Research Article

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Weighted composition operators on bicomplex Lorentz spaces with their characterization and properties

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Abstract: This article presents a characterization of non-singular measurable transformation, denoted as T, mapping from Ω to itself, along with bicomplex-valued $\mathbb{BC}$-measurable function u defined on Ω, which induces a weighted composition operator. The study then proceeds to fully identify their $\mathbb{D}$-compactness and $\mathbb{D}$-closedness within the range of bicomplex Lorentz spaces denoted as $L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$, where $(\Omega, \mathfrak{M}, \vartheta)$ represents a σ-finite complete $\mathbb{BC}$-measure space, $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$ is a $\mathbb{BC}$-measure, and the parameters satisfy $1 < p \leq \infty$, $1 \leq q \leq \infty$.

Keywords: bicomplex numbers, BC-valued functions, hyperbolic norm, $\mathbb{D}$-distribution function, $\mathbb{D}$-rearrangement, multiplication operator, composition operator

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1 Introduction and preliminaries on $\mathbb{BC}$

Bicomplex ($\mathbb{BC}$)-valued functions are used in many areas of mathematics, including probability theory and mathematical analysis. Vector spaces are usually taken into account over real or complex numbers in conventional functional analysis. Bicomplex scalars, on the other hand, provide a deeper framework that allows additional possibilities for applications. The study of modules with bicomplex scalars within the framework of functional analysis has garnered a lot of attention recently. The book by Alpay et al. [1] is a noteworthy addition to this subject, which can present new ideas and viewpoints on this subject. It offers remarkable findings, methods, and uses related to in the structure of functional analysis, the study of modules with bicomplex scalars. Among other things, these conclusions cover several aspects of functional analysis, including operator theory, function spaces, and spectral theory.

The Hahn-Banach theorem for bicomplex modules and hyperbolic modules is investigated in [2]. Topological bicomplex modules, exploring their topological properties and investigating concepts such as convergence, continuity, and compactness in the sense of $\mathbb{BC}$, are done in [3]. Fundamental theorems such as the principle of uniform boundedness, open mapping theorem, interior mapping theorem for bicomplex modules, and closed graph theorem are studied in [4]. $\mathbb{BC}$-bounded linear operators and bicomplex functional calculus are examined in [5].

In [6], in collaboration with [4], the authors delved further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity, and related topological concepts specific to these settings.

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The book authored by Luna-Elizarrarás et al. provides an in-depth exploration of bicomplex analysis and geometry [7]. It covers holomorphic functions, integration, differential equations, and geometric properties specific to the bicomplex domain.

Besides these, bicomplex Lebesgue spaces and some of their geometric and topological properties are defined in [8,9] and [10]. Bicomplex sequence spaces $l_p(\mathbb{BC})$ are defined and examined with various properties in [11] and [12].

These references show the exploration of properties, the development of new theorems, and the application of functional analysis techniques in the context of bicomplex numbers. Researchers and readers interested in these topics can refer to these articles and the books for detailed insights into the respective areas of study.

We now summarize bicomplex numbers with some basic properties. The set of bicomplex numbers $\mathbb{BC}$, which is a four-dimensional extension of the real numbers, is defined as

$$\mathbb{BC} \coloneqq \{W = w_1 + jw_2 | w_1, w_2 \in \mathbb{C}(i)\},$$

where i and j are the imaginary units satisfying $ij = ji$, $i^2 = j^2 = -1$. Here, $\mathbb{C}(i)$ is the field of complex numbers with the imaginary unit i. According to ring structure, for any $Z = z_1 + jz_2$, $W = w_1 + jw_2$ in $\mathbb{BC}$, usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j(z_2 + w_2),$$
$$ZW = (z_1w_1 - z_2w_2) + j(z_2w_1 + z_1w_2).$$

The set $\mathbb{BC}$ forms a commutative ring under the usual addition and multiplication. It has a unit element denoted as $1_{\mathbb{BC}} \coloneqq 1$ and is a module over itself.

The product of the imaginary units i and j bring out a hyperbolic unit k, such that $k^2 = 1$. This implies that k is a square root of 1 and is distinct from i and j. The product operation of all units i, j, and k in the bicomplex numbers is commutative and

$$ij = k, \quad jk = -i, \quad \text{and} \quad ik = -j.$$

Hyperbolic numbers $\mathbb{D}$ are two-dimensional extensions of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\beta = \beta_1 + k\beta_2$, where β_1 and β_2 are the real numbers, and k is the hyperbolic unit. In the hyperbolic number system, for any two hyperbolic numbers $\beta = \beta_1 + k\beta_2$ and $\gamma = \delta_1 + k\delta_2$, addition and multiplication are defined as follows:

$$\beta + \gamma = (\beta_1 + \delta_1) + k(\beta_2 + \delta_2),$$
$$\beta\gamma = (\beta_1\delta_1 + \beta_2\delta_2) + k(\beta_1\delta_2 + \beta_2\delta_1).$$

Furthermore, $\mathbb{BC}$ is a normed space with the norm $\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$ for any $W = w_1 + jw_2$ in $\mathbb{BC}$. In light of this, $\|W_1W_2\|_{\mathbb{BC}} \leq \sqrt{2}\,\|W_1\|_{\mathbb{BC}}\|W_2\|_{\mathbb{BC}}$ for every $W_1, W_2 \in \mathbb{BC}$, and finally, $\mathbb{BC}$ is a modified Banach algebra [13].

If the hyperbolic numbers e_1 and e_2 are defined as

$$e_1 = \frac{1 + k}{2} \quad \text{and} \quad e_2 = \frac{1 - k}{2},$$

then it is easy to see that

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad (e_1)^* = e_1, \quad (e_2)^* = e_2, \quad e_1 + e_2 = 1, \quad e_1 \cdot e_2 = 0$$

are satisfied and $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$, where $W^* = \overline{w_1} - j\overline{w_2}$ is *-conjugate of W. Using this linearly independent set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely, i.e., $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1z_1 + e_2z_2, \tag{1}$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ [1]. Here z_1 and z_2 are the elements of $\mathbb{C}(i)$ and the formula in (1) is called the *idempotent representation* of the bicomplex number W.

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named with ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1z_1 + e_2z_2$ is defined as

$$|W|_k = e_1|z_1| + e_2|z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as $\mathbb{D} \subset \mathbb{BC}$. Thus, $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$ can be written as

$$\alpha = e_1\alpha_1 + e_2\alpha_2,$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are the real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 = e_1\alpha_1 + e_2\alpha_2 \in \mathbb{D}$, then we say that α is a positive hyperbolic number. Thus, the set of non-negative hyperbolic numbers $\mathbb{D}^+ \cup \{0\}$ is defined by

$$\begin{aligned}\mathbb{D}^+ \cup \{0\} &= \{\alpha = \beta_1 + k\beta_2 : \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\}\\ &= \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.\end{aligned}$$

Now, let α and γ be any two elements of $\mathbb{D}$. In [1,2] and [7], a relation $\preccurlyeq$ is defined on $\mathbb{D}$ by

$$\alpha \preccurlyeq \gamma \iff \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

It is shown in [1] that this relation "$\preccurlyeq$" defines a partial order on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α and γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preccurlyeq \gamma \iff \alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$. Any function f defined on $\mathbb{D}$ is called $\mathbb{D}$-increasing if $f(\alpha) \prec f(\gamma)$, $\mathbb{D}$-decreasing if $f(\alpha) \succ f(\gamma)$, $\mathbb{D}$-nonincreasing if $f(\alpha) \succcurlyeq f(\gamma)$ and $\mathbb{D}$-nondecreasing if $f(\alpha) \preccurlyeq f(\gamma)$ whenever $\alpha \prec \gamma$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preccurlyeq$" one can refer to [1, Section 1.5] and [7].

Definition 1.1. Let A be a subset of $\mathbb{D}$. A is called a $\mathbb{D}$-bounded above set if there is a hyperbolic number δ such that $\delta \succcurlyeq \alpha$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is $\mathbb{D}$-bounded from above, then the $\mathbb{D}$-supremum of A is defined as the smallest member of the set of all upper bounds of A [6].

Remark 1.1. [1, Remark 1.5.2] Let A be a $\mathbb{D}$-bounded above subset of $\mathbb{D}$, $A_1 \coloneqq \{\lambda_1 : e_1\lambda_1 + e_2\lambda_2 \in A\}$ and $A_2 \coloneqq \{\lambda_2 : e_1\lambda_1 + e_2\lambda_2 \in A\}$. Then, the $\sup_{\mathbb{D}}A$ is given by

$$\sup_{\mathbb{D}}A \coloneqq e_1 \sup A_1 + e_2 \sup A_2.$$

Similarly, for any $\mathbb{D}$-bounded below set A, $\mathbb{D}$-infimum of A is defined as

$$\inf_{\mathbb{D}}A = e_1 \inf A_1 + e_2 \inf A_2.$$

Remark 1.2. A $\mathbb{BC}$-module space or $\mathbb{D}$-module space Y can be decomposed as

$$Y = e_1Y_1 + e_2Y_2, \tag{2}$$

where $Y_1 = e_1Y$ and $Y_2 = e_2Y$ are $\mathbb{R}$-vector or $\mathbb{C}(i)$-vector spaces. The spelling in (2) is called as the idempotent decomposition of the space Y [1,6,7].

Definition 1.2. Let $\mathfrak{M}$ be a σ-algebra on a set Ω. A bicomplex-valued function $\mu = \mu_1e_1 + \mu_2e_2$ defined on Ω is called a $\mathbb{BC}$-measure on $\mathfrak{M}$ if μ_1 and μ_2 are complex measures on $\mathfrak{M}$. In particular if μ_1 and μ_2 are the positive measures on $\mathfrak{M}$, then μ is called a $\mathbb{D}$-measure on $\mathfrak{M}$. Also, if μ_1 and μ_2 are the real measures on $\mathfrak{M}$, then μ is called a $\mathbb{D}^+$-measure on $\mathfrak{M}$ [14,15].

Assume that $\Omega = (\Omega, \mathfrak{M}, \mu)$ is a σ-finite complete measure space and f_1 and f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called as a $\mathbb{BC}$-measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is a $\mathbb{D}$-valued measurable function on Ω [14,15].

For any $\mathbb{BC}$-valued measurable function $f = f_1 e_1 + f_2 e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$ -valued measurable. Because if $f = f_1 e_1 + f_2 e_2$ is a $\mathbb{BC}$-valued measurable function, then f_1, f_2 are $\mathbb{C}$-measurable functions and real, imaginary parts of f_1 and f_2 are $\mathbb{R}$-valued measurable. Also for any two $\mathbb{BC}$-valued measurable functions f and g, it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions [14,15]. More results on $\mathbb{D}$-topology such as $\mathbb{D}$-limit, $\mathbb{D}$-continuity, $\mathbb{D}$-Cauchy, and $\mathbb{D}$-convergence can be found in [2–4,6,7,15] and references therein.

Definition 1.3. Let $\mu = \mu_1 e_1 + \mu_2 e_2$ be a $\mathbb{D}$-measure and $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ be a $\mathbb{BC}$-measure on $\mathfrak{M}$. Then, λ is said to be absolutely $\mathbb{BC}$-continuous with respect to μ, and denoted by $\lambda \ll_{\mathbb{BC}} \mu$, if λ_i is absolutely continuous with respect to μ_i for $i = 1, 2$ [15].

If for $A \in \mathfrak{M}$, λ_i is concentrated on A for $i = 1, 2$, then λ is said to be $\mathbb{BC}$-concentrated on A. Any two $\mathbb{BC}$-measures $\lambda' = \lambda_1' e_1 + \lambda_2' e_2$, $\lambda'' = \lambda_1'' e_1 + \lambda_2'' e_2$ on $\mathfrak{M}$ are called mutually $\mathbb{BC}$-singular and denoted by $\lambda' \perp_{\mathbb{BC}} \lambda''$ if λ_i' and λ_i'' are mutually singular for $i = 1, 2$ [15].

Theorem 1.1. (Lebesgue-Radon-Nikodym theorem) *Let $\mathfrak{M}$ be a σ-algebra on Ω. Let μ be a σ-finite $\mathbb{D}$-measure on $\mathfrak{M}$, and let λ be $\mathbb{BC}$-measure on $\mathfrak{M}$.*

(a) *There is a unique pair of $\mathbb{BC}$-measures λ' and λ'' on $\mathfrak{M}$ such that*

$$\lambda = \lambda' + \lambda'',$$

where $\lambda' \ll_{\mathbb{BC}} \mu$ and $\lambda'' \perp_{\mathbb{BC}} \mu$. If λ is $\mathbb{D}$-finite measure on $\mathfrak{M}$, then λ' and λ'' are also so.

(b) *There exists a unique $h \in L^1_{\mathbb{BC}}(\mu)$ such that*

$$\lambda'(E) = \int_E h \mathrm{d}\mu,$$

for all $E \in \mathfrak{M}$ [15, Theorem 3.13].

Definition 1.4. [15] Let $(\Omega, \mathfrak{M}, \vartheta)$ be a measure space with $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$, $\mathfrak{F}(\Omega, \mathfrak{M})$ indicate the set of all $\mathfrak{M}$-measurable functions on Ω, and $u \in \mathfrak{F}(\Omega, \mathfrak{M})$ be a $\mathbb{BC}$-valued, measurable function. Let $E_M = \{x \in \Omega : |u(x)|_k > M\}$ for any $M \geqslant 0$. Since u is a $\mathfrak{M}$-measurable function, $|u|_k = |u_1|e_1 + |u_2|e_2$ is $\mathbb{D}$-valued measurable, i.e., $E_M \in \mathfrak{M}$ for any $M \geqslant 0$. If the set A is defined as $A = \{M > 0 : \vartheta(E_M) = 0\} = \{M \in \mathbb{D}^+ : |u(x)|_k \leqslant M\vartheta - a.\ e.\}$, then essential $\mathbb{D}$-supremum of u, denoted by $\mathrm{essup}_{\mathbb{D}} u$ or $\|u\|_\infty^{\mathbb{D}}$, is defined by

$$\|u\|_\infty^{\mathbb{D}} = \mathrm{essup}_{\mathbb{D}} u = \inf_{\mathbb{D}}(A).$$

2 $\mathbb{D}$-distribution and $\mathbb{D}$-rearrangement functions

Now suppose that $(\Omega, \mathfrak{M}, \vartheta)$ is a σ-finite complete $\mathbb{BC}$-measure space and $\mathfrak{F}(\Omega, \mathfrak{M})$ is the set of all $\mathbb{BC}$-measurable $\mathbb{BC}$-valued functions on Ω. In a manuscript conducted recently [16], fundamental properties and related theorems are given related to $\mathbb{D}$-distribution and $\mathbb{D}$-rearrangement functions. Therefore, we did not examine these functions in this article.

Definition 2.1. Let $u = u_1 e_1 + u_2 e_2$ be an element of $\mathfrak{F}(\Omega, \mathfrak{M})$ and $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$ be a $\mathbb{BC}$-measure. Then, $\mathbb{BC}$-distribution function $D_u^{\mathbb{BC}} : \mathbb{D}^+ \cup \{0\} \to \mathbb{D}^+ \cup \{0\}$ of u is given by

$$\begin{aligned} D_u^{\mathbb{BC}}(\lambda) &= D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2 \\ &= \vartheta_1\{x \in \Omega : |u_1(x)| > \lambda_1\}e_1 + \vartheta_2\{x \in \Omega : |u_2(x)| > \lambda_2\}e_2, \end{aligned} \tag{3}$$

for all $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \geqslant 0$.

Definition 2.2. Let $\lambda \in \mathbb{D}^+ \cup \{0\}$ and u be a $\mathbb{BC}$ valued, measurable function in $\mathfrak{F}(\Omega, \mathfrak{M})$. The $\mathbb{D}$-decreasing rearrangement of u is the function $u^*_{\mathbb{BC}} : \mathbb{D}^+ \cup \{0\} \to \mathbb{D}^+ \cup \{0\}$ defined by

$$\begin{aligned} u^*_{\mathbb{BC}}(t) &= \inf_{\mathbb{D}}\{\alpha \geqslant 0 : D_u^{\mathbb{BC}}(\alpha) \leqslant t\} \\ &= \inf\{\alpha_1 \geq 0 : D_{u_1}(\alpha_1) \leq t_1\}e_1 + \inf\{\alpha_2 \geq 0 : D_{u_2}(\alpha_2) \leq t_2\}e_2 \\ &= u_1^*(t_1)e_1 + u_2^*(t_2)e_2, \end{aligned} \tag{4}$$

where $\inf_{\mathbb{D}}\emptyset = \infty_{\mathbb{D}}$.

According to [14, Example 2.2], since

$$\|u\|_\infty^{\mathbb{D}} = \inf_{\mathbb{D}}\{\alpha \geqslant 0 : \vartheta\{x \in \Omega : |u(x)|_k > \alpha\} = 0\},$$

and $\|u_1\|_\infty, \|u_2\|_\infty \leqslant \|u\|_\infty^{\mathbb{D}}$, one can write $\|u\|_\infty^{\mathbb{D}} = \|u_1\|_\infty e_1 + \|u_2\|_\infty e_2$ and so

$$\begin{aligned} u^*_{\mathbb{BC}}(0) &= \inf_{\mathbb{D}}\{\alpha \geqslant 0 : D_u^{\mathbb{BC}}(\alpha) = 0\} \\ &= \inf_{\mathbb{D}}\{\alpha \geqslant 0 : \vartheta_j\{x \in \Omega : |u_j(x)| > \alpha_j\} = 0, j = 1, 2\} \\ &= \|u\|_\infty^{\mathbb{D}}. \end{aligned} \tag{5}$$

Definition 2.3. The function $u^{**}_{\mathbb{BC}} : \mathbb{D}^+ \to \mathbb{D}^+ \cup \{0\}$ is defined as

$$u^{**}_{\mathbb{BC}}(t) = \left(\frac{1}{t_1}\int_0^{t_1} u_1^*(s)ds\right)e_1 + \left(\frac{1}{t_2}\int_0^{t_2} u_2^*(s)ds\right)e_2 = u_1^{**}(t_1)e_1 + u_2^{**}(t_2)e_2, \tag{6}$$

where $t = t_1e_1 + t_2e_2$ and $u^*_{\mathbb{BC}} = u_1^*e_1 + u_2^*e_2$. This function $u^{**}_{\mathbb{BC}}(\cdot)$ is called the $\mathbb{D}$-maximal function of u since it is the $\mathbb{D}$-largest of all $\mathbb{D}$-average values over $u^*_{\mathbb{BC}}$.

Remark 2.1. Even if the value of $u^{**}_{\mathbb{BC}}(t)$ at $t = 0$ is not included in the aforementioned definition, the $\mathbb{D}$-limit as t_1 and t_2 approach zero from the right for $t = t_1e_1 + t_2e_2$ is defined for all rearrangements. In fact,

$$\begin{aligned} \lim_{t_1,t_2\to 0^+}{}_{\mathbb{D}} u^{**}_{\mathbb{BC}}(t) &= \lim_{t_1,t_2\to 0^+}{}_{\mathbb{D}}(u_1^{**}(t_1)e_1 + u_2^{**}(t_2)e_2) \\ &= \lim_{t_1\to 0^+} u_1^{**}(t_1)e_1 + \lim_{t_2\to 0^+} u_2^{**}(t_2)e_2 \\ &= u_1^*(0)e_1 + u_2^*(0)e_2 = u^*_{\mathbb{BC}}(0) \\ &= \|u\|_\infty^{\mathbb{D}}, \end{aligned}$$

where the last equality is from (5).

Theorem 2.1. *Let $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$ be two elements of $\mathfrak{F}(\Omega, \mathfrak{M})$ and $\vartheta = \vartheta_1e_1 + \vartheta_2e_2$ be a $\mathbb{BC}$-measure with resonant measures ϑ_1 and ϑ_2. Then,*

$$(u + v)^{**}_{\mathbb{BC}}(t) \leqslant u^{**}_{\mathbb{BC}}(t) + v^{**}_{\mathbb{BC}}(t),$$

for all $t \in \mathbb{D}^+$.

Definition 2.4. Let $\vartheta = \vartheta_1e_1 + \vartheta_2e_2$ be a $\mathbb{BC}$-measure, $(\Omega, \mathfrak{M}, \vartheta)$ be a σ-finite complete $\mathbb{BC}$-measurable space, and $\mathfrak{F}(\Omega, \mathfrak{M})$ be the set of all measurable $\mathbb{BC}$-valued functions on Ω. For $0 < p \leq \infty$ and $0 < q \leq \infty$, bicomplex Lorentz space, $L^{\mathbb{BC}}_{p,q}(\Omega) = L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$ is the set of all equivalence classes of $\mathbb{BC}$-measurable functions $f = f_1e_1 + f_2e_2 \in \mathfrak{F}(\Omega, \mathfrak{M})$ such that the functional $\|f\|^{\mathbb{BC}}_{p,q}$ is $\mathbb{D}$-finite, where

$$\|f\|^{\mathbb{BC}}_{p,q} = e_1\,\|f_1\|_{p,q} + e_2\,\|f_2\|_{p,q}$$

and

$$\|f_i\|_{p,q} = \begin{cases} \left(\frac{q}{p} \int\limits_0^\infty (t^{1/p} f_i^*(t))^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & \text{if } 0 < p < \infty, 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p} f_i^*(t), & \text{if } 0 < p \le \infty, q = \infty \end{cases}$$

for $i = 1, 2$.

Remark 2.2. For the $\mathbb{BC}$-Lorentz $L_{p,q}^{\mathbb{BC}}(\Omega)$ space, the case $p = \infty$ and $0 < q < \infty$ is not of any interest. The reason for this is that $\|f\|_{\infty,q}^{\mathbb{BC}} < \infty_{\mathbb{D}}$ says that $f = 0$ (ϑ-a.e) on Ω. The $\mathbb{BC}$-Lorentz $L_{p,q}^{\mathbb{BC}}(\Omega)$ spaces can be seen as generalizations of the ordinary $\mathbb{BC}$-Lebesgue spaces, $L_{\mathbb{BC}}^{p}(\Omega)$, which are examined in [10]. The reason for this is that if one writes $q = p$, one can obtain $L_{p,p}^{\mathbb{BC}}(\Omega) = L_{\mathbb{BC}}^{p}(\Omega)$ for $0 < p \le \infty$.

Example 2.1. For any $\mathfrak{M}$-measurable set E of finite measure according to ϑ_1 and ϑ_2, we have

$$\begin{aligned} \|\chi_E\|_{p,q}^{\mathbb{BC}} &= e_1 \|\chi_E\|_{p,q} + e_2 \|\chi_E\|_{p,q} \\ &= e_1 \left(\frac{p}{q}\right)^{\frac{1}{q}} \vartheta_1(E)^{\frac{1}{p}} + e_2 \left(\frac{p}{q}\right)^{\frac{1}{q}} \vartheta_2(E)^{\frac{1}{p}} \\ &= \left(\frac{p}{q}\right)^{\frac{1}{q}} \vartheta(E)^{\frac{1}{p}}, \end{aligned}$$

for $0 < p, q < \infty$ by [11, Definition 2.2]. If $q = \infty$, then

$$\begin{aligned} \|\chi_E\|_{p,\infty}^{\mathbb{BC}} &= e_1 \sup_{t>0} t^{1/p} u_1^*(t) + e_2 \sup_{t>0} t^{1/p} u_2^*(t) \\ &= e_1 \vartheta_1(E)^{\frac{1}{p}} + e_2 \vartheta_2(E)^{\frac{1}{p}} \\ &= \vartheta(E)^{\frac{1}{p}} \end{aligned}$$

since $e_1 \cdot e_2 = 0$ in $\mathbb{BC}$.

Theorem 2.2. *The $\mathbb{BC}$-Lorentz $L_{p,q}^{\mathbb{BC}}$ space is a quasi-normed linear space.*

Remark 2.3. The functional $\|\cdot\|_{p,q}^{\mathbb{BC}}$ is a norm if and only if $1 \le q \le p < \infty$ or the trivial case $p = \infty = q$.

Now, we investigate the conditions under which $\mathbb{BC}$-Lorentz spaces $L_{p,q}^{\mathbb{BC}}(\Omega)$ are normed spaces. To achieve this, we must propose a new functional, $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$, which, for particular values of p and q, is equivalent to $\|\cdot\|_{p,q}^{\mathbb{BC}}$. The triangle inequality will be satisfied by the new functional, which is defined by maximal function instead of rearrangement, for more values of p and q rather than $\|\cdot\|_{p,q}^{\mathbb{BC}}$. The conclusions in this part will be important for the discussion of the topological characteristics of the normed space $(L_{p,q}^{\mathbb{BC}}(\Omega), \|\cdot\|_{(p,q)}^{\mathbb{BC}})$ in the section that follows.

Definition 2.5. For any $f \in L_{p,q}^{\mathbb{BC}}(\Omega)$, the functional $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ defined by

$$\|f\|_{(p,q)}^{\mathbb{BC}} = e_1 \|f_1\|_{(p,q)} + e_2 \|f_2\|_{(p,q)}$$

induces a norm on $L_{p,q}^{\mathbb{BC}}(\Omega)$, where

$$\|f_i\|_{(p,q)} = \begin{cases} \left(\frac{q}{p} \int\limits_0^\infty (t^{1/p} f_i^{**}(t))^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & \text{if } 0 < p < \infty, 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p} f_i^{**}(t), & \text{if } 0 < p \le \infty, q = \infty. \end{cases}$$

Using Theorem 2.1 and the Minkowski inequality, it is easy to see that $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ satisfy the triangle inequality for $1 \le p < \infty$ and $1 \le q \le \infty$. Therefore, $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ is a norm on $L_{p,q}^{\mathbb{BC}}$, and hence, $(L_{p,q}^{\mathbb{BC}}(\Omega), \|\cdot\|_{(p,q)}^{\mathbb{BC}})$ is a $\mathbb{D}$-normed space if $1 < p < \infty$, $1 \le q \le \infty$, or $p = \infty = q$. Moreover, the norm $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ and the quasi-norm $\|\cdot\|_{p,q}^{\mathbb{BC}}$ are $\mathbb{D}$-equivalent, i.e.,

$$\|\cdot\|_{p,q}^{\mathbb{BC}} \preccurlyeq \|\cdot\|_{(p,q)}^{\mathbb{BC}} \preccurlyeq \frac{p}{p-1} \|\cdot\|_{p,q}^{\mathbb{BC}},$$

where the first inequality is an immediate consequence of the fact that $u_{\mathbb{BC}}^*(\cdot) \preccurlyeq u_{\mathbb{BC}}^{**}(\cdot)$, and the second follows from the bicomplex version of the Hardy inequality.

Theorem 2.3. (Completeness). *The* $\mathbb{BC}$*-Lorentz space* $L_{p,q}^{\mathbb{BC}}(\Omega)$ *with the quasinorm* $\|\cdot\|_{p,q}^{\mathbb{BC}}$ *is* $\mathbb{D}$*-complete for all* $0 < p < \infty$, $0 < q \le \infty$. *Nevertheless, if* $1 < p < \infty$, $1 \le q \le \infty$, $p = q = 1$, *or* $p = q = \infty$, *then the normed space* $(L_{p,q}^{\mathbb{BC}}(\Omega), \|\cdot\|_{(p,q)}^{\mathbb{BC}})$ *is a bicomplex Banach space.*

Theorem 2.4. *Let S be the set of all simple integrable functions. Then, the set* $\mathfrak{S} = \{e_1 s_1 + e_2 s_2 : s_1, s_2 \in S\}$ *is dense in* $L_{p,q}^{\mathbb{BC}}(\Omega)$ *for* $0 < p < \infty$ *and* $0 < q < \infty$.

3 Characterizations of weighted composition operators

Let $(\Omega, \mathfrak{M}, \vartheta)$ be a σ-finite complete $\mathbb{BC}$-measure space, $T : \Omega \to \Omega$ be a $\mathbb{BC}$-measurable ($T^{-1}(E) \in \mathfrak{M}$, for any $E \in \mathfrak{M}$), non-singular ($\vartheta(T^{-1}(E)) = 0$ whenever $\vartheta(E) = 0$) transformation, and $u = e_1 u_1 + e_2 u_2$ be a $\mathbb{BC}$-valued $\mathbb{BC}$-measurable function defined on Ω. We establish a linear transformation $W = W_{u,T}$ mapping the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega)$ to the linear space encompassing all $\mathbb{BC}$-valued $\mathbb{BC}$-measurable functions, defined as

$$\begin{aligned} W(f)(x) = W_{u,T}(f)(x) &= u(T(x))f(T(x)) \\ &= e_1 u_1(T(x)) f_1(T(x)) + e_2 u_2(T(x)) f_2(T(x)) \\ &= \sum_{i=1}^{2} e_i u_i(T(x)) f_i(T(x)), \end{aligned}$$

for all $x \in \Omega$ and $f = e_1 f_1 + e_2 f_2 \in L_{p,q}^{\mathbb{BC}}(\Omega)$.

If W is $\mathbb{D}$-bounded with range in $L_{p,q}^{\mathbb{BC}}(\Omega)$ subsequently, this transformation is termed a weighted composition operator on $L_{p,q}^{\mathbb{BC}}(\Omega)$. If $u = e_1 + e_2$, then

$$W_{u,T}(f)(x) = e_1 f_1(T(x)) + e_2 f_2(T(x)) = (f \circ T)(x)$$

is called a composition operator C_T induced by T. If T is the identity mapping, then $W \equiv M_u$ will be a multiplication operator induced by u. The study of these operators on Lebesgue spaces has been carried out in [17–22] and references therein. Composition and multiplication operators on the Lorentz spaces, weighted Lorentz spaces, Lorentz-Karamata spaces were studied in [23–28] and [29].

This study characterizes non-singular measurable transformations T from Ω into itself, along with $\mathbb{BC}$-valued $\mathbb{BC}$-measurable functions u on Ω, which induce weighted composition operators. Subsequently, their compactness and closedness within the range of $\mathbb{BC}$-Lorentz spaces $L_{p,q}^{\mathbb{BC}}(\Omega)$, where $1 < p \le \infty, 1 \le q \le \infty$, are fully identified.

Theorem 3.1. *Let* $(\Omega, \mathfrak{M}, \vartheta)$ *be a* σ*-finite complete* $\mathbb{BC}$*-measure space and* $u : \Omega \to \mathbb{BC}$ *be a* $\mathbb{BC}$*-measurable function. Suppose that* $T : \Omega \to \Omega$ *is a* $\mathbb{BC}$*-measurable, non-singular transformation such that the Lebesgue-Radon-Nikodym derivative* $f_T = e_1 f_T^1 + e_2 f_T^2 = d(\vartheta T^{-1})/d\vartheta$ *is in* $L_{\mathbb{BC}}^{\infty}(\vartheta)$. *Then,*

$$W_{u,T} : f \mapsto u \circ T \cdot f \circ T$$

is $\mathbb{D}$*-bounded on* $L_{p,q}^{\mathbb{BC}}(\Omega)$ *for* $1 < p \le \infty, 1 \le q \le \infty$ *if* $u \in L_{\mathbb{BC}}^{\infty}(\vartheta)$.

Proof. Assume that f_T is in $L^{\infty}_{\mathbb{BC}}(\vartheta)$ and $\|f_T\|^{\mathbb{D}}_{\infty} = \gamma$. Then, $\|f^1_T\|_\infty = \gamma_1$ and $\|f^2_T\|_\infty = \gamma_2$ for $\gamma = e_1\gamma_1 + e_2\gamma_2 > 0$. For any $f = e_1 f_1 + e_2 f_2 \in L^{\mathbb{BC}}_{p,q}(\Omega)$, the $\mathbb{D}$-distribution function of $W(f)$ satisfies

$$\begin{aligned}
D^{\mathbb{BC}}_{W(f)}(\lambda) &= D_{u_1(T)f_1(T)}(\lambda_1)e_1 + D_{u_2(T)f_2(T)}(\lambda_2)e_2\\
&= \sum_{i=1}^{2}\vartheta_i\{x\in\Omega : |u_i(T(x))f_i(T(x))| > \lambda_i\}e_i\\
&= \sum_{i=1}^{2}\vartheta_i T^{-1}\{x\in\Omega : |u_i(x)f_i(x)| > \lambda_i\}e_i\\
&\leqslant \sum_{i=1}^{2}\vartheta_i T^{-1}\{x\in\Omega : \|u_i\|_\infty |f_i(x)| > \lambda_i\}e_i\\
&\leqslant \sum_{i=1}^{2}\gamma_i\vartheta_i\{x\in\Omega : \|u_i\|_\infty |f_i(x)| > \lambda_i\}e_i\\
&= (e_1\gamma_1 + e_2\gamma_2)(D_{\|u_1\|_\infty f_1}(\lambda_1)e_1 + D_{\|u_2\|_\infty f_2}(\lambda_2)e_2)\\
&= \gamma D^{\mathbb{BC}}_{\|u\|^{\mathbb{D}}_\infty f}(\lambda),
\end{aligned}$$

for all $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \geqslant 0$, where $Wf = W_{u,T}(f) = e_1 u_1(T)f_1(T) + e_2 u_2(T)f_2(T)$. Therefore, for each $t = e_1 t_1 + e_2 t_2 \geqslant 0$, we have

$$\left\{\lambda > 0 : D^{\mathbb{BC}}_{\|u\|^{\mathbb{D}}_\infty f}(\lambda) \leqslant \frac{t_1}{\gamma_1}e_1 + \frac{t_2}{\gamma_2}e_2\right\} \subset \{\lambda > 0 : D^{\mathbb{BC}}_{W(f)}(\lambda) \leqslant e_1 t_1 + e_2 t_2\}.$$

This inclusion says that

$$\begin{aligned}
W(f)^*_{\mathbb{BC}}(t) &= \inf\nolimits_{\mathbb{D}}\{\lambda \geqslant 0 : D^{\mathbb{BC}}_{W(f)}(\lambda) \leqslant t\}\\
&\leqslant \inf\nolimits_{\mathbb{D}}\left\{\lambda \geqslant 0 : D^{\mathbb{BC}}_{\|u\|^{\mathbb{D}}_\infty f}(\lambda) \leqslant \frac{t_1}{\gamma_1}e_1 + \frac{t_2}{\gamma_2}e_2\right\}\\
&= \sum_{i=1}^{2}\inf\left\{\lambda_i \geq 0 : D_{\|u_i\|_\infty f_i}(\lambda_i) \leq \frac{t_i}{\gamma_i}\right\}e_i\\
&= \|u_1\|_\infty (f_1)^*_{\mathbb{BC}}\left(\frac{t_1}{\gamma_1}\right)e_1 + \|u_2\|_\infty (f_2)^*_{\mathbb{BC}}\left(\frac{t_2}{\gamma_2}\right)e_2\\
&= (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\left((f_1)^*_{\mathbb{BC}}\left(\frac{t_1}{\gamma_1}\right)e_1 + (f_2)^*_{\mathbb{BC}}\left(\frac{t_2}{\gamma_2}\right)e_2\right)\\
&= \|u\|^{\mathbb{D}}_\infty f^*_{\mathbb{BC}}\left(\frac{t}{\gamma}\right)
\end{aligned}$$

and

$$W(f)^{**}_{\mathbb{BC}}(t) \leqslant \|u\|^{\mathbb{D}}_\infty f^{**}_{\mathbb{BC}}\left(\frac{t}{\gamma}\right) = (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\left((f_1)^{**}_{\mathbb{BC}}\left(\frac{t_1}{\gamma_1}\right)e_1 + (f_2)^{**}_{\mathbb{BC}}\left(\frac{t_2}{\gamma_2}\right)e_2\right).$$

Therefore, for $1 \leq q < \infty$,

$$\begin{aligned}
\|Wf\|^{\mathbb{BC}}_{(p,q)} &= e_1\, \|u_1(T)f_1(T)\|_{(p,q)} + e_2\, \|u_2(T)f_2(T)\|_{(p,q)}\\
&= \sum_{i=1}^{2} e_i\left(\frac{q}{p}\int_0^\infty (t_i^{1/p}(u_i(T)f_i(T))^{**}(t_i))^q \frac{\mathrm{d}t_i}{t_i}\right)^{1/q} \qquad (7)\\
&\leqslant \sum_{i=1}^{2} e_i\, \|u_i\|^{1/q}_\infty\left(\frac{q}{p}\int_0^\infty \left(t_i^{1/p} f_i(T)^{**}\left(\frac{t_i}{\gamma_i}\right)\right)^q \frac{\mathrm{d}t_i}{t_i}\right)^{1/q}
\end{aligned}$$

$$\leqslant \sum_{i=1}^{2} e_i \, \|u_i\|_\infty \sum_{i=1}^{2} e_i(\gamma_i)^{1/p} \|f_i\|_{(p,q)}$$
$$= \|u\|_\infty^{\mathbb{D}} \gamma^{1/p} \, \|f\|_{(p,q)}^{\mathbb{BC}}$$

and

$$\begin{aligned}
\|Wf\|_{(p,\infty)}^{\mathbb{BC}} &= e_1 \, \|u_1(T)f_1(T)\|_{(p,\infty)} + e_2 \, \|u_2(T)f_2(T)\|_{(p,\infty)} \\
&= \sum_{i=1}^{2} e_i \sup_{t_i>0} t_i^{1/p}(u_i(T)f_i(T))^{**}(t_i) \\
&\leqslant \sum_{i=1}^{2} e_i \, \|u_i\|_\infty \sup_{t_i>0} t_i^{1/p}(f_i)^{**}\left(\frac{t_i}{\gamma_i}\right) \\
&= \left(\sum_{i=1}^{2} e_i \, \|u_i\|_\infty\right)\left(\sum_{i=1}^{2} e_i(\gamma_i)^{1/p} \|f_i\|_{(p,\infty)}\right) \\
&= \|u\|_\infty^{\mathbb{D}} \gamma^{1/p} \, \|f\|_{(p,\infty)}^{\mathbb{BC}}
\end{aligned} \tag{8}$$

can be written. As a result, W is a $\mathbb{D}$-bounded operator on $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$, and

$$\|W\|_{(p,q)}^{\mathbb{BC}} \leqslant \|u\|_\infty^{\mathbb{D}} \gamma^{1/p}$$

by (7) and (8). □

Theorem 3.2. *Let u be a $\mathbb{BC}$-valued measurable function and $T : \Omega \to \Omega$ be a non-singular measurable transformation such that $T(E_\varepsilon) \subset E_\varepsilon$ for each $e_1\varepsilon_1 + e_2\varepsilon_2 = \varepsilon > 0$, where $E_\varepsilon = \{x \in \Omega : |u(x)|_k > \varepsilon\}$. If W is $\mathbb{D}$-bounded on $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$, then $u \in L_{\mathbb{BC}}^\infty(\vartheta)$.*

Proof. Assume that $u \notin L_{\mathbb{BC}}^\infty(\vartheta)$. Then, for each $N > 0$, the set $E_N = \{x \in \Omega : |u(x)|_k > N\}$ has a $\mathbb{D}$-positive measure. It means there exist $N_1, N_2 > 0$ such that $|u_1(x)| > N_1$ and $|u_2(x)| > N_2$ for all $x \in E_N$ where $N = N_1e_1 + N_2e_2$ and $\vartheta(E_N) > 0$. Using the definition of $\mathbb{D}$-distribution function and the property $\chi_{E_N}e_1 + \chi_{E_N}e_2 \leqslant \chi_{T^{-1}(E_N)}e_1 + \chi_{T^{-1}(E_N)}e_2$, one obtain

$$\begin{aligned}
D_{\chi_{E_N}}^{\mathbb{BC}}(\lambda) &= D_{\chi_{E_N}}(\lambda_1)e_1 + D_{\chi_{E_N}}(\lambda_2)e_2 \\
&= \sum_{i=1}^{2} \vartheta_i\{x \in \Omega : |\chi_{E_N}(x)| > \lambda_i\}e_i \\
&\leqslant \sum_{i=1}^{2} \vartheta_i\{x \in \Omega : |\chi_{T^{-1}(E_N)}| > \lambda_i\}e_i \\
&\leqslant \sum_{i=1}^{2} \vartheta_i\{x \in \Omega : |u_i(T(x))\chi_{T^{-1}(E_N)}| > \lambda_i N_i\}e_i.
\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}
(W\chi_{E_N})_{\mathbb{BC}}^*(t) &= \inf_{\mathbb{D}}\{\lambda \geqslant 0 : D_{W\chi_{E_N}}^{\mathbb{BC}}(\lambda) \leqslant t\} = (u(T(x))\chi_{E_N}(T(x)))_{\mathbb{BC}}^*(t) \\
&= \sum_{i=1}^{2} \inf\{\lambda_i \geq 0 : D_{u_i(T(x))\chi_{E_N}(T(x))}(\lambda_i) \leq t_i\}e_i \\
&= \sum_{i=1}^{2} N_i \inf\{\lambda_i \geq 0 : D_{u_i(T(x))\chi_{E_N}(T(x))}(\lambda_i N_i) \leq t_i\}e_i \\
&\geqslant (N_1e_1 + N_2e_2)\sum_{i=1}^{2} \inf\{\lambda_i \geq 0 : D_{\chi_{E_N}}(\lambda_i) \leq t_i\}e_i \\
&= N(\chi_{E_N})_{\mathbb{BC}}^*(t)
\end{aligned}$$

by (9). Thus, we have

$$(W\chi_{E_N})^{**}_{\mathbb{BC}}(t) = \sum_{i=1}^{2}\left(\frac{1}{t_i}\int_0^{t_i}(u_i(T(x))\chi_{E_N}(T(x)))^*(s)ds\right)e_i$$
$$\geqslant \sum_{i=1}^{2}N_i\left(\frac{1}{t_i}\int_0^{t_i}(\chi_{E_N})^*(s)ds\right)e_i$$
$$= (N_1e_1 + N_2e_2)(\chi_{E_N})^{**}_{\mathbb{BC}}(t)$$

and $\|W\chi_{E_N}\|^{\mathbb{BC}}_{(p,q)} \geqslant N\,\|\chi_{E_N}\|^{\mathbb{BC}}_{(p,q)}$. This contradicts the boundedness of $W = W_{u,T}$. □

Combining the previous two theorems, we obtain the following.

Theorem 3.3. *Let u be a $\mathbb{BC}$-measurable, $\mathbb{BC}$-valued function, and T be a non-singular measurable transformation on Ω such that the Lebesgue-Radon-Nikodym derivative $f_T = d(\vartheta T^{-1})/d\vartheta$ is in $L^{\infty}_{\mathbb{BC}}(\vartheta)$ and $T(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 > 0$, where $E_\varepsilon = \{x : |u(x)|_k > \varepsilon\}$. Then, $W = W_{u,T}$ is $\mathbb{D}$-bounded on $L^{\mathbb{BC}}_{p,q}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$ if and only if $u \in L^{\infty}_{\mathbb{BC}}(\vartheta)$.*

4 Compactness and closed range

We are prepared to examine the compactness and closed range properties of the weighted composition operator $W(f) = W_{u,T}(f) = u(T)\cdot f(T)$ on the $\mathbb{BC}$-Lorentz spaces $L^{\mathbb{BC}}_{p,q}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$.

Let $T : \Omega \to \Omega$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1f_T^1 + e_2f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L^{\infty}_{\mathbb{BC}}(\vartheta)$ with $\|f_T\|^{\mathbb{D}}_{\infty} = e_1\,\|f_T^1\|_\infty + e_2\,\|f_T^2\|_\infty = \gamma$. Then, $\|f_T^1\|_\infty = \gamma_1$ and $\|f_T^2\|_\infty = \gamma_2$ for $\gamma = e_1\gamma_1 + e_2\gamma_2 > 0$. For each $f = e_1f_1 + e_2f_2 \in L^{\mathbb{BC}}_{p,q}(\Omega)$ and $t = e_1t_1 + e_2t_2 > 0$,

$$(Wf)^*_{\mathbb{BC}}(\gamma t) = \inf\nolimits_{\mathbb{D}}\{\alpha \geqslant 0 : D^{\mathbb{BC}}_{Wf}(\alpha) \leqslant \gamma t\}$$
$$= \sum_{i=1}^{2}\inf\{\alpha_i \geq 0 : D_{u_i(T(x))\cdot f_i(T(x))}(\alpha_i) \leq \gamma_i t_i\}e_i$$
$$= \sum_{i=1}^{2}\inf\{\alpha_i \geq 0 : \vartheta_i\{x \in \Omega : |u_i(T(x))\cdot f_i(T(x))| > \alpha_i\} \leq \gamma_i t_i\}e_i$$
$$= \sum_{i=1}^{2}\inf\{\alpha_i \geq 0 : \vartheta_i T^{-1}\{x \in \Omega : |(u_i\cdot f_i)(x)| > \alpha_i\} \leq \gamma_i t_i\}e_i$$
$$\leqslant \sum_{i=1}^{2}\inf\{\alpha_i \geq 0 : \vartheta_i\{x \in \Omega : |(u_i\cdot f_i)(x)| > \alpha_i\} \leq t_i\}e_i$$
$$= (u\cdot f)^*_{\mathbb{BC}}(t) = (M_uf)^*_{\mathbb{BC}}(t)$$

can be written. Therefore,

$$\|Wf\|^{\mathbb{BC}}_{(p,q)} \leqslant \gamma_1^{1/p}\,\|M_{u_1}f_1\|^{\mathbb{BC}}_{(p,q)}e_1 + \gamma_2^{1/p}\,\|M_{u_2}f_2\|^{\mathbb{BC}}_{(p,q)}e_2 = \gamma^{1/p}\,\|M_uf\|^{\mathbb{BC}}_{(p,q)}, \tag{10}$$

by [11, Definition 2.2]. Now, let $U = \{x : u(x) \neq 0\}$ and assume that f_T is bounded away from zero in the $\mathbb{D}$-metric on U. It means $f_T{>}\delta$ ($\vartheta-$a.e.) for some $\delta = e_1\delta_1 + e_2\delta_2 > 0$. Then, for all $E \in \mathfrak{M}$ with $E \subset U$, we obtain

$$\vartheta T^{-1}(E) = \vartheta_1(T^{-1}(E))e_1 + \vartheta_2(T^{-1}(E))e_2$$
$$= e_1\int_E f_T^1 d\vartheta_1 + e_2\int_E f_T^1 d\vartheta_1$$
$$\geqslant \delta_1\vartheta_1(E)e_1 + \delta_2\vartheta_2(E)e_2 = \delta\vartheta(E)$$

and so $\|Wf\|_{(p,q)}^{\mathbb{BC}} \succcurlyeq \delta^{1/p}\,\|M_u f\|_{(p,q)}^{\mathbb{BC}}$. As a result, with (10),

$$\delta^{1/p}\,\|M_u f\|_{(p,q)}^{\mathbb{BC}} \preccurlyeq \|Wf\|_{(p,q)}^{\mathbb{BC}} \preccurlyeq \gamma^{1/p}\,\|M_u f\|_{(p,q)}^{\mathbb{BC}} \tag{11}$$

can be written for each $f \in L_{p,q}^{\mathbb{BC}}(\Omega)$ whenever $f_T \in L_{\mathbb{BC}}^{\infty}(\vartheta)$ is bounded away from zero in the $\mathbb{D}$-metric and $1 < p \leq \infty, 1 \leq q \leq \infty$. By (11) and [23, Theorem 3.1], we can write the following theorem.

Theorem 4.1. *Let $T : \Omega \to \Omega$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1 f_T^1 + e_2 f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L_{\mathbb{BC}}^{\infty}(\vartheta)$ and bounded away from zero in the $\mathbb{D}$-metric. Let u be a $\mathbb{BC}$-valued, measurable function on Ω such that $W_{u,T}$ is bounded on the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$. Then, the following are equivalent*:

(i) *$W_{u,T}$ is compact,*
(ii) *M_u is compact,*
(iii) *$L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ are finite dimensional for each $\delta = e_1\delta_1 + e_2\delta_2 > 0$, where*

$$L_{p,q}^{\mathbb{BC}}(H_{u,\delta}) = \{f\chi_{H_{u,\delta}} : f \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)\} \quad \text{and} \quad H_{u,\delta} = \{x \in \Omega : |u(x)|_k \succcurlyeq \delta\}.$$

Given that $W_{u,T} = C_T M_u$, a condition sufficient for the compactness of the weighted composition operator $W_{u,T}$ on $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$ can be inferred using [28, Theorem 3.1].

Theorem 4.2. *Let $T : \Omega \to \Omega$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1 f_T^1 + e_2 f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L_{\mathbb{BC}}^{\infty}(\vartheta)$ and u be a $\mathbb{BC}$-valued, $\mathbb{BC}$-measurable function on Ω such that $u \in L_{\mathbb{BC}}^{\infty}(\vartheta)$. Let $\{U_n\}$ be the set of all atoms of Ω with $\vartheta(U_n) = \vartheta_1(U_n)e_1 + \vartheta_2(U_n)e_2 > 0$ for each n. Then, $W_{u,T}$ is compact on the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$ if ϑ_1 and ϑ_2 are purely atomic measures and*

$$\gamma_j^n = \frac{\vartheta_j T^{-1}(U_n)}{\vartheta_j(U_n)} \to 0,$$

for $j = 1, 2$.

Theorem 4.3. *If ϑ is non-atomic measure, i.e., ϑ_1 and ϑ_2 are non-atomic measures and $W_{u,T}$ is bounded on the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$, then $W_{u,T}$ is compact if and only if $u \cdot f_T = 0$ (ϑ-a.e.).*

Proof. It is easy to see that when $u \cdot f_T = 0$ (ϑ-a.e.), $W_{u,T}$ is compact. Now, let $W = W_{u,T}$ be compact and assume that $u \cdot f_T \neq 0$. Then, there exists a unit positive hyperbolic number $\alpha = e_1\alpha_1 + e_2\alpha_2 > 0$ such that the set

$$U = \{x \in \Omega : |u(x)|_k > \alpha\} \cap \{x \in \Omega : |f_T(x)|_k > \alpha\}$$

has $\mathbb{D}$-positive measure. Since ϑ_1 and ϑ_2 are non-atomic measures, there exists a decreasing sequence $\{U_n\}$ of $\mathbb{BC}$-measurable subsets of U such that

$$\vartheta_j(U_n) = \frac{a_j}{2^n}, 0 < a_j < \vartheta_j(U),$$

for $j = 1, 2$. Let $v_n = e_1 v_n^{(1)} + e_2 v_n^{(2)} = \frac{\chi_{U_n}}{\|\chi_{U_n}\|_{(p,q)}^{\mathbb{BC}}}(e_1 + e_2)$. Then, $\{v_n\}$ is a $\mathbb{D}$-bounded sequence in $L_{p,q}^{\mathbb{BC}}(\Omega)$. For any $n \in \mathbb{N}$, let $m = 2n$. Then, for $t = e_1 t_1 + e_2 t_2 \succcurlyeq 0$,

$$\begin{aligned}(W_{v_n - v_m})^*_{\mathbb{BC}}\left(\frac{t}{\alpha}\right) &= \inf\nolimits_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_{(Wv_n - Wv_m)}^{\mathbb{BC}}(\lambda) \preccurlyeq \frac{t}{\alpha}\right\} \\ &= \sum_{i=1}^{2} \inf\left\{\lambda_i \geq 0 : D_{u_i(T(x))\cdot(v_n^{(i)} - v_m^{(i)})(T(x))}(\lambda_i) \leq \frac{t_i}{\alpha_i}\right\} e_i \\ &= \sum_{i=1}^{2} \inf\left\{\lambda_i \geq 0 : \vartheta_i\{x \in \Omega : |u_i(T(x)) \cdot (v_n^{(i)} - v_m^{(i)})(T(x))| > \lambda_i\} \leq \frac{t_i}{\alpha_i}\right\} e_i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \inf\left\{\lambda_i \geq 0 : \vartheta_i T^{-1}\{y \in U_n : |u_i(y)\cdot(v_n^{(i)} - v_m^{(i)})(y)| > \lambda_i\} \leq \frac{t_i}{\alpha_i}\right\} e_i \\
&\geqslant \sum_{i=1}^{2} \inf\{\lambda_i \geq 0 : \vartheta_i\{y \in U_n : |(v_n^{(i)} - v_m^{(i)})(y)| > \lambda_i\alpha_i\} \leq t_i\} e_i \\
&= \sum_{i=1}^{2} \frac{1}{\alpha_i} \inf\{\lambda_i \geq 0 : \vartheta_i\{y \in U_n : |(v_n^{(i)} - v_m^{(i)})(y)| > \lambda_i\} \leq t_i\} e_i \\
&\geqslant \sum_{i=1}^{2} \frac{1}{\alpha_i} \inf\{\lambda_i \geq 0 : \vartheta_i\{y \in U_n \backslash U_m : |(v_n^{(i)} - v_m^{(i)})(y)| > \lambda_i\} \leq t_i\} e_i \\
&= \left(\frac{1}{\alpha_1} e_1 + \frac{1}{\alpha_2} e_2\right) \frac{(\chi_{U_n \backslash U_m})^{*}_{\mathbb{BC}}(t)}{\|\chi_{U_n}\|^{\mathbb{BC}}_{(p,q)}}
\end{aligned}
$$

can be written. Hence,

$$
\begin{aligned}
\|Wv_n - Wv_m\|^{\mathbb{BC}}_{(p,q)} &\geqslant \left(\frac{1}{\alpha_1} e_1 + \frac{1}{\alpha_2} e_2\right) \frac{1}{\|\chi_{U_n}\|^{\mathbb{BC}}_{(p,q)}} \left(\frac{p}{p-1}\right)^{1/q} \vartheta(U_n \backslash U_m)^{1/p} \\
&= \left(\frac{1}{\alpha_1} e_1 + \frac{1}{\alpha_2} e_2\right) \left(\frac{\vartheta(U_n \backslash U_m)}{\vartheta(U_n)}\right)^{1/p} > \varepsilon,
\end{aligned}
$$

for some $\varepsilon > 0$ and large values of n. The absence of a convergent subsequence in $\{Wv_n\}$ contradicts the compactness of W. Hence, $u \cdot f_T = 0$ (ϑ-a.e.). □

The subsequent theorem provides a characterization for a weighted composition operator to possess a closed range on $L^{\mathbb{BC}}_{p,q}(\Omega)$.

Theorem 4.4. *Let $T : \Omega \to \Omega$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1 f_T^1 + e_2 f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L^{\infty}_{\mathbb{BC}}(\vartheta)$ and bounded away from zero in the $\mathbb{D}$-metric. Let u be a $\mathbb{BC}$-valued, measurable function on Ω such that $W_{u,T}$ is bounded on the $\mathbb{BC}$-Lorentz space $L^{\mathbb{BC}}_{p,q}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$. Then, $W_{u,T}$ has $\mathbb{D}$-closed range if and only if there exists a $\beta = e_1\beta_1 + e_2\beta_2 > 0$ such that $|u(x)|_k \geqslant \beta$ (ϑ-a.e.) on $U = \{x \in \Omega : u(x) \neq 0\}$.*

Proof. Let $L^{\mathbb{BC}}_{p,q}(U) = \{f_1\chi_U e_1 + f_2\chi_U e_2 : f \in L^{\mathbb{BC}}_{p,q}(\Omega)\}$, where $U = \{x \in \Omega : u(x) \neq 0\}$. First, assume that $W = W_{u,T}$ has $\mathbb{D}$-closed range. Then, there exists an $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 > 0$ such that

$$
\|Wf\|^{\mathbb{BC}}_{(p,q)} \geqslant (e_1\varepsilon_1 + e_2\varepsilon_2)\|f\|^{\mathbb{BC}}_{(p,q)},
$$

for all $f \in L^{\mathbb{BC}}_{p,q}(U)$. Let $\beta > 0$ such that $\gamma_j^{1/p}\beta_j < \varepsilon_j$ for $j = 1, 2$, where $\gamma = \|f_T\|^{\mathbb{D}}_{\infty}$. If possible, $E = \{x \in \Omega : |u(x)|_k < \beta\}$ has $\mathbb{D}$-positive measure, i.e., $0 < \vartheta(E) < \infty_{\mathbb{D}}$, then $\chi_E \in L^{\mathbb{BC}}_{p,q}(U)$, and so

$$
\begin{aligned}
\|W\chi_E\|^{\mathbb{BC}}_{(p,q)} &\leqslant \gamma^{1/p}\, \|M_u\chi_E\|^{\mathbb{BC}}_{(p,q)} = \gamma^{1/p}\, \|u \cdot \chi_E\|^{\mathbb{BC}}_{(p,q)} \\
&\leqslant \gamma^{1/p}(e_1\beta_1 + e_2\beta_2)\|\chi_E\|^{\mathbb{BC}}_{(p,q)} \\
&= (e_1\gamma_1^{1/p} + e_2\gamma_2^{1/p})(e_1\beta_1 + e_2\beta_2)\|\chi_E\|^{\mathbb{BC}}_{(p,q)} \\
&< \varepsilon\, \|\chi_E\|^{\mathbb{BC}}_{(p,q)},
\end{aligned}
$$

by (10). This contradiction implies that $|u(x)|_k \geqslant \beta$ (ϑ-a.e.) for all $x \in U$.

Conversely, if $|u(x)|_k \geqslant \beta$ (ϑ-a.e.) for all $x \in U$, then using the property of $f_T > \delta$ (ϑ-a.e.) for some $\delta = e_1\delta_1 + e_2\delta_2 > 0$, bounded away from zero in the $\mathbb{D}$-metric on U, we obtain

$$
\begin{aligned}
\|Wf\|^{\mathbb{BC}}_{(p,q)} &\geqslant \delta^{1/p}\, \|M_u f\|^{\mathbb{BC}}_{(p,q)} = \delta^{1/p}\, \|u \cdot f\|^{\mathbb{BC}}_{(p,q)} \\
&\geqslant (e_1\delta_1^{1/p} + e_2\delta_2^{1/p})(e_1\beta_1 + e_2\beta_2)\|f\|^{\mathbb{BC}}_{(p,q)} \\
&= \delta^{1/p}\beta\, \|f\|^{\mathbb{BC}}_{(p,q)},
\end{aligned}
$$

for any $f \in L^{\mathbb{BC}}_{p,q}(U)$ by (11). As a result, $W_{u,T}$ has $\mathbb{D}$-closed range being $\ker(W_{u,T}) = L^{\mathbb{BC}}_{p,q}(\Omega \backslash U)$. □

Corollary 4.1. *If $T^{-1}(E_\varepsilon) \subset E_\varepsilon$ for each $\varepsilon > 0$ and $W_{u,T}$ has $\mathbb{D}$-closed range, then $|u(x)|_k \geqslant \beta$ (ϑ-a.e.) on $U = \{x \in X : u(x) \neq 0\}$ for some $\beta = e_1\beta_1 + e_2\beta_2 > 0$.*

The subsequent theorem can be directly derived utilizing observation (11) and using the modified version of [23, Theorem 4.1].

Theorem 4.5. *Let $T : \Omega \to \Omega$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1 f_T^1 + e_2 f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L^\infty_{\mathbb{BC}}(\vartheta)$ and bounded away from zero in the $\mathbb{D}$-metric. Let u be a $\mathbb{BC}$-valued, measurable function on Ω such that $W_{u,T}$ is bounded on the $\mathbb{BC}$-Lorentz space $L^{\mathbb{BC}}_{p,q}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$. Then, the following are equivalent*:

(1) *$W_{u,T}$ has $\mathbb{D}$-closed range,*
(2) *M_u has $\mathbb{D}$-closed range,*
(3) *$|u(x)|_k \geqslant \beta$ (ϑ-a.e.) for some $\beta = e_1\beta_1 + e_2\beta_2 > 0$ on $U = \{x \in X : u(x) \neq 0\}$.*

Theorem 4.6. *If ϑ is non-atomic measure, i.e., ϑ_1 and ϑ_2 are non-atomic measures and $W_{u,T}$ is bounded on the $\mathbb{BC}$-Lorentz space $L^{\mathbb{BC}}_{p,q}(\Omega)$ for $1 < p \leq \infty, 1 \leq q \leq \infty$, then $W_{u,T}$ is injective if and only if $u \circ T \neq 0$ (ϑ-a.e.) and T is surjective.*

Proof. First, let $W_{u,T}$ be injective. Assume that T is not surjective. Then, there exists a measurable set $F \subset \Omega \backslash T(\Omega)$ such that $0 \neq \chi_F \in L^{\mathbb{BC}}_{p,q}(\Omega)$ and $W_{u,T}(\chi_F) = 0$. This means $W_{u,T}$ is not injective, which is a contradiction. Furthermore, suppose that there exists a measurable set $E = \{x \in \Omega : |u(T(x))| = 0\}$ such that $\vartheta(E) > 0$. Then, a measurable set E_1 can be found such that $T^{-1}(E_1) \subset E$ and $\vartheta(E_1) < \infty_{\mathbb{D}}$. Then, $\chi_{E_1} \in L^{\mathbb{BC}}_{p,q}(\Omega)$ and $(u \circ T \cdot \chi_{E_1} \circ T)^*_{\mathbb{BC}}(t) = 0$ for all $t > 0$. This gives a non-trivial kernel of $W_{u,T}$, which is a contradiction. Hence, $u \circ T \neq 0$ (ϑ-a.e.). The converse is easy. □

5 Conclusion

In this article, we study weighted composition operators on $\mathbb{BC}$-Lorentz spaces, characterizing their behavior and exploring their key properties. Our work highlights the relationship between bicomplex numbers, Lorentz space geometry, and the action of these operators.

A key result is identifying conditions under which these operators are bounded, compact, or exhibit other important properties. We demonstrate that their behavior depends heavily on the weight functions and the structure of the underlying function spaces. Additionally, we expose connections to areas such as operator theory and harmonic analysis.

This research advances the understanding of operators on function spaces with $\mathbb{BC}$-valued functions and Lorentz norms, offering insights that could benefit fields such as signal processing, quantum mechanics, and mathematical physics. After reading this article, one can examine the spectral properties of such operators and obtain more results. Additionally, he can explore connections between weighted composition operators and other operator classes, such as Toeplitz, Hankel, or Lambert operators.

Further research could also be focusing on the injectivity and Fredholm properties of these operators. Examining conditions for injectivity, characterizing their kernels and ranges, and studying their Fredholmness would be interesting problems. These investigations could connect to integral equations and other function spaces.

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