted considerable interest among researchers [9–11], which generalize the Riemann-Liouville derivative. There are some equations derived from practical problems that have been studied, such as Hilfer-Hadamard type FDEs [12] and Hilfer-Hadamard FDEs in resistor, inductor, capacitor circuit models [13], etc. For recent developments in Hilfer-Hadamard FDEs, readers can explore [14–17] and its related references.

Fractional Langevin equations (FLEs) serve as a valuable mathematical framework for characterizing various phenomena within dynamic systems of complex media. Existing research has concentrated on exploring FLEs incorporating the Caputo or Riemann-Liouville fractional derivative under diverse boundary conditions. These FDEs have found applications in disciplines including thermoelasticity, groundwater systems, and blood flow dynamics [18,19].

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This study investigates a specific FLE incorporating Hilfer-Hadamard derivatives and a variable coefficient:

$$\begin{cases} {}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda]x(t) + \eta(t){}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) = f(t, x(t)), \quad t \in (a, b], & (1) \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c, & (2) \end{cases}$$

where $0 < \alpha_2 < \gamma_2 < \alpha_1 < \gamma_1$, $a, b > 0$, and $\lambda > 0$, the functions $\eta(t)$ and $f(t, x(t))$ will be defined subsequently.

The complexity of variable-coefficient functions leads to challenges in directly obtaining a representation of solutions to (1) and (2), very few papers have considered FLEs with Hilfer-Hadamard fractional derivatives and variable coefficients. Consequently, to address this gap in the literature, we focus on investigating a specific nonlinear FLE with Hilfer-Hadamard fractional derivatives:

$$\begin{cases} {}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda]x(t) = g(t, x(t), {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)), \quad t \in (a, b], & (3) \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c. & (4) \end{cases}$$

Clearly, (1) is the special case of (3).

Integral transforms and fixed point theorems are commonly used methods for dealing with nonlinear FDEs. For example, in [19], the authors investigated the following nonlinear FLE with antiperiodic boundary conditions

$$\begin{cases} D^\beta(D^\alpha + \lambda)x(t) = f(t, x(t)), \quad t \in (0, 1), 0 < \alpha < 1, 1 \le \beta \le 2, \\ x(0) + x(1) = 0, D^\alpha x(0) + D^\alpha x(1) = 0, D^{2\alpha}x(0) + D^{2\alpha}x(1) = 0, \end{cases}$$

where D^α is the Caputo fractional derivative of order α. They transformed the aforementioned problem into a nonlinear mixed Fredholm-Volterra integral equation, gave the following solution:

$$x(t) = I^{\alpha+\beta}f(\cdot, x(\cdot))(t) - \lambda I^\alpha x(\cdot)(t) + c_0 + \frac{c_1 t^\alpha}{\Gamma(\alpha + 1)} + \frac{c_2 t^{\alpha+1}}{\Gamma(\alpha + 2)}, \tag{5}$$

and decided the constants c_0, c_1 and c_2 by boundary value conditions [19, Lemma 3.1]. Moreover, they obtained the existence and uniqueness of solution under appropriate assumptions on f and a contractive assumption $\Lambda_{\alpha,\beta,L} < 1$ [19, Theorem 3.1].

In our paper, we utilize the generalized Mittag-Leffler function $E_{\mu,\nu}$ [20] and the Prabhakar integral operator [21] to define

$$\mathcal{E}_{\mu,\nu;\,\lambda}(t, s) = \left(\log\frac{t}{s}\right)^{\nu-1} E_{\mu,\nu}\left[-\lambda\left(\log\frac{t}{s}\right)^\mu\right], \tag{6}$$

$$(\mathbf{E}_{\mu,\nu;\,\lambda}\phi)(t) = \int_a^t \left(\log\frac{t}{s}\right)^{\nu-1} E_{\mu,\nu}\left[-\lambda\left(\log\frac{s}{a}\right)^\mu\right]\frac{\phi(s)}{s}\mathrm{d}s, \tag{7}$$

and obtain a unique solution for problems (3) and (4) by

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t, a) + [\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}g(s, x(s), {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(s))](t). \tag{8}$$

An analytical solution for problems (1) and (2) is derived as a specific instance in Section 3. This approach differs from the techniques presented in [19]. Our approach enables us to derive a novel representation of the solution, which can describe more clearly the structure of the solution (Remark 3.1), and brings convenience for further study of the other properties of the solution, including Hyers-Ulam stability and attractivity, etc. Moreover, by utilizing the properties of Mittag-Leffler functions, we can study boundary value problems for corresponding FLEs.

The main contributions of our article are given as follows:

- The general solution for equation (3) is provided using the generalized Mittag-Leffler function, as outlined in Theorem 4.1.

- In the absence of a contractive assumption, the existence and uniqueness of solutions are established, as indicated in Theorem 4.1. This technique can help us remove an indispensable hypothesis similar to $\Lambda_{\alpha,\beta,L} < 1$ [19, Theorem 3.1].
- Explicit solutions for two types of initial value problems (IVPs) with a variable coefficient are derived as specific instances of problems (3) and (4), as demonstrated in problems (22) and (23) and problems (27) and (28).

This article is organized as follows: Section 2 includes definitions and properties that will be referenced in succeeding sections. Sections 3 and 4 delve into the existence and uniqueness of solutions. Section 5 outlines the conclusions drawn from the analysis of two types of linear equations with variable coefficients. Finally, two examples are provided in Section 6 to demonstrate the findings.

2 Preliminaries

Definition 2.1. [4] Let $\alpha > 0$ and $n = [\alpha] + 1$. The left-sided Hadamard fractional integral and fractional derivative of order α for a function h are defined as follows:

$$(\mathcal{J}_{a^+}^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{h(\tau)}{\tau}\mathrm{d}\tau, \quad t > a,$$

$$(\mathcal{D}_{a^+}^{\alpha}h)(t) = \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n(\mathcal{J}_{a^+}^{n-\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha+1}\frac{h(\tau)}{\tau}\mathrm{d}\tau, \quad t > a.$$

Lemma 2.1. [4] *If* $0 < \alpha < 1$, $\beta > 0$, $0 < a < b < \infty$, $t \in (a, b]$, *then the following statements hold:*

$$\mathcal{J}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\log\frac{t}{a}\right)^{\alpha+\beta-1},$$

$$\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1}, \quad \beta > \alpha,$$

$$\mathcal{D}_{a^+}^{\alpha}1 = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha}, \qquad \mathcal{D}_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\alpha-1} = 0.$$

Let $C[a, b]$ be the space of continuous functions y on $[a, b]$ with the norm $\|y\|_C = \max_{t\in[a,b]}|y(t)|$. For $0 \le \nu < 1$, we denote the weighted space

$$C_{\nu,\log}[a, b] \coloneqq \left\{y \in C(a, b]; \left(\log\frac{t}{a}\right)^{\nu} y(t) \in C[a, b]\right\} \quad \text{and} \quad \|y\|_{C_{\nu,\log}} = \max_{t\in[a,b]}\left|\left(\log\frac{t}{a}\right)^{\nu} y(t)\right|. \tag{9}$$

Clearly, $C_{\nu,\log}[a, b]$ is a Banach space [4], and $C_{0,\log}[a, b] = C[a, b]$. The notations C and $C_{\nu,\log}$ are used as abbreviations for $C[a, b]$ and $C_{\nu,\log}[a, b]$, respectively.

We introduce the weighted space

$$C_{1-\gamma,\log}^{\alpha,\beta}[a, b] = \{y(t) \in C_{1-\gamma,\log}[a, b];\ {}_H\mathcal{D}_{a^+}^{\alpha,\beta}y(t) \in C_{1-\gamma,\log}[a, b]\},$$

with the norm $\|y\|_{C_{1-\gamma,\log}^{\alpha,\beta}} = \max\{\|y\|_{C_{1-\gamma,\log}}, \|{}_H\mathcal{D}_{a^+}^{\alpha,\beta}y\|_{C_{1-\gamma,\log}}\}$. We abbreviate $C_{1-\gamma,\log}^{\alpha,\beta}[a, b]$ by $C_{1-\gamma,\log}^{\alpha,\beta}$. Clearly,

$$C_{1-\gamma,\log}^{\alpha,\beta}[a, b] \subset C_{1-\gamma,\log}[a, b].$$

Lemma 2.2. [4] *Let* $\omega_1, \omega_2 > 0$, *and* $h \in C_{\mu,\log}$. *Then the following statements hold*:

$$(\mathcal{J}_{a^+}^{\omega_1}\mathcal{J}_{a^+}^{\omega_2}h)(t) = (\mathcal{J}_{a^+}^{\omega_1+\omega_2}h)(t),$$

$$(\mathcal{D}_{a^+}^{\omega_1}\mathcal{J}_{a^+}^{\omega_1}h)(t) = h(t),$$

$$(\mathcal{D}_{a^+}^{\omega_2}\mathcal{J}_{a^+}^{\omega_1}h)(t) = (\mathcal{J}_{a^+}^{\omega_1-\omega_2}h)(t), \quad \textit{for}\ \ \omega_1 > \omega_2.$$

Lemma 2.3. [4] *Let* $\omega > 0$ *and* $y \in C_{\nu,\log}$.
(i) *If* $\nu \leq \omega$, *then* $(\mathcal{J}^{\omega}_{a^+}y)(t) \in C$.
(ii) *If* $\nu > \omega$, *then* $(\mathcal{J}^{\omega}_{a^+}y)(t) \in C_{\nu-\omega,\log}$.

Lemma 2.4. *Let* $\omega \in (0,1)$ *and* $y \in C_{\nu,\log}$, *if* $\omega > \nu$, *then*

$$(\mathcal{J}^{\omega}_{a^+}y)(a^+) = \lim_{t\to a^+}(\mathcal{J}^{\omega}_{a^+}y)(t) = 0.$$

Proof. From the definition of $\mathcal{J}^{\omega}_{a^+}$, we have

$$\begin{aligned} |(\mathcal{J}^{\omega}_{a^+}y)(t)| &\leq \frac{\|y\|_{C_{\nu,\log}}}{\Gamma(\omega)}\int_a^t\left(\log\frac{t}{s}\right)^{\omega-1}\left(\log\frac{s}{a}\right)^{-\nu}\frac{ds}{s} \\ &= (\log t - \log a)^{\omega-\nu}\cdot\frac{\Gamma(1-\nu)}{\Gamma(\omega+1-\nu)}\|y\|_{C_{\nu,\log}} \\ &\to 0, \quad t\to a^+. \end{aligned} \tag{10}$$

□

Lemma 2.5. [4] *Let* $\omega \in (0,1)$. *If* $y \in C_{\nu,\log}$ *and* $\mathcal{J}^{1-\omega}_{a^+}y \in C^1_{\nu,\log}$, *then*

$$(\mathcal{J}^{\omega}_{a^+}\mathcal{D}^{\omega}_{a^+}y)(t) = y(t) - \frac{(\mathcal{J}^{1-\omega}_{a^+}y)(a^+)}{\Gamma(\omega)}\left(\log\frac{t}{a}\right)^{\omega-1}.$$

2.1 Hilfer-Hadamard fractional derivative

Similar to [22], we present a definition of modified Hilfer-Hadamard derivative.

Definition 2.2. The left-sided Hilfer-Hadamard fractional derivative of order $\alpha \in (0,1)$, $\beta \in [0,1]$ for $h(t)$ is defined by

$${}_H\mathcal{D}^{\alpha,\beta}_{a^+}h(t) = \mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)], \quad t \in (a,b],$$

where $\gamma = \alpha + \beta(1-\alpha)$.

Lemma 2.6. *If* $0 < \alpha, \omega < 1$, $\beta \in [0,1]$, *then*

$${}_H\mathcal{D}^{\alpha,\beta}_{a^+}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right](t) = 0, \quad t > a,$$

$${}_H\mathcal{D}^{\alpha,\beta}_{a^+}\left[\left(\log\frac{s}{a}\right)^{\omega-1}\right](t) = \frac{\Gamma(\omega)}{\Gamma(\omega-\alpha)}\left(\log\frac{t}{a}\right)^{\omega-\alpha-1}, \quad t > a, \omega > \gamma.$$

Proof. From Lemma 2.1, we deduce that

$${}_H\mathcal{D}^{\alpha,\beta}_{a^+}\left[\left(\log\frac{s}{a}\right)^{\gamma-1}\right](t) = \mathcal{D}^{1-\gamma+\alpha}_{a^+}\left[\mathcal{J}^{1-\gamma}_{a^+}\left(\log\frac{t}{a}\right)^{\gamma-1} - \left(\mathcal{J}^{1-\gamma}_{a^+}\left(\log\frac{t}{a}\right)^{\gamma-1}\right)(a^+)\right] = 0,$$

$$\begin{aligned} {}_H\mathcal{D}^{\alpha,\beta}_{a^+}\left[\left(\log\frac{s}{a}\right)^{\omega-1}\right](t) &= \mathcal{D}^{1-\gamma+\alpha}_{a^+}\left[\mathcal{J}^{1-\gamma}_{a^+}\left(\log\frac{t}{a}\right)^{\omega-1} - \left(\mathcal{J}^{1-\gamma}_{a^+}\left(\log\frac{t}{a}\right)^{\omega-1}\right)(a^+)\right] \\ &= \frac{\Gamma(\omega)}{\Gamma(\omega+1-\gamma)}\mathcal{D}^{1-\gamma+\alpha}_{a^+}\left(\log\frac{t}{a}\right)^{\omega-\gamma} \\ &= \frac{\Gamma(\omega)}{\Gamma(\omega-\alpha)}\left(\log\frac{t}{a}\right)^{\omega-\alpha-1}. \end{aligned}$$

□

In addition, Lemma 2.6 implies the following result.

Lemma 2.7. *For $\alpha \in (0,1)$ and $\beta \in [0,1)$, if $h \in C^{\alpha,\beta}_{1-\gamma,\log}$, then ${}_H\mathcal{D}^{\alpha,\beta}_{a^+}h(t) = 0$ if and only if $h(t) = d\left(\log\frac{t}{a}\right)^{\gamma-1}$, where d is an arbitrary constant.*

Lemma 2.8. *For $h \in C_{1-\gamma,\log}$, ${}_H\mathcal{D}^{\alpha,\beta}_{a^+}\mathcal{J}^{\alpha}_{a^+}h(t) = h(t)$.*

Proof. By Lemmas 2.4 and 2.2, we infer that

$$
\begin{aligned}
{}_H\mathcal{D}^{\alpha,\beta}_{a^+}\mathcal{J}^{\alpha}_{a^+}h(t) &= \mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma+\alpha}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma+\alpha}_{a^+}h)(a^+)] \\
&= \mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma+\alpha}_{a^+}h)(t)] = h(t).
\end{aligned}
$$

□

Lemma 2.9. *If $h \in C^{\alpha,\beta}_{1-\gamma,\log}$, then*

$$
\mathcal{J}^{\alpha}_{a^+}{}_H\mathcal{D}^{\alpha,\beta}_{a^+}h(t) = h(t) - \frac{(\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)}{\Gamma(\gamma)}\left(\log\frac{t}{a}\right)^{\gamma-1}.
$$

Proof. Since $h \in C^{\alpha,\beta}_{1-\gamma,\log}$, we can see that $h \in C_{1-\gamma,\log}$ and ${}_H\mathcal{D}^{\alpha,\beta}_{a^+}h(t) \in C_{1-\gamma,\log}$, then

$$
\mathcal{J}^{\gamma-\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] \in C^{1}_{1-\gamma,\log}
$$

and

$$
\{\mathcal{J}^{\gamma-\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)]\}(a^+) = 0. \tag{11}
$$

From Lemma 2.5, it follows that

$$
\mathcal{J}^{1-\gamma+\alpha}_{a^+}\mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] = (\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+),
$$

thus

$$
\begin{aligned}
\mathcal{J}^{\alpha}_{a^+}{}_H\mathcal{D}^{\alpha,\beta}_{a^+}h(t) &= \mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] \\
&= \mathcal{D}^{1-\gamma}_{a^+}\mathcal{J}^{1-\gamma}_{a^+}\mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] \\
&= \mathcal{D}^{1-\gamma}_{a^+}\mathcal{J}^{1-\gamma+\alpha}_{a^+}\mathcal{D}^{1-\gamma+\alpha}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] \\
&= \mathcal{D}^{1-\gamma}_{a^+}[(\mathcal{J}^{1-\gamma}_{a^+}h)(t) - (\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)] \\
&= h(t) - \frac{(\mathcal{J}^{1-\gamma}_{a^+}h)(a^+)}{\Gamma(\gamma)}\left(\log\frac{t}{a}\right)^{\gamma-1}.
\end{aligned}
$$

□

Theorem 2.1. *Let $\alpha \in (0,1)$, $\beta \in [0,1)$. Then $g \in C^{\alpha,\beta}_{1-\gamma,\log}$ if and only if*

$$
g(t) = (\mathcal{J}^{\alpha}_{a^+}\varphi)(t) + c\left(\log\frac{t}{a}\right)^{\gamma-1}, \tag{12}
$$

where $\varphi \in C_{1-\gamma,\log}$ and c is an arbitrary constant.

Proof. Let $g \in C^{\alpha,\beta}_{1-\gamma,\log}$, there exists $\varphi(t) \in C_{1-\gamma,\log}$ such that $({}_H\mathcal{D}^{\alpha,\beta}_{a^+}g)(t) = \varphi(t)$, then $\mathcal{J}^{\alpha}_{a^+}{}_H\mathcal{D}^{\alpha,\beta}_{a^+}g(t) = \mathcal{J}^{\alpha}_{a^+}\varphi(t)$. By Lemma 2.9, we have

$$
\mathcal{J}^{\alpha}_{a^+}{}_H\mathcal{D}^{\alpha,\beta}_{a^+}g(t) = g(t) - \frac{(\mathcal{J}^{1-\gamma}_{a^+}g)(a^+)}{\Gamma(\gamma)}\left(\log\frac{t}{a}\right)^{\gamma-1}.
$$

Thus $g(t) = \mathcal{J}^{\alpha}_{a^+}\varphi(t) + c\left(\log\frac{t}{a}\right)^{\gamma-1}$, where $c = \frac{(\mathcal{J}^{1-\gamma}_{a^+}g)(a^+)}{\Gamma(\gamma)}$.

If g satisfies (12), then $({}_H\mathcal{D}^{\alpha,\beta}_{a^+}g)(t) = \varphi(t)$ and $\frac{(\mathcal{J}^{1-\gamma}_{a^+}g)(a^+)}{\Gamma(\gamma)} = c$. □

2.2 Mittag-Leffler functions

Definition 2.3. [4,20] Let $\rho, \sigma > 0$, $z \in \mathbb{C}$, the generalized Mittag-Leffler function $E_{\rho,\sigma}$ is defined by:

$$E_{\rho,\sigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \sigma)}.$$

Lemma 2.10. [4,20] *Let $\rho \in (0,1)$, $\sigma > 0$, $E_{\rho,\sigma}(\cdot)$ is nonnegative and for any $z \le 0$:*

$$E_{\rho,\rho}(z) \le \frac{1}{\Gamma(\rho)}, \quad E_{\rho,\sigma}(z) \le \frac{1}{\Gamma(\sigma)}.$$

Similar to [20], we obtain the following conclusions.

Lemma 2.11. *For $\rho, \sigma, \omega > 0$, and $\lambda > 0$,*

$$\mathcal{J}_{a^+}^{\omega}[\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a)](t) = \mathcal{E}_{\rho,\sigma+\omega;\,\lambda}(t,a),$$
$$\mathcal{D}_{a^+}^{\omega}[\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a)](t) = \mathcal{E}_{\rho,\sigma-\omega;\,\lambda}(t,a), \quad \sigma > \omega.$$

Theorem 2.2. *For $\rho, \sigma > 0$, and $\omega > 0$, the following formulas hold:*
(i) ${}_H\mathcal{D}_{a^+}^{\alpha,\beta}[\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a)](t) = \mathcal{E}_{\rho,\sigma-\alpha;\,\lambda}(t,a)$, *for* $\sigma > \alpha$,
(ii) ${}_H\mathcal{D}_{a^+}^{\alpha,\beta}[\mathcal{E}_{\rho,\gamma;\,\lambda}(s,a)](t) = -\lambda\mathcal{E}_{\rho,\gamma;\,\lambda}(t,a)$.

Proof. (i) Note that $\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a))(t) = \mathcal{E}_{\rho,\sigma+1-\gamma;\,\lambda}(t,a)$, then $\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a))(a^+) = \mathcal{E}_{\rho,\sigma+1-\gamma;\,\lambda}(a,a) = 0$. Now we arrive at

$$\begin{aligned}{}_H\mathcal{D}_{a^+}^{\alpha,\beta}(\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a))(t) &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a))(t) - (\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\sigma;\,\lambda}(s,a))(a^+))]\\ &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}(\mathcal{E}_{\rho,\sigma+1-\gamma;\,\lambda}(s,a))\\ &= \mathcal{E}_{\rho,\sigma-\alpha;\,\lambda}(t,a).\end{aligned}$$

(ii) From Definition 2.2, we infer that

$$\begin{aligned}{}_H\mathcal{D}_{a^+}^{\alpha,\beta}(\mathcal{E}_{\rho,\gamma;\,\lambda}(s,a))(t) &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\gamma;\,\lambda}(s,a))(t) - (\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\gamma;\,\lambda}(s,a))(a^+))]\\ &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[\mathcal{J}_{a^+}^{1-\gamma}(\mathcal{E}_{\rho,\gamma;\,\lambda}(s,a))(t) - E_{\alpha,1}(0)]\\ &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}(\mathcal{E}_{\alpha,1;\,\lambda}(t,a) - 1)\\ &= \mathcal{E}_{\alpha,\gamma-\alpha;\,\lambda}(t,a) - \frac{1}{\Gamma(\gamma-\alpha)}\left(\log\frac{t}{a}\right)^{\gamma-\rho-1}\\ &= -\lambda\mathcal{E}_{\rho,\gamma;\,\lambda}(t,a).\end{aligned}$$

□

Theorem 2.3. *Let $\rho, \sigma, \omega \in (0,1)$, for $\phi \in C_{1-\gamma,\log}$, then*
(i) $\mathcal{J}_{a^+}^{\omega}[(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(s)](t) = (\mathbf{E}_{\rho,\sigma+\omega;\,\lambda}\phi)(t) = (\mathbf{E}_{\rho,\sigma;\,\lambda}\mathcal{J}_{a^+}^{\omega}\phi)(t)$,
(ii) $\mathcal{D}_{a^+}^{\omega}[(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(s)](t) = (\mathbf{E}_{\rho,\sigma-\omega;\,\lambda}\phi)(t)$, $\sigma > \omega$,
(iii) ${}_H\mathcal{D}_{a^+}^{\alpha,\beta}[(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(s)](t) = (\mathbf{E}_{\rho,\sigma-\alpha;\,\lambda}\phi)(t)$, $\sigma > \alpha$,
(iv) ${}_H\mathcal{D}_{a^+}^{\alpha,\beta}[(\mathbf{E}_{\rho,\alpha;\,\lambda}\phi)(s)](t) = \phi(t) - \lambda(\mathbf{E}_{\rho,\rho;\,\lambda}\phi)(t)$.

Proof. Similar to [23, Theorem 6], (i), (ii) can be proved. By (ii), we conclude that

$$\begin{aligned}{}_H\mathcal{D}_{a^+}^{\alpha,\beta}[(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(s)](t) &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[\mathcal{J}_{a^+}^{1-\gamma}(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(t) - \mathcal{J}_{a^+}^{1-\gamma}(\mathbf{E}_{\rho,\sigma;\,\lambda}\phi)(a^+)] \\ &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[(\mathbf{E}_{\rho,\sigma+1-\gamma;\,\lambda}\phi)(t)] \\ &= (\mathbf{E}_{\rho,\sigma-\alpha;\,\lambda}\phi)(t).\end{aligned}$$

(iv)

$$\begin{aligned}{}_H\mathcal{D}_{a^+}^{\alpha,\beta}[(\mathbf{E}_{\rho,\alpha;\,\lambda}\phi)(s)](t) &= \mathcal{D}_{a^+}^{1-\gamma+\alpha}[\mathcal{J}_{a^+}^{1-\gamma}(\mathbf{E}_{\rho,\alpha;\,\lambda}\phi)(t) - \mathcal{J}_{a^+}^{1-\gamma}(\mathbf{E}_{\rho,\alpha;\,\lambda}\phi)(a^+)] \\ &= \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{J}_{a^+}^{1-\alpha}(\mathbf{E}_{\rho,\alpha;\,\lambda}\phi)(t) \\ &= \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_a^t E_\rho\left[-\lambda\left(\log\frac{t}{s}\right)^\rho\right]\frac{\phi(s)}{s}\mathrm{d}s \\ &= \phi(t) - \lambda(\mathbf{E}_{\rho,\rho;\,\lambda}\phi)(t).\end{aligned}$$

□

3 Representation of solutions

In this section, we will study the representation of solutions to IVP (3) and (4). First, similar to [24], we obtain the following result.

Theorem 3.1. *For $\lambda \in \mathbb{R}$, $\alpha > 0$, and $0 < \gamma < 1$, the following assertions hold:*
(i) for $h \in C_{1-\gamma,\log}$, the series $\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t)$ is convergent and

$$\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t) = h(t) - \lambda(\mathbf{E}_{\alpha,\alpha;\,\lambda}h)(t),$$

(ii) the operator $I + \lambda\mathcal{J}_{a^+}^{\alpha} : C_{1-\gamma,\log} \to C_{1-\gamma,\log}$ is invertible and

$$(I + \lambda\mathcal{J}_{a^+}^{\alpha})^{-1}h(t) = \sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t). \tag{13}$$

Proof. (i) For $h \in C_{1-\gamma,\log}$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t) &= h(t) + \sum_{k=1}^{\infty}(-\lambda)^k\mathcal{J}_{a^+}^{k\alpha}h(t) \\ &= h(t) + \int_a^t \sum_{k=1}^{\infty}\frac{(-\lambda)^k\left(\log\frac{t}{s}\right)^{k\alpha-1}}{\Gamma(k\alpha)}\frac{h(s)}{s}\mathrm{d}s \\ &= h(t) - \lambda\int_a^t \sum_{k=0}^{\infty}\frac{(-\lambda)^k\left(\log\frac{t}{s}\right)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\frac{h(s)}{s}\mathrm{d}s \\ &= h(t) - \lambda(\mathbf{E}_{\alpha,\alpha;\,\lambda}h)(t).\end{aligned}$$

(ii) For $h \in C_{1-\gamma,\log}$, we obtain

$$(I + \lambda\mathcal{J}_{a^+}^{\alpha})\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t) = \sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k h(t) + \sum_{k=0}^{\infty}(-1)^k(\lambda\mathcal{J}_{a^+}^{\alpha})^{k+1}h(t) = h(t).$$

Similarly, we deduce

$$\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha})^k(I+\lambda\mathcal{J}_{a^+}^{\alpha})h(t)=h(t);$$

therefore, the operator $(I+\lambda\mathcal{J}_{a^+}^{\alpha}):C_{1-\gamma,\log}\to C_{1-\gamma,\log}$ is invertible and (13) holds. □

Theorem 3.2. *Let* $g:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $g(\cdot,u(\cdot),v(\cdot))\in C_{1-\gamma_1,\log}$ *for any* $u,v\in C_{1-\gamma_1,\log}$, *then* $x(t)\in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ *satisfies* (3) *and* (4) *if and only if* $x(t)$ *satisfies*

$$x(t)=c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a)+[\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}g(s,x(s),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(s))](t). \tag{14}$$

Proof. Let $x\in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ satisfy (3) and (4), then ${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)\in C_{1-\gamma_1,\log}^{\alpha_2,\beta_2}\subset C_{1-\gamma_2,\log}^{\alpha_2,\beta_2}$ and from Lemma 2.4, one obtains

$$\{\mathcal{J}_{a^+}^{1-\gamma_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)]\}(a^+)=0. \tag{15}$$

Denote

$$G_x(t):=g(t,x(t),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)), \tag{16}$$

applying $\mathcal{J}_{a^+}^{\alpha_2}$ to both sides of (3) and taking Lemma 2.9 and (15) into account, we obtain

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)-(\mathcal{J}_{a^+}^{\alpha_2}G_x)(t)=0.$$

By Lemma 2.8, we find

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}[x(t)+\lambda(\mathcal{J}_{a^+}^{\alpha_1}x)(t)-(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t)]=0. \tag{17}$$

By using Lemma 2.4, we have

$$\{\mathcal{J}_{a^+}^{1-\gamma_1}[x(t)+\lambda(\mathcal{J}_{a^+}^{\alpha_1}x)(t)-(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t)]\}(a^+)=c. \tag{18}$$

By applying $\mathcal{J}_{a^+}^{\alpha_1}$ to both sides of (17) and combining with Lemma 2.9 and (18), we obtain

$$x(t)=\frac{c}{\Gamma(\gamma_1)}\left(\log\frac{t}{a}\right)^{\gamma_1-1}-\lambda(\mathcal{J}_{a^+}^{\alpha_1}x)(t)+(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t). \tag{19}$$

If $x(t)$ satisfies (19), then $x(t)\in C_{1-\gamma_1,\log}$ and

$$(\mathcal{J}_{a^+}^{1-\gamma_1}x)(t)=c-\lambda(\mathcal{J}_{a^+}^{1-\gamma_1+\alpha_1}x)(t)+(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2+1-\gamma_1}G_x)(t).$$

From Lemma 2.4, it follows that $(\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+)=c$, that is (4) holds, thus,

$$\begin{aligned}\mathcal{D}_{a^+}^{1-\gamma_1+\alpha_1}[(\mathcal{J}_{a^+}^{1-\gamma_1}x)(t)-(\mathcal{J}_{a}^{1-\gamma_1}x)(a^+)]&=\mathcal{D}_{a^+}^{1-\gamma_1+\alpha_1}[-\lambda(\mathcal{J}_{a^+}^{1-\gamma_1+\alpha_1}x)(t)+(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2+1-\gamma_1}G_x)(t)]\\&=-\lambda x(t)+(\mathcal{J}_{a^+}^{\alpha_2}G_x)(t)\in C_{1-\gamma_1,\log}.\end{aligned}$$

This means $x\in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ and

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)=(\mathcal{J}_{a^+}^{\alpha_2}G_x)(t).$$

Furthermore,

$$\mathcal{D}_{a^+}^{1-\gamma_2+\alpha_2}\mathcal{J}_{a^+}^{1-\gamma_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)]=G_x(t)\in C_{1-\gamma_1,\log}.$$

From Lemma 2.4, it follows that $\mathcal{J}_{a^+}^{1-\gamma_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+\lambda x(t)](a^+)=0$. Therefore,

$${}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}+\lambda]x(t)=G_x(t),$$

which means that $x(t)$ satisfies (3). Now, we prove the equivalence of (3) and (4) with (19).

Set $\widetilde{\phi}(t) := \frac{c}{\Gamma(\gamma_1)}\left(\log\frac{t}{a}\right)^{\gamma_1-1}$, then from (19), we have

$$(I+\lambda\mathcal{J}_{a^+}^{\alpha_1})x(t)=\widetilde{\phi}(t)+(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t).$$

By Theorem 3.1 (ii), we infer that

$$\begin{aligned}(I+\lambda\mathcal{J}_{a^+}^{\alpha_1})^{-1}\widetilde{\phi}(t)&=\sum_{k=0}^{\infty}(-\lambda)^k(\mathcal{J}_{a^+}^{k\alpha_1})\widetilde{\phi}(t)\\&=c\left(\log\frac{t}{a}\right)^{\gamma_1-1}\sum_{k=0}^{\infty}\frac{\left[-\lambda\left(\log\frac{t}{a}\right)^{\alpha_1}\right]^k}{\Gamma(k\alpha_1+\gamma_1)}\\&=c\left(\log\frac{t}{a}\right)^{\gamma_1-1}E_{\alpha_1,\gamma_1}\left[-\lambda\left(\log\frac{t}{a}\right)^{\alpha_1}\right],\end{aligned}$$

and

$$\begin{aligned}(I+\lambda\mathcal{J}_{a^+}^{\alpha_1})^{-1}(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t)&=\sum_{k=0}^{\infty}(-\lambda\mathcal{J}_{a^+}^{\alpha_1})^k(\mathcal{J}_{a^+}^{\alpha_1+\alpha_2}G_x)(t)\\&=\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2-1}E_{\alpha_1,\alpha_1+\alpha_2}[-\lambda\left(\log\frac{t}{s}\right)^{\alpha_1}]g(s,x(s),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(s))\frac{\mathrm{d}s}{s}.\end{aligned}$$

Thus, the conclusion holds. □

By Theorems 2.2 (ii) and 2.3 (iii), it is evident that the following conclusions are valid and can be easily observed.

Remark 3.1.
(i) The function $c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a)$ is a general solution to the homogeneous equation:

$$_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}+\lambda]x(t)=0,\quad t\in(a,b].$$

(ii) The function $\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}g(t,x(t),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t))$ is a particular solution to the inhomogeneous equation

$$_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}+\lambda]x(t)=g(t,x(t),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)).$$

Equation (14) shows the structure of solution more clearly than Fredholm-Volterra integral solution expressed by (5).

4 Existence and uniqueness of solutions

In this section, we denote $\Omega := C_{1-\gamma,\log}^{\alpha,\beta}$.

Theorem 4.1. *Let* $g(t,u(t),v(t))\in C_{1-\gamma_1,\log}$ *for any* $u(t),v(t)\in C_{1-\gamma_1,\log}$ *and satisfy*

$$|g(t,u(t),v(t))-g(t,\tilde{u}(t),\tilde{v}(t))|\le l_1(t)|u(t)-\tilde{u}(t)|+l_2(t)|v(t)-\tilde{v}(t)|,$$

where $\tilde{u},\tilde{v}\in C_{1-\gamma_1,\log}$ *and* $l_i(t)\in C$ $(i=1,2)$. *Then IVP* (3) *and* (4) *has a unique solution* $x(t)\in\Omega$ *given by* (14).

Proof. We define an operator $\mathcal{F}:\Omega\to\Omega$ by

$$(\mathcal{F}x)(t)=c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a)+[\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}g(s,x(s),{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(s))](t).\tag{20}$$

Obviously, by Theorem 3.2, $\mathcal{F}$ is well defined and its fixed point serves as a solution to IVP (3) and (4).

Note that (18) and by Theorem 2.3 (iii) and Lemma 2.2, we obtain

$$ {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}[c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a)] = -\lambda[c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a)] \in C_{1-\gamma_1,\log}, $$

and

$$ {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}[\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}G_x(t)] = \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}G_x(t). $$

Choosing

$$ \Lambda \coloneqq (\|l_1\|_C + \|l_2\|_C)\max\left\{\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\log\frac{b}{a}\right)^{\alpha_1}, 1\right\}, $$

we can see

$$ \begin{aligned} \left(\log\frac{t}{a}\right)^{\gamma_1-1}|G_x(t)-G_{\tilde{x}}(t)| &\le \left(\log\frac{t}{a}\right)^{\gamma_1-1}\{l_1(t)|x(t)-\tilde{x}(t)| + l_2(t)|\,{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) - {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}\tilde{x}(t)|\} \\ &\le (\|l_1\|_C + \|l_2\|_C)\|x-\tilde{x}\|_\Omega, \end{aligned} $$

$$ \begin{aligned} &\left(\log\frac{t}{a}\right)^{1-\gamma_1}|(\mathcal{F}x)(t)-(\mathcal{F}\tilde{x})(t)| \\ &\le \left(\log\frac{t}{a}\right)^{1-\gamma_1}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2-1}E_{\alpha_1,\alpha_1+\alpha_2}\left[-\lambda\left(\log\frac{t}{s}\right)^{\alpha_1}\right]\cdot|G_x(s)-G_{\tilde{x}}(s)|\frac{ds}{s} \\ &\le \frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}\left(\log\frac{b}{a}\right)^{\alpha_1}(\|l_1\|_C+\|l_2\|_C)}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\left(\log\frac{s}{a}\right)^{\gamma_1-1}\frac{ds}{s}\cdot\|x-\tilde{x}\|_\Omega \\ &\le \frac{(\|l_1\|_C+\|l_2\|_C)\Gamma(\alpha_2)\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2)}\left(\log\frac{b}{a}\right)^{\alpha_1}\left(\log\frac{t}{a}\right)^{\alpha_2}\cdot\|x-\tilde{x}\|_\Omega \\ &\le \frac{\Lambda\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\left(\log\frac{t}{a}\right)^{\alpha_2}\|x-\tilde{x}\|_\Omega, \end{aligned} $$

and

$$ \begin{aligned} &\left(\log\frac{t}{a}\right)^{1-\gamma_1}|{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}[(\mathcal{F}x)(t)-(\mathcal{F}\tilde{x})(t)]| \\ &\le \frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}}{\Gamma(\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}|G_x(s)-G_{\tilde{x}}(s)|\frac{ds}{s} \\ &\le \frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}(\|l_1\|_C+\|l_2\|_C)}{\Gamma(\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\left(\log\frac{s}{a}\right)^{\gamma_1-1}\frac{ds}{s}\cdot\|x-\tilde{x}\|_\Omega \\ &= \frac{(\|l_1\|_C+\|l_2\|_C)\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\left(\log\frac{t}{a}\right)^{\alpha_2}\cdot\|x-\tilde{x}\|_\Omega \\ &\le \frac{\Lambda\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\left(\log\frac{t}{a}\right)^{\alpha_2}\|x-\tilde{x}\|_\Omega. \end{aligned} $$

Hence,

$$ \|\mathcal{F}x-\mathcal{F}\tilde{x}\|_\Omega \le \frac{\Lambda\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\left(\log\frac{b}{a}\right)^{\alpha_2}\|x-\tilde{x}\|_\Omega. $$

Furthermore, we have

$$
\begin{aligned}
&\left(\log\frac{t}{a}\right)^{1-\gamma_1}|(\mathcal{F}^2x)(t)-(\mathcal{F}^2\tilde{x})(t)| \\
&\le \left(\log\frac{t}{a}\right)^{1-\gamma_1}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2-1}E_{\alpha_1,\alpha_1+\alpha_2}[-\lambda\left(\log\frac{t}{s}\right)^{\alpha_1}]\cdot|G_{\mathcal{F}x}(s)-G_{\mathcal{F}\tilde{x}}(s)|\frac{ds}{s} \\
&\le \frac{\Lambda\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}\left(\log\frac{b}{a}\right)^{\alpha_1}(\|l_1\|_C+\|l_2\|_C)}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\left(\log\frac{s}{a}\right)^{\alpha_2+\gamma_1-1}ds\|x-\tilde{x}\|_\Omega \\
&= \frac{\Lambda\Gamma(\alpha_2)\left(\log\frac{b}{a}\right)^{\alpha_1}(\|l_1\|_C+\|l_2\|_C)}{\Gamma(\alpha_1+\alpha_2)}\frac{\Gamma(\gamma_1)}{\Gamma(2\alpha_1+\gamma_1)}\left(\log\frac{t}{a}\right)^{2\alpha_2}\|x-\tilde{x}\|_\Omega \\
&\le \frac{\Lambda^2\Gamma(\gamma_1)}{\Gamma(2\alpha_1+\gamma_1)}\left(\log\frac{t}{a}\right)^{2\alpha_2}\|x-\tilde{x}\|_\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\log\frac{t}{a}\right)^{1-\gamma_1}|{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}[(\mathcal{F}^2x)(t)-(\mathcal{F}^2\tilde{x})(t)]| \\
&\le \frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}}{\Gamma(\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\cdot|G_{\mathcal{F}x}(s)-G_{\mathcal{F}\tilde{x}}(s)|\frac{ds}{s} \\
&\le \frac{\Lambda\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\frac{\left(\log\frac{t}{a}\right)^{1-\gamma_1}(\|l_1\|_C+\|l_2\|_C)}{\Gamma(\alpha_2)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\left(\log\frac{s}{a}\right)^{\alpha_2+\gamma_1-1}ds\|x-\tilde{x}\|_\Omega \\
&\le \frac{\Lambda(\|l_1\|_C+\|l_2\|_C)\Gamma(\gamma_1)}{\Gamma(2\alpha_1+\gamma_1)}\left(\log\frac{t}{a}\right)^{2\alpha_2}\|x-\tilde{x}\|_\Omega \\
&\le \frac{\Lambda^2\Gamma(\gamma_1)}{\Gamma(2\alpha_1+\gamma_1)}\left(\log\frac{t}{a}\right)^{2\alpha_2}\|x-\tilde{x}\|_\Omega.
\end{aligned}
$$

Hence,

$$\|\mathcal{F}^2x-\mathcal{F}^2\tilde{x}\|_\Omega \le \frac{\Lambda^2\Gamma(\gamma_1)}{\Gamma(2\alpha_1+\gamma_1)}\left(\log\frac{b}{a}\right)^{2\alpha_2}\|x-\tilde{x}\|_\Omega.$$

Through the process of induction, it can be inferred that

$$\|\mathcal{F}^kx-\mathcal{F}^k\tilde{x}\|_\Omega \le \frac{\Lambda^k\Gamma(\gamma_1)}{\Gamma(k\alpha_1+\gamma_1)}\left(\log\frac{b}{a}\right)^{k\alpha_2}\|x-\tilde{x}\|_\Omega. \tag{21}$$

When k is large enough, the right-hand side of (21) is less than $L\|x-\tilde{x}\|_\Omega$ ($L\in(0,1)$). In view of the generalized Banach fixed point theorem, $\mathcal{F}$ possesses a unique fixed point $x\in\Omega$ satisfying (14). □

Set $G_x(t)=-\eta(t)_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)+f(t,x(t))$ and $\eta(t)\in C$, we conclude an immediate consequence of Theorem 4.1.

Theorem 4.2. *Let $f(t,x(t))\in C_{1-\gamma_1,\log}$ for any $x(t)\in C_{1-\gamma_1,\log}$ and satisfy*

$$|f(t,x(t))-f(t,\tilde{x}(t))|\le l_1(t)|x(t)-\tilde{x}(t)|,\quad x,\tilde{x}\in C_{1-\gamma_1,\log},$$

where $l_1(t)\in C$. Then IVP (1) and (2) *has a unique solution $x(t)\in\Omega$.*

Corollary 4.1. *Let $f(t) \in C_{1-\gamma_1,\log}$. Then IVP*

$$\begin{cases} {}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda]x(t) = f(t), & t \in (a,b], \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c, \end{cases}$$

has a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ given via

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + [\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}f(s)](t).$$

For $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$, note that the fact $[\mathcal{J}_{a^+}^{1-\alpha_2}({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x](a^+) = 0$, it is evident that

$${}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda]x(t) = f(t), \quad t \in (a,b]$$

and

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) + \lambda x(t) = \mathcal{J}_{a^+}^{\alpha_2}f(t), \quad t \in (a,b]$$

are equivalent, and this leads to the following conclusion.

Corollary 4.2. *Let $f(t) \in C_{1-\gamma_1,\log}$. Then IVP*

$$\begin{cases} {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) + \lambda x(t) = \mathcal{J}_{a^+}^{\alpha_2}f(t), & t \in (a,b], \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c, \end{cases}$$

has a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ given by

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + [\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}f(s)](t).$$

5 Special cases

By the results from Section 4, explicit solutions can be derived for the following two kinds of Hilfer-Hadamard-type IVPs with a variable coefficient.

Theorem 5.1. *Let $\eta(t) \in C$ and $f(t) \in C_{1-\gamma_1,\log}$. Then problem*

$$\begin{cases} {}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}[{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda]x(t) + \eta(t)\,{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) = f(t), & t \in (a,b], \quad (22) \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c, & (23) \end{cases}$$

has a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ represented by

$$x(t) = \frac{c}{\Gamma(\gamma_1)}\left(\log\frac{t}{a}\right)^{\gamma_1-1} + \mathcal{J}_{a^+}^{\alpha_1}\sum_{k=0}^{\infty}(-1)^{k+1}[(\mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}\eta)(\cdot)]^k\{\lambda c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) - (\mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}f)(t)\}. \quad (24)$$

Proof. It follows from Theorem 4.1 that

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}[-\eta(t)\,{}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) + f(t)] \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1} \quad (25)$$

is a unique solution to problem (22) and (23).

By applying ${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}$ to (25) and writing $y(t) = {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t)$, one obtains

$$y(t) = {}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}\{c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda}[-\eta(\cdot)y(\cdot) + f(\cdot)](t)\}.$$

By Theorems 2.2 and 2.3, we obtain

$$y(t) = -\lambda c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}[-\eta(\cdot)y(\cdot) + f(\cdot)](t). \tag{26}$$

Let

$$\begin{cases} y_0(t) = -\lambda c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda} f(t) \\ y_n(t) = y_0(t) - \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}[\eta(\cdot)y_{n-1}(\cdot)](t), \quad n = 1, 2, \dots, \end{cases}$$

then we find

$$y_n(t) = \sum_{k=0}^{n}(-1)^k[(\mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}\eta)(\cdot)]^k\{-\lambda c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda} f(t)\}.$$

Hence,

$$y(t) = \lim_{n\to\infty} y_n(t) = \sum_{k=0}^{\infty}(-1)^k[(\mathbf{E}_{\alpha_1,\alpha_2;\,\lambda}\eta)(\cdot)]^k\{-\lambda c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_2;\,\lambda} f(t)\},$$

which gives an explicit solution to (26). This combining with (25) leads to (24). □

Theorem 5.2. *Let $\delta(t) \in C$ and $f(t) \in C_{1-\gamma_1,\log}$. Then problem*

$$\begin{cases} ({}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2} + \delta(t))({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t) = f(t), \quad t \in (a,b], & (27) \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c, & (28) \end{cases}$$

has a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ of the form

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1;\,\lambda}\sum_{k=0}^{\infty}(-\mathcal{J}_{a^+}^{\alpha_2}\delta(\cdot))^k\mathcal{J}_{a^+}^{\alpha_2}f(t). \tag{29}$$

Proof. Clearly, (27) can be rewritten as follows:

$${}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2}({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t) + \delta(t){}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) + \lambda\delta(t)x(t) = f(t), \quad t \in (a,b].$$

Setting

$$g(t, x(t), ({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x)(t)) := -\delta(t)({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x)(t) - \lambda\delta(t)x(t) + f(t),$$

it follows from (19) and Theorem 4.1 that problems (27) and (28) have a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ given by

$$x(t) = \frac{c}{\Gamma(\gamma_1)}\left(\log\frac{t}{a}\right)^{\gamma_1-1} - \lambda(\mathcal{J}_{a^+}^{\alpha_1}x)(t) + \mathcal{J}_{a^+}^{\alpha_1+\alpha_2}[-\delta(t)({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t) + f(t)]. \tag{30}$$

By applying ${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}$ to both sides of (30), one obtains

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1}x(t) + \lambda x(t) = \mathcal{J}_{a^+}^{\alpha_2}[-\delta(t)({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t) + f(t)]. \tag{31}$$

Let $y(t) = ({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t)$, then y belongs to $C_{1-\gamma_1,\log}$ and satisfies

$$y(t) = \mathcal{J}_{a^+}^{\alpha_2}[-\delta(t)y(t) + f(t)]. \tag{32}$$

Let

$$\begin{cases} y_0(t) = (\mathcal{J}_{a^+}^{\alpha_2}f)(t), \\ y_n(t) = y_0(t) + \mathcal{J}_{a^+}^{\alpha_2}[-\delta(\cdot)y_{n-1}(\cdot)](t), \quad n = 1, 2, \dots, \end{cases}$$

we find

$$y_n(t) = \sum_{k=0}^{n} (-1)^k (\mathcal{J}_{a^+}^{\alpha_2}\delta(\cdot))^k \mathcal{J}_{a^+}^{\alpha_2} f(t).$$

Then the explicit solution $y(t)$ to equation (32) is obtained as a limit of $\{y_n(t)\}$:

$$y(t) = \sum_{k=0}^{\infty} (-1)^k (\mathcal{J}_{a^+}^{\alpha_2}\delta(\cdot))^k \mathcal{J}_{a^+}^{\alpha_2} f(t).$$

Hence,

$${}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} x(t) + \lambda x(t) = \mathcal{J}_{a^+}^{\alpha_2} \sum_{k=0}^{\infty} (-\delta(\cdot)\mathcal{J}_{a^+}^{\alpha_2})^k f(t).$$

It follows from Corollary 4.2 and Theorem 2.3 that

$$\begin{aligned} x(t) &= c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\,\lambda} \sum_{k=0}^{\infty} (-\delta(\cdot)\mathcal{J}_{a^+}^{\alpha_2})^k f(t) \\ &= c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1;\,\lambda} \sum_{k=0}^{\infty} (-1)^k (\mathcal{J}_{a^+}^{\alpha_2}\delta(\cdot))^k \mathcal{J}_{a^+}^{\alpha_2} f(t). \end{aligned}$$

□

When $\delta(t) \equiv \delta$ ($\neq 0$) is a constant, we obtain the following conclusion.

Theorem 5.3. *Let $f(t) \in C_{1-\gamma_1,\log}$. Then problem*

$$\begin{cases} ({}_H\mathcal{D}_{a^+}^{\alpha_2,\beta_2} + \delta)({}_H\mathcal{D}_{a^+}^{\alpha_1,\beta_1} + \lambda)x(t) = f(t), & t \in (a,b], \\ (\mathcal{J}_{a^+}^{1-\gamma_1}x)(a^+) = c \end{cases} \tag{33}$$

has a unique solution $x(t) \in C_{1-\gamma_1,\log}^{\alpha_1,\beta_1}$ represented by

$$x(t) = c\mathcal{E}_{\alpha_1,\gamma_1;\,\lambda}(t,a) + \mathbf{E}_{\alpha_1,\alpha_1;\,\lambda}(\mathbf{E}_{\alpha_2,\alpha_2;\delta} f)(t).$$

Proof. By using Theorem 3.1(i), we have

$$\sum_{k=0}^{\infty} (-\delta)^k (\mathcal{J}_{a^+}^{\alpha_2})^{k+1} f(t) = -\frac{1}{\delta} \sum_{k=1}^{\infty} (-\delta\mathcal{J}_{a^+}^{\alpha_2})^k f(t) = (\mathbf{E}_{\alpha_2,\alpha_2;\delta} f)(t).$$

Now the result follows from Theorem 5.2 ($\delta(t) = \delta$). □

6 Applications

Example 6.1. Consider the following IVP:

$$\begin{cases} {}_H\mathcal{D}_{1^+}^{\frac{1}{10},\frac{1}{5}} \left[{}_H\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{3}} + 3 \right] x(t) = \dfrac{t^2 |{}_H\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{3}} x(t)|}{1 + |{}_H\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{3}} x(t)|} + \sqrt{t} \cdot x(t) + (\log t)^{-\frac{1}{3}}, & t \in (1,b], \quad (34) \\ (\mathcal{J}_{1^+}^{\frac{1}{3}} x)(1^+) = 1. & (35) \end{cases}$$

Taking $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{10}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{5}$, $\lambda = 3$, $c = 1$, and

$$g(t,u(t),v(t)) = \frac{t^2|v(t)|}{1+|v(t)|} + \sqrt{t}\,u(t) + (\log t)^{-\frac{1}{3}}, \quad t \in (1,b], \ u,v \in \mathbb{R},$$

we can see

$$|g(t,u(t),v(t)) - g(t,\tilde{u}(t),\tilde{v}(t))| \le \sqrt{t}\,|u(t)-\tilde{u}(t)| + t^2|v(t)-\tilde{v}(t)|, \quad u,v,\tilde{u},\tilde{v}\in\mathbb{R}.$$

By Theorem 4.1, problems (34) and (35) has a unique solution $x(t) \in C^{\frac{1}{2},\frac{1}{3}}_{\frac{1}{3},\log}$.

Example 6.2. Consider the following IVP:

$$\begin{cases} {}_H\mathcal{D}^{\frac{1}{5},\frac{1}{8}}_{1^+}\left[{}_H\mathcal{D}^{\frac{1}{3},\frac{1}{4}}_{1^+} + 2\right]x(t) + t\,{}_H\mathcal{D}^{\frac{1}{3},\frac{1}{4}}_{1^+}x(t) = 2\cos((\log t)^{\frac{1}{2}}x(t)+1) + (\log t)^{-\frac{1}{4}}, \quad t\in(1,b], & (36)\\ (\mathcal{J}^{\frac{1}{2}}_{1^+}x)(1^+) = 2. & (37)\end{cases}$$

Taking $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{1}{8}$, $\eta(t) = t$, $\lambda = 2$, $c = 2$, and

$$f(t,x(t)) = 2\cos((\log t)^{\frac{1}{2}}x(t)+1) + (\log t)^{-\frac{1}{4}}, \quad t\in(1,b],\ \ x\in\mathbb{R},$$

we can see

$$|f(t,x(t)) - f(t,\tilde{x}(t))| \le 2(\log t)^{\frac{1}{2}}|x(t)-\tilde{x}(t)|, \quad x,\tilde{x}\in\mathbb{R}.$$

Hence, from Theorem 4.2, problems (36) and (37) has a unique solution $x(t) \in C^{\frac{1}{3},\frac{1}{4}}_{\frac{1}{2},\log}$.

Acknowledgments: The authors wish to express their gratitude to the referees for their helpful suggestions. This work was supported by the Natural Science Foundation of China (11971329).

Funding information: This work was supported by the Natural Science Foundation of China (11971329).

Author contributions: All authors read and approved the final manuscript.

Conflict of interest: The authors declare no conflict of interest.

Ethical approval: The conducted research is not related to either human or animals use.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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