DE GRUYTER Demonstratio Mathematica 2025; 58: 20250099

Research Article

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Nonlinear heat equation with viscoelastic term: Global existence and blowup in finite time

https://doi.org/10.1515/dema-2025-0099
received October 4, 2023; accepted January 23, 2025

Abstract: In this study, we investigate a nonlinear heat equation incorporating both a viscoelastic term and a reaction-diffusion term that depends on space-time variables. Initially, we establish the local Hadamard well-posedness results using the standard Faedo-Galerkin method. Subsequently, we demonstrate that the solution exhibits finite-time blowup for initial energy values that are both negative and nonnegative. Finally, we establish the global existence of the solution and provide general decay estimates for the energy functions with small initial energy, utilizing Martinez's inequality.

Keywords: nonlinear heat equation, viscoelastic, global existence, decay rate, blowup in finite time

MSC 2020: 35B40, 35K51

1 Introduction

In this study, we consider the following nonlinear parabolic equation:

$$u_t - \frac{\partial}{\partial x}(\mu_1(x,t)u_x) + \int_0^t g(t-s)\frac{\partial}{\partial x}(\mu_2(x,s)u_x(x,s))\,\mathrm{d}s = f(x,t,u), \quad (x,t) \in (0,1) \times (0,\infty), \tag{1.1}$$

associated with the homogeneous Robin boundary conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.2}$$

and supplemented with initial condition

$$u(0,t) = u_0(x), \tag{1.3}$$

where $h_0, h_1 \geqslant 0$ are given constants such that $h_0 + h_1 > 0$, and μ_1, μ_2, g, f, u_0 are given functions satisfying conditions to be specified later.

Problem (1.1)–(1.3) without viscoelastic term and $\mu_1 = 1$ is called heat equation or parabolic equation and has the simple form

$$u_t - \Delta u = f(u), \quad (x,t) \in \Omega \times (0,\infty). \tag{1.4}$$

Problem (1.1)–(1.3) along with (1.4), together with appropriate boundary and initial conditions, arises naturally in engineering and physical sciences. Such problems have been extensively investigated, and numerous results

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regarding existence, nonexistence, regularity, and asymptotic behavior have garnered significant attention from mathematicians. For instance, Gazzola and Weth [1] considered the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u, & (x,t) \in \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$, and $1 < p < \frac{n+2}{n-2}$. By using potential well method, which was first introduced by Payne and Sattinger [2], the authors studied the dichotomy between global existence and blowup for the resulting solutions via the initial energy. Roughly speaking, the natural phase space can be divided into three parts corresponding to three initial energy levels. Since the initial energy level is over the mountain pass level, by exploiting comparison principle as well as strong parabolic maximum principle, the authors gave us some properties of the resulting solution. Inspired by this celebrated research, a vast number of research works related to parabolic equations, coupled parabolic equation or pseudo-parabolic equation have been investigated. Interested readers can find it in [3–10]. The common point of these studies is that the authors just consider the equations containing the coefficient independence on spatial-time variables. Problem (1.1)–(1.3) n is the mathematical model of many natural phenomena in physical science and engineering. For example, in the study of heat conduction in materials with memory, to our best knowledge, the first study that treats the heat equation with coefficient dependence on spatial-time variables is [11]. In this study, the authors consider the following nonlinear heat equation:

$$u_t - \frac{\partial}{\partial x}(\mu_1(x,t)u_x) + \int_0^t g(t-s)\frac{\partial}{\partial x}(\mu_2(x,s)u_x(x,s))\,\mathrm{d}s = f(u), \quad (x,t) \in (0,1) \times (0,\infty), \tag{1.6}$$

associated with the homogeneous Robin boundary conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.7}$$

and supplemented with initial condition.

$$u(0,t) = u_0(x), \tag{1.8}$$

First, by adopting the Faedo-Galerkin method, the authors established the existence and uniqueness of the weak solutions. Subsequently, by employing certain differential inequalities, the authors provided a sufficient condition for finite-time blowup as well as exponential decay estimates. With the inclusion of the term μ_1 and the viscoelastic term $\int_0^t g(t-s)\frac{\partial}{\partial x}(\mu_2(x,s)u_x(x,s))\,\mathrm{d}s$ in our equation, we cannot employ the potential well method to identify invariant sets under the flow of our problem. Consequently, obtaining *a priori* estimates for the solution becomes challenging. To address this issue, we need to utilize an alternative method first introduced by Vitillaro [12]. This method has been widely employed to investigate problems involving viscoelastic terms, as in [13–15]. However, some open questions remain related to this problem. These questions can be stated as follows:

(1) In order to prove the blowup results, the authors just focused on the negative initial energy, i.e., $E(0) < 0$. Thus, a natural question arises: can we prove that the solution blows up in finite time with nonnegative initial energy?

(2) In order to obtain the decay estimate for solution at infinity, the authors required that the relaxation function g satisfy the inequality

$$g'(t) \leqslant -\xi_1 g(t), \quad \forall t \in [0,\infty),$$

where ξ_1 is a positive constant. This estimate, combined with the positively of the function g, yields $g(t) \lesssim \exp(-\xi_1 t)$. This means that the relaxation function g must be dominated by an exponential function. Thus, a natural question arises: can we consider the generalized case

$$g'(t) \leqslant -\xi(t)g(t), \quad \forall t \in [0,\infty).$$

This work is organized as follows.

(1) In Section 2, we introduce some notations, preliminaries, and present the local Hadamard well-posedness result.
(2) In Section 3, we introduce a novel method for proving the blowup result with both negative and non-negative initial energy. The innovation in this section lies in our ability to address both cases simultaneously using the same auxiliary function;
(3) Section 4 is dedicated to proving a sufficient condition for the global existence and decay of weak solutions.

2 Preliminary results and notations

For the sake of convenience, we put $\Omega = (0,1)$. We omit the definitions of the usual function spaces such that $L^p = L^p(\Omega)$ and $H^m = H^m(\Omega)$. Let $\langle \cdot,\cdot \rangle$ be the scalar product in L^2 and $\langle \cdot,\cdot \rangle_{H^1}$ be the scalar product in H^1. The notation $\|\cdot\|_p$ stands for the L^p norm, $\|\cdot\| = \|\cdot\|_2$, and $\|\cdot\|_X$ for the norm in the Banach space X. We denote by X' the dual space of X. We denote by $L^p(0, T; X)$ with $p \in [1, \infty]$ for the Banach space of measurable functions $u : (0, T) \to \mathbb{R}$, such that

$$\|u\|_{L^p(0,T;\,X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty),$$

and

$$\|u\|_{L^\infty(0,T;\,X)} = \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X .$$

Let $\mu_1, \mu_2 \in C(\overline{\Omega} \times [0, T])$ such that $\mu_i(x, t) \geq \underline{\mu}_i > 0$ for all $(x, t) \in \Omega \times [0, T]$ and for all $i \in \{1, 2\}$. We consider two families of symmetric bilinear forms $\{a_i(t;\cdot,\cdot)\}_{0 \leq t \leq T}$ on $H^1 \times H^1$ defined by

$$a_i(t;\, u, v) = \langle \mu_i(t) u_x, v_x \rangle + h_0 \mu_i(0, t) u(0) v(0) + h_1 \mu_i(1, t) u(1) v(1), \tag{2.1}$$

for all $u, v \in H^1$, $t \in [0, T]$, and $i \in \{1, 2\}$. We have the following lemmas. The proofs are straightforward, so we omit the details.

Lemma 2.1. *The embedding $H^1 \hookrightarrow C(\overline{\Omega})$ is compact, and we have the following estimate*:

$$\|v\|_{C(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1}, \quad \forall v \in H^1.$$

Lemma 2.2. *Let $\mu_1, \mu_2 \in C(\overline{\Omega} \times [0, T])$ with $\mu_i(x, t) \geq \underline{\mu}_i > 0$ for all $(x, t) \in \overline{\Omega} \times [0, T]$ and for all $i \in \{1, 2\}$. Then, the symmetric bilinear forms $a_1(t;\cdot,\cdot)$, $a_2(t;\cdot,\cdot)$ are continuous on $H^1 \times H^1$ and coercive on H^1. Furthermore, we have*

$$|a_i(t;\, u, v)| \leq a_{iT}\, \|u\|_{H^1} \|v\|_{H^1}, \quad a_i(t;\, u, u) \geq a_{0i}\, \|v\|_{H^1}^2, \tag{2.2}$$

for all $u, v \in H^1$, $t \in [0, T]$, and $i \in \{1, 2\}$, where

$$a_{iT} = (1 + 2h_0 + 2h_1) \max_{(x,t) \in \overline{\Omega} \times [0,T]} \mu_i(x, t), \quad a_{0i} = \frac{1}{3} \underline{\mu}_i \min\{1, \max\{h_0, h_1\}\}, \quad i \in \{1, 2\}. \tag{2.3}$$

Lemma 2.3. *Let $E : [0, \infty) \to [0, \infty)$ be a decreasing function and $\varphi : [0, \infty) \to [0, \infty)$ a strictly increasing function of class $C^1([0, \infty))$ such that*

$$\varphi(0) = 0, \quad \text{and} \quad \lim_{t \to \infty} \varphi(t) = \infty. \tag{2.4}$$

Assume that there exist $\sigma \geqslant 0$ and $\omega > 0$ such that

$$\int_S^\infty E^{1+\sigma}(t)\varphi'(t)\mathrm{d}t \leqslant \frac{1}{\omega}E^\sigma(0)E(S), \quad \forall S \in (0,\infty). \tag{2.5}$$

Then, the function $t \longmapsto E(t)$ has the following decay property: for all $t \in [0,\infty)$, we have

(1) *If $\sigma = 0$, then*

$$E(t) \lesssim \exp(-\omega\varphi(t)), \quad \forall t \in [0,\infty); \tag{2.6}$$

(2) *If $\sigma > 0$, then*

$$E(t) \lesssim (1 + \omega\sigma\varphi(t))^{-\frac{1}{\sigma}}, \quad \forall t \in [0,\infty). \tag{2.7}$$

We conclude this section by presenting the local Hadamard well-posedness results. To achieve this, we first provide a precise definition of the weak solution to problem (1.1)–(1.3).

Definition 2.1. A function u is called a weak solution of problem (1.1)–(1.3) on the open interval $(0, T)$ if and only if the function u belongs to the following functional space:

$$W_T \coloneqq \{u \in L^\infty(0,T;H^1) : u_t \in L^2(0,T;L^2)\} \tag{2.8}$$

satisfying the following distributional identity

$$\langle u'(t), v\rangle + a_1(t; u(t), v) - \int_0^t g(t-s)a_2(s; u(s), v)\mathrm{d}s = \langle f[u](t), v\rangle, \tag{2.9}$$

for all test function $v \in H^1$, and the initial condition

$$u(0) = u_0, \tag{2.10}$$

where

$$f[u](x,t) \coloneqq f(x,t,u(x,t)). \tag{2.11}$$

To ensure the existence results, we must stipulate the following hypotheses:

[A_1] $u_0 \in H^1$;
[A_2] $\mu_1, \mu_1' \in C(\overline{\Omega} \times [0,\infty))$ and there exists the constant $\underline{\mu}_1$ such that $\mu_1(x,t) \geqslant \underline{\mu}_1 > 0$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$;
[A_3] $\mu_2 \in C(\overline{\Omega} \times [0,\infty))$;
[A_4] $f \in C(\overline{\Omega} \times [0,\infty) \times \mathbb{R})$;
[A_5] $g \in C^1([0,\infty))$.

Employing the standard Faedo-Galerkin method, we can readily establish the following result.

Theorem 2.1. *Let* [A_1]–[A_5] *be invalid. For any $T \in (0,\infty)$, there exists $T_* \in (0,T]$ such that problem* (1.1)–(1.3) *admit a local weak solution $u \in W_{T_*}$.*

Furthermore, if in addition

[A_4] *For all $M > 0$, there exists $\ell_M > 0$ such that*

$$|f(x,t,u_1) - f(x,t,u_2)| \leqslant \ell_M|u_1 - u_2|, \quad \forall x \in \overline{\Omega}, t \in [0,\infty), u_1, u_2 \in [-M,M],$$

then the solution is unique.

3 Finite time blowup and lifespan estimates

The main goal of this section is to show that with some suitable conditions, every weak solution to problem (1.1)–(1.3) blows up in finite time. In this section, we aim to consider problem (1.1)–(1.3) in a specific case $f(x,t,u) = K(x,t)f(u)$, $\mu_2(x,t) = \mu_2(x)$. First, we posit the following assumptions:

[A_2'] $\mu_1, \mu_1' \in C(\overline{\Omega} \times [0,\infty))$, and there exists the constant $\underline{\mu}_1$ such that $\mu_1(x,t) \geqslant \underline{\mu}_1 > 0$, and $\frac{\partial \mu_1}{\partial t}(x,t) \leqslant 0$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$;

[A_3'] $\mu_2 \in C(\overline{\Omega})$ and there exists the constant $\underline{\mu}_2$ such that $\mu_2(x) \geqslant \underline{\mu}_2 > 0$ for all $x \in \overline{\Omega}$.

Now, on the product space $H^1 \times H^1$, we consider the following symmetric bilinear forms:

$$a(u,v) = \langle u_x, v_x \rangle + h_0 u(0)v(0) + h_1 u(1)v(1), \quad \forall u, v \in H^1,$$
$$b(u,v) = \langle \mu_2 u_x, v_x \rangle + h_0 u(0)v(0) + h_1 u(1)v(1), \quad \forall u, v \in H^1,$$
$$\frac{\partial a_1}{\partial t}(t; u,v) = \langle \mu_1'(t) u_x, v_x \rangle + h_0 \mu_1'(0,t) u(0)v(0) + h_1 \mu_1'(1,t) u(1)v(1), \quad \forall u, v \in H^1.$$

Furthermore, it is easy to prove that the forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are continuous on $H^1 \times H^1$ and coercive on H^1. On the other hand, the norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$, and $v \longmapsto \|v\|_b = \sqrt{b(v,v)}$ are equivalent. In fact, we have the following lemmas.

Lemma 3.1. *There exist positive constants $\underline{a}, \bar{a}, \bar{\mu}_1, \bar{\mu}_2$ such that*
(1) $a(v,v) \geqslant \underline{a}\ \|v\|_{H^1}^2$ *for all* $v \in H^1$;
(2) $|a(u,v)| \leqslant \bar{a}\ \|u\|_{H^1} \|v\|_{H^1}$ *for all* $u, v \in H^1$;
(3) $b(v,v) \geqslant \underline{\mu}_2\ \|v\|_a^2$ *for all* $v \in H^1$;
(4) $|b(u,v)| \leqslant \bar{\mu}_2\ \|u\|_a \|v\|_a$ *for all* $u, v \in H^1$;
(5) $a_1(t; v,v) \geqslant \underline{\mu}_1\ \|v\|_a^2$ *for all* $v \in H^1$;
(6) $a_1(t; u,v) \leqslant \bar{\mu}_1\ \|u\|_a \|v\|_a$ *for all* $u, v \in H^1$;
(7) $\frac{\partial a_1}{\partial t}(t; v,v) \leqslant 0$ *for all* $v \in H^1$, $t \in [0,\infty)$.

Lemma 3.2. *On H^1, the norms $v \longmapsto \|v\|_a$ and $v \longmapsto \|v\|_b$ are equivalent and*

$$\sqrt{\underline{\mu}_2}\ \|v\|_a \leqslant \|v\|_b \leqslant \sqrt{\bar{\mu}_2}\ \|v\|_a, \quad \forall v \in H^1.$$

To derive blowup results, we also need to specify the following set of assumptions:

[K_1] $K, K_t \in C(\overline{\Omega} \times [0,\infty))$ satisfying
 (i) $0 \leqslant K(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$;
 (ii) $K_t(x,t) \geqslant 0$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$.

[F_1] $f \in C^1(\mathbb{R})$ and there exists a constant $p > 2$ such that

$$uf(u) \geqslant p\int_0^u f(z)\mathrm{d}z = pF(u) \geqslant 0, \quad \forall u \in \mathbb{R}.$$

[G_1] $g \in C^1([0,\infty))$ with the following properties:
 (i) $g(t) > 0$ for all $t \in [0,\infty)$;
 (ii) $g'(t) \leqslant 0$ for all $t \in [0,\infty)$;
 (iii) $G_\infty \leqslant \frac{p(p-2)}{(p-1)^2}\frac{\underline{\mu}_1}{\bar{\mu}_2}$ with $G_\infty = \lim_{t\to\infty} G(t)$, where $G(t) = \int_0^t g(s)\mathrm{d}s$.

Let us introduce the modified energy functional

$$E(t) = \frac{1}{2}a_1(t;\, u(t), u(t)) - \frac{G(t)}{2}\,\|u(t)\|_b^2 + \frac{(g\star u)(t)}{2} - \int_0^1 K(x,t)F(u(x,t))\mathrm{d}x, \tag{3.1}$$

where

$$(g\star u)(t) = \int_0^t g(t-s)\|u(t)-u(s)\|_b^2\mathrm{d}s. \tag{3.2}$$

Lemma 3.3. *Let* $[A_1]$, $[A_2']$ $[A_3']$, $[K_1]$, $[F_1]$, *and* $[G_1]$ *be in force. Then, we have the following estimate*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t) + \int_0^t \|u'(s)\|^2\mathrm{d}s\right) \leqslant 0, \quad \forall t \in [0, T_\infty). \tag{3.3}$$

Moreover, the following energy inequality holds

$$E(t) + \int_0^t \|u'(s)\|^2\mathrm{d}s \leqslant E(0), \quad \forall t \in [0, T_\infty). \tag{3.4}$$

Proof of Lemma 3.3. By using u_t as a test function to equation (1.1) and integrating both sides on Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t) + \int_0^t \|u'(s)\|^2\mathrm{d}s\right) = \frac{1}{2}\frac{\partial a_1}{\partial t}(t;\, u(t), u(t)) + \frac{(g'\star u)(t)}{2} - \frac{g(t)}{2}\,\|u(t)\|_b^2 - \int_0^1 K_t(x,t)F(u(x,t))\mathrm{d}x, \tag{3.5}$$

for any smooth solution u. We can extend (3.5) to weak solutions by employing density arguments. By combining (3.5) with $[A_2']$, $[K_1]$, $[F_1]$, and $[G_1]$, we completely obtain the result of Lemma 3.3. □

In the next theorem, we will prove that the weak solution u blows up in finite time, provided the initial energy is negative.

Theorem 3.1. *Assume that* $[A_1]$, $[A_2']$ $[A_3']$, $[K_1]$, $[F_1]$, *and* $[G_1]$ *hold. If the initial energy* $E(0) < 0$, *then the weak solution of problem* (1.1)–(1.3) *blows up at finite time.*

Proof of Theorem 3.1. By the last statement in Theorem 2.1, it is enough to prove that no global solution in $[0,\infty)$ can exist. Therefore, we proceed by assuming, for contradiction, that weak solutions exist across the entire time domain $[0,\infty)$. Next with $T_0 > 0$, $\beta > 0$ and $\tau > 0$ specified later, we define the auxiliary functional

$$M(t) = \int_0^t \|u(s)\|^2\,\mathrm{d}s + (T_0 - t)\|u_0\|^2 + \beta(t+\tau)^2, \quad \forall t \in [0, T_0]. \tag{3.6}$$

By direct computation, we find that

$$M'(t) = \|u(t)\|^2 - \|u_0\|^2 + 2\beta(t+\tau) = 2\int_0^t \langle u'(s), u(s)\rangle\mathrm{d}s + 2\beta(t+\tau). \tag{3.7}$$

From (3.6) and (3.7), it is easy to check that $M(t) > 0$ for all $t \in [0, T_0]$ and $M'(0) = 2\beta\tau > 0$. By direct calculation, it follows from (3.7) that

$$M''(t) = 2\langle u'(t), u(t)\rangle + 2\beta. \tag{3.8}$$

On the other hand, testing the equation in (1.1) with u, integrating both sides on Ω and plugging the previous result into the expression of $M''(t)$, we discover

$$M''(t) = 2\beta - 2a_1(t;\, u(t), u(t)) + 2\int_0^t g(t-s)b(u(s), u(t))\mathrm{d}s + 2\langle K(t)f(u(t)), u(t)\rangle. \tag{3.9}$$

Now, we put

$$\vartheta(t) = \left(\int_0^t \|u(s)\|^2\,\mathrm{d}s + \beta(t+\tau)^2\right)\left(\int_0^t \|u'(s)\|^2\,\mathrm{d}s + \beta\right). \tag{3.10}$$

Note that by utilizing Cauchy-Schwarz inequality, we can assert that

$$\begin{aligned}\frac{1}{4}[M'(t)]^2 &= \left(\int_0^t \langle u'(s), u(s)\rangle\,\mathrm{d}s + \beta(t+\tau)\right)^2\\ &\leqslant \left(\sqrt{\int_0^t \|u'(s)\|^2\,\mathrm{d}s}\sqrt{\int_0^t \|u(s)\|^2\,\mathrm{d}s} + \beta(t+\tau)\right)^2\\ &\leqslant \left(\int_0^t \|u'(s)\|^2\,\mathrm{d}s + \beta\right)\left(\int_0^t \|u(s)\|^2\,\mathrm{d}s + \beta(t+\tau)^2\right).\end{aligned}$$

This estimate, combined with (3.6) and (3.10), entails

$$\frac{1}{4}[M'(t)]^2 \leqslant \vartheta(t) \leqslant M(t)\left(\int_0^t \|u'(s)\|^2\mathrm{d}s + \beta\right). \tag{3.11}$$

Hence, it follows from (3.6), (3.9), and (3.11) that

$$M''(t)M(t) - \frac{p}{2}[M'(t)]^2 \geqslant 2M(t)D(t), \tag{3.12}$$

where $D : [0, T_0] \longrightarrow \mathbb{R}$ is the function defined by

$$D(t) = \beta - a_1(t;\, u(t), u(t)) + \int_0^t g(t-s)b(u(s), u(t))\mathrm{d}s + \langle K(t)f(u(t)), u(t)\rangle - p\left(\int_0^t \|u'(s)\|^2\mathrm{d}s + \beta\right). \tag{3.13}$$

By utilizing the Cauchy-Schwarz inequality, with $\delta > 0$, we obtain

$$\begin{aligned}\int_0^t g(t-s)b(u(s), u(t))\,\mathrm{d}s &= G(t)\|u(t)\|_b^2 + \int_0^t g(t-s)b(u(s)-u(t), u(t))\,\mathrm{d}s\\ &\geqslant \left(1 - \frac{1}{2\delta}\right)G(t)\|u(t)\|_b^2 - \frac{\delta}{2}(g\star u)(t),\end{aligned} \tag{3.14}$$

and by $[K_1]$ and $[F_1]$, we also deduce that

$$\langle K(t)f(u(t)), u(t)\rangle \geqslant p\int_0^1 K(x,t)F(u(x,t))\mathrm{d}x. \tag{3.15}$$

Combining (3.1), (3.4), and (3.13)–(3.15), we may conclude that

$$\begin{aligned}D(t) \geqslant \beta(1-p) - pE(0) &+ \left(\frac{\delta}{2} - 1\right)a_1(t;\, u(t), u(t)) - \left(\frac{\delta}{2} + \frac{1}{2\delta} - 1\right)G(t)\|u(t)\|_b^2\\ &+ \left(\frac{p}{2} - \frac{\delta}{2}\right)(a_1(t;\, u(t), u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t)).\end{aligned} \tag{3.16}$$

Now, we estimate the right-hand side of (3.16) as follows. By recalling Lemmas 3.1 and 3.3, we easily obtain

$$\left(\frac{p}{2} - 1\right)a_1(t;\, u(t), u(t)) \geqslant \frac{(p-2)\underline{\mu}_1}{2\overline{\mu}_2}\|u(t)\|_b^2, \quad \forall t \in [0, \infty).$$

Thus, it follows from the fifth property of Assumption $[G_1]$ that

$$\left(\frac{p}{2}+\frac{1}{2p}-1\right)G(t)\|u(t)\|_b^2 \leqslant \frac{(p-1)^2}{2p}G_\infty\,\|u(t)\|_b^2 \leqslant \frac{(p-2)\underline{\mu}_1}{2\overline{\mu}_2}\,\|u(t)\|_b^2$$
$$\leqslant \left(\frac{p}{2}-1\right)a_1(t;\,u(t),u(t)).$$

Thus, by choosing $\delta = p, \beta \in \left(0, -\frac{pE(0)}{p-1}\right]$, from (3.12) and (3.16), we obtain

$$M(t) \geqslant \left[\left(1-\frac{p}{2}\right)M^{-\frac{p}{2}}(0)M'(0)t + M^{1-\frac{p}{2}}(0)\right]^{-\frac{2}{p-2}}, \quad \forall t \in [0, T_0]. \tag{3.17}$$

If we choose $\tau \in \left(\frac{2\Psi(0)}{(p-2)\beta}, \infty\right)$ and $T_0 \in \left[\frac{\beta\tau^2}{(p-2)\beta\tau - 2\Psi(0)}, \infty\right)$, then we will have

$$T_* = \frac{M^{1-\frac{p}{2}}(0)}{\left(\frac{p}{2}-1\right)\frac{M'(0)}{M^{\frac{p}{2}}(t)}} = \frac{2M(0)}{(p-2)M'(0)} = \frac{2T_0\Psi(0)+\beta\tau^2}{(p-2)\beta\tau} \in (0, T_0]. \tag{3.18}$$

Therefore, from (3.17) and (3.18), we deduce that $\lim_{t\nearrow T_*}\|u(t)\|_{H^1} = \infty$. This is a contradiction with the fact that the solution is global and it shows that the solution blows up at finite time. This completes the proof of Theorem 3.1. □

Next we state the blowup result when the initial energy is nonnegative and small enough. In this case, we make the following assumptions:

$[K_2]$ $K, K_t \in C(\overline{\Omega}\times[0,\infty))$ such that
- (i) $0 \leqslant K(x,t) \leqslant K_2$ for all $(x,t) \in \overline{\Omega}\times[0,\infty)$ with K_2 being a positive constant;
- (ii) $K_t(x,t) \geqslant 0$ for all $(x,t) \in \overline{\Omega}\times[0,\infty)$.

$[F_2]$ $f \in C^1(\mathbb{R})$ and there exist constants $\overline{d}_2 > 0$, $d_2 > p > 2$, $q_i > p$ for all $i \in \{1, \dots, N\}$ such that
- (i) $uf(u) \geqslant p\int_0^u f(z)\mathrm{d}z = pF(u) \geqslant 0$ for all $u \in \mathbb{R}$;
- (ii) $uf(u) \leqslant d_2F(u) \leqslant \overline{d}_2 d_2\left(|u|^p + \sum_{i=1}^N |u|^{q_i}\right)$ for all $u \in \mathbb{R}$.

First, we put

$$y(t) = \sqrt{a_1(t;\,u(t),u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t)}, \quad \forall t \in [0, T_\infty).$$

and note that if $[G_1]$ holds, then

$$y(t) \geqslant \sqrt{\ell}\,\|u(t)\|_b, \quad \forall t \in [0, T_\infty),$$

where

$$\ell = \frac{\underline{\mu}_1}{\overline{\mu}_2} - G_\infty > \frac{p(p-2)}{(p-1)^2}\frac{\underline{\mu}_1}{\overline{\mu}_2} - G_\infty \geqslant 0.$$

On the other hand, we have

$$\begin{aligned}
E(t) &= \frac{1}{2}a_1(t;\,u(t),u(t)) - \frac{G(t)}{2}\,\|u(t)\|_b^2 + \frac{(g\star u)(t)}{2} - \int_0^1 K(x,t)F(u(x,t))\mathrm{d}x\\
&\geqslant \frac{1}{2}y^2(t) - K_2\overline{d}_2\left(\|u(t)\|_p^p + \sum_{i=1}^N \|u(t)\|_{q_i}^{q_i}\right)\\
&\geqslant \frac{1}{2}y^2(t) - K_2\overline{d}_2\left(S_p^p\ell^{-\frac{p}{2}}y^p(t) + \sum_{i=1}^N S_{q_i}^{q_i}\ell^{-\frac{q_i}{2}}y^{q_i}(t)\right)\\
&= \mathscr{H}(y(t)),
\end{aligned}$$

where $S_\theta = \sup_{v \in H^1 \setminus \{0\}} \frac{\| v \|_\theta}{\| v \|_b}$ for all $\theta \in [1, \infty]$ and $\mathscr{H} : [0, \infty) \longrightarrow \mathbb{R}$ is the function defined by

$$\mathscr{H}(\lambda) = \frac{\lambda^2}{2} - K_2 \bar{d}_2 \left(S_p^p e^{-\frac{p}{2}} \lambda^p + \sum_{i=1}^N S_{q_i}^{q_i} e^{-\frac{q_i}{2}} \lambda^{q_i} \right).$$

Before stating our main results, we give a useful lemma as follows. The proof of this lemma is not difficult, so we omit it.

Lemma 3.4. *Let*

$$\mathscr{H}(\lambda) = \frac{\lambda^2}{2} - K_2 \bar{d}_2 \left(S_p^p e^{-\frac{p}{2}} \lambda^p + \sum_{i=1}^N S_{q_i}^{q_i} e^{-\frac{q_i}{2}} \lambda^{q_i} \right).$$

Then, we have

(i) *The equation $\mathscr{H}'(\lambda) = 0$ has a unique positive solution λ_0 satisfying*

$$1 - K_2 \bar{d}_2 \left(p S_p^p e^{-\frac{p}{2}} \lambda_0^{p-2} + \sum_{i=1}^N q_i S_{q_i}^{q_i} e^{-\frac{q_i}{2}} \lambda_0^{q_i - 2} \right) = 0;$$

(ii) *$\mathscr{H}(0) = 0$ and $\lim_{\lambda \to \infty} \mathscr{H}(\lambda) = -\infty$;*

(iii) *$\mathscr{H}'(\lambda) > 0$ if and only if $\lambda \in (0, \lambda_0)$, and $\mathscr{H}'(\lambda) < 0$ if and only if $\lambda \in (\lambda_0, \infty)$.*

The following lemma will play an essential role in this study, and it is similar to the lemma used first by Vitillaro in [12].

Lemma 3.5. *Suppose that the assumptions $[A_1]$, $[A_2']$, $[A_3']$, $[K_2]$, $[F_2]$, $[G_1]$ hold and*

$$E(0) < \mathscr{H}(\lambda_0) = K_2 \bar{d}_2 \left[\left(\frac{p}{2} - 1 \right) S_p^p e^{-\frac{p}{2}} + \sum_{i=1}^N \left(\frac{q_i}{2} - 1 \right) S_{q_i}^{q_i} e^{-\frac{q_i}{2}} \right].$$

We have

(i) *If $a_1(0; u_0, u_0) < \lambda_0^2$, then there exists $\lambda_1 \in [0, \lambda_0)$ such that*

$$y(t) \leqslant \lambda_1, \quad \forall t \in [0, T_\infty).$$

(ii) *If $a_1(0; u_0, u_0) > \lambda_0^2$ and $E(0) \geqslant 0$, then there exists $\lambda_2 \in [\lambda_0, \infty)$ such that*

$$y(t) \geqslant \lambda_2, \quad \forall t \in [0, T_\infty).$$

Proof of Lemma 3.5. Since $0 \leqslant E(0) < \mathscr{H}(0)$, there exist two constants $\lambda_1 \in [0, \lambda_0)$ and $\lambda_2 \in (\lambda_0, \infty)$ such that

$$E(0) = \mathscr{H}(\lambda_1) = \mathscr{H}(\lambda_2).$$

First, we assume that $a_1(0; u_0, u_0) < \lambda_0^2$. We have

$$\mathscr{H}(\lambda_1) = E(0) \geqslant \mathscr{H}(y(0)) = \mathscr{H}(\sqrt{a_1(0; u_0, u_0)}). \tag{3.19}$$

By Lemma 3.5, (3.19) leads to $a_1(0; u_0, u_0) \leqslant \lambda_1^2$. We claim that $y(t) \leqslant \lambda_1$ for all $t \in [0, T_\infty)$. Suppose, by contradiction, there exists $t_0 \in (0, T_\infty)$ such that $y(t_0) > \lambda_1$. By the continuity of y, without loss of generality, we may assume that $y(t_0) \in (\lambda_1, \lambda_0)$. By Lemmas 3.1 and 3.4, we obtain

$$E(t_0) \geqslant \mathscr{H}(y(t_0)) > \mathscr{H}(\lambda_1) = E(0),$$

which is a contradiction, because of $E(t_0) \leqslant E(0)$ for all $t \in [0, T_\infty)$. Using the same argument as above we obtain the second statement. This completes the proof of Lemma 3.5. □

Theorem 3.2. *Assume that the assumptions* $[A_1]$, $[A_2']$, $[A_3']$, $[K_2]$, $[F_2]$, *and* $[G_1]$ *hold. For any initial conditions* $u_0 \in H^1$ *such that* $a_1(0; u_0, u_0) > \lambda_0^2$ *and*

$$0 \leqslant E(0) < \min\left\{\mathscr{H}(0), \frac{(p-\delta_*)\lambda_2^2}{2p}\right\},$$

where $\delta_* \in (2, p)$ *is a unique solution of equation* $\frac{\mu_1}{\overline{\mu}_2}\frac{\delta(\delta-2)}{(\delta-1)^2} = G_\infty$, *then the weak solution of problem* (1.1)–(1.3) *blows up at finite time.*

Proof of Theorem 3.2. We will assume that the weak solution of problem (1.1)–(1.3) exists in the whole interval $[0, \infty)$. By Lemma 3.5, we have

$$\left(\frac{p}{2} - \frac{\delta}{2}\right)(a_1(t; u(t), u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t)) \geqslant \left(\frac{p}{2} - \frac{\delta}{2}\right)\lambda_2^2, \quad \forall t \in [0, T_\infty).$$

With the same notation and calculation in the proof of Theorem 3.1, estimate (3.16) holds. Since $\varphi(x) = \frac{\mu_1}{\overline{\mu}_2}\frac{x(x-2)}{(x-1)^2}$ is continuous and strictly increasing in $[2, p]$, so that

$$0 = \varphi(2) < G_\infty < \varphi(p) = \frac{\mu_1}{\overline{\mu}_2}\frac{p(p-2)}{(p-1)^2},$$

it follows that there exists a unique constant $\delta_* \in (2, p)$ such that $\varphi(\delta_*) = G_\infty$.

Choose $\delta = \delta_*$ and $\beta \in \left(0, \frac{(p-\delta_*)\lambda_2^2 - 2pE(0)}{2(p-1)}\right]$, we deduce from (3.16) that

$$M''(t)M(t) - \frac{p}{2}[M'(t)]^2 \geqslant 0, \quad \forall t \in [0, \infty).$$

This is also a contradiction, hence the solution blows up at finite time. Theorem 3.2 is proved completely. □

4 Global solution and decay estimates

First, we make the following assumptions:

$[G_2]$ $g, \sigma \in C^1([0, \infty))$ with the following properties:

(i) $g(t) > 0$ for all $t \in [0, \infty)$;

(ii) $g'(t) \leqslant 0$ for all $t \in [0, \infty)$;

(iii) $\ell > 0$.

According to (3.1), we have

$$E(t) = \left(\frac{1}{2} - \frac{1}{p}\right)(a_1(t; u(t), u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t)) + \frac{I(t)}{p}, \tag{4.1}$$

where

$$I(t) = a_1(t; u(t), u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t) - p\int_0^1 K(x, t)F(u(x, t))\mathrm{d}x. \tag{4.2}$$

We are now in a position to prove the global existence of solutions starting with suitable initial data.

Theorem 4.1. *Assume that* $[A_1]$, $[A_2']$, $[A_3']$, $[K_2]$, $[F_2]$, *and* $[G_2]$ *hold. For any initial condition* $u_0 \in H^1$ *such that* $a_1(0; u_0, u_0) < \lambda_0^2$ *and* $E(0) < \mathscr{H}(\lambda_0)$, *the weak solution of problem* (1.1)–(1.3) *is global.*

Proof of Theorem 4.1. First, we have

$$\begin{aligned} I(t) &= a_1(t;\, u(t), u(t)) - G(t)\|u(t)\|_b^2 + (g\star u)(t) - p\int_0^1 K(x,t)F(u(x,t))\mathrm{d}x \\ &\geqslant y^2(t) - pK_2\overline{d}_2\left(S_p^p \ell^{-\frac{p}{2}} y^p(t) + \sum_{i=1}^N S_{q_i}^{q_i} \ell^{-\frac{q_i}{2}} y^{q_i}(t)\right) \\ &\geqslant \left[1 - pK_2\overline{d}_2\left(S_p^p \ell^{-\frac{p}{2}} \lambda_1^{p-2} + \sum_{i=1}^N S_{q_i}^{q_i} \ell^{-\frac{q_i}{2}} \lambda_1^{q_i-2}\right)\right] y^2(t) \\ &\geqslant \left[1 - pK_2\overline{d}_2\left(S_p^p \ell^{-\frac{p}{2}} \lambda_1^{p-2} + \sum_{i=1}^N S_{q_i}^{q_i} \ell^{-\frac{q_i}{2}} \lambda_1^{q_i-2}\right)\right] \ell\, \|u(t)\|_b^2. \end{aligned} \tag{4.3}$$

We note that $\left[1 - pK_2\overline{d}_2\left(S_p^p \ell^{-\frac{p}{2}} \lambda_1^{p-2} + \sum_{i=1}^N S_{q_i}^{q_i} \ell^{-\frac{q_i}{2}} \lambda_1^{q_i-2}\right)\right]\ell > 0$. It follows from (4.1) and (4.3) that

$$\frac{2p}{p-2}E(0) \geqslant \frac{2p}{p-2}E(t) \geqslant y^2(t) \geqslant \ell\,\|u(t)\|_b^2, \quad \forall t \in [0, T_\infty). \tag{4.4}$$

By the last statement in Theorem 2.1, the solution has to be globally defined. The proof of Theorem 4.1 is now complete. □

We provide the decay results of solution of problem (1.1)–(1.3) with conditions
[G2′] $g, \sigma \in C^1([0,\infty))$ with the following properties:

(i) $g(t) > 0$;

(ii) $g'(t) \leqslant 0$ for all $t \in [0,\infty)$;

(iii) $\ell > 0$;

(iv) there exists a function $\xi : [0,\infty) \longrightarrow (0,\infty)$ such that

$$g'(t) \leqslant -\xi(t)g(t), \quad \xi'(t) \leqslant 0, \quad \forall t \in [0,\infty), \int_0^\infty \xi(t)\mathrm{d}t = \infty.$$

Theorem 4.2. *Assume that* $[A_1]$, $[A_2']$, $[A_3']$, $[K_2]$, $[F_2]$, *and* $[G_2']$ *hold. For any initial condition* $u_0 \in H^1$ *such that* $a_1(0;\, u_0, u_0) < \lambda_0^2$, $E(0) < \mathscr{H}(\lambda_0)$, *and*

$$1 - \frac{p(d_2-2)\overline{d}_2K_2}{p-2}\left[S_p^p \ell^{-\frac{p}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2}{2}} + \sum_{i=1}^N S_{q_i}^{q_i} \ell^{-\frac{q_i}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{q_i-2}{2}}\right] > 0, \tag{4.5}$$

there exists $\gamma_* > 0$ *such that*

$$E(t) \lesssim \exp\left(-\gamma_*\int_0^t \xi(s)\mathrm{d}s\right), \quad \forall s \in [0,\infty). \tag{4.6}$$

Proof of Theorem 4.2. The global existence result is obtained from Theorem 4.1 directly. We only need to prove the decay estimate (4.6). Multiplying equation (1.1) by $\xi(t)u$ and then integrating it over the spatial-time cylinder $\Omega \times (t_1, t_2)$, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\xi(t)\langle u'(t), u(t)\rangle dt + \int_{t_1}^{t_2}\xi(t)a_1(t; u(t), u(t))dt\\ = \int_{t_1}^{t_2}\xi(t)\int_0^t g(t-s)b(u(s), u(t))dsdt + \int_{t_1}^{t_2}\xi(t)\langle K(t)f(u(t)), u(t)\rangle dt.\end{aligned} \tag{4.7}$$

We have

$$\begin{aligned}\int_{t_1}^{t_2}\xi(t)\int_0^t g(t-s)b(u(s), u(t))dsdt\\ = \int_{t_1}^{t_2}\xi(t)\int_0^t g(t-s)b(u(s)-u(t), u(t))dsdt + \int_{t_1}^{t_2}\xi(t)G(t)\|u(t)\|_b^2 dt.\end{aligned} \tag{4.8}$$

Combining (4.7), (4.8), and (3.1), we obtain

$$\begin{aligned}2\int_{t_1}^{t_2}\xi(t)E(t)dt = -\int_{t_1}^{t_2}\xi(t)\langle u'(t), u(t)\rangle dt + \int_{t_1}^{t_2}\xi(t)\int_0^t g(t-s)b(u(s)-u(t), u(t))dsdt\\ + \int_{t_1}^{t_2}\xi(t)(g\star u)(t)dt + \int_{t_1}^{t_2}\xi(t)\langle K(t)f(u(t)), u(t)\rangle dt - 2\int_{t_1}^{t_2}\xi(t)\int_0^1 K(x,t)F(u(x,t))dxdt.\end{aligned} \tag{4.9}$$

Now, we estimate the terms of right-hand side of (4.9) as follows:

Estimate for $I_1 = -\int_{t_1}^{t_2}\xi(t)\langle u'(t), u(t)\rangle dt$.

By utilizing Cauchy-Schwarz inequality and (3.4), (4.4), for any positive constant $\varepsilon > 0$, we obtain

$$\begin{aligned}I_1 = -\int_{t_1}^{t_2}\xi(t)\langle u'(t), u(t)\rangle dt \le \varepsilon\int_{t_1}^{t_2}\xi(t)E(t)dt - C(\varepsilon)\int_{t_1}^{t_2}\xi(t)E'(t)dt\\ \le \varepsilon\int_{t_1}^{t_2}\xi(t)E(t)dt + C(\varepsilon)E(t_1).\end{aligned} \tag{4.10}$$

Estimate for $I_2 = \int_{t_1}^{t_2}\xi(t)\int_0^t g(t-s)b(u(s)-u(t), u(t))dsdt$.

First, by utilizing Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_0^t g(t-s)b(u(s)-u(t), u(t))ds &\le \int_0^t g(t-s)\|u(s)-u(t)\|_b ds\, \|u(t)\|_b\\ &\le \varepsilon\, \|u(t)\|_b^2 + C(\varepsilon)\left(\int_0^t g(t-s)\|u(s)-u(t)\|_b ds\right)^2\\ &\le \varepsilon E(t) + C(\varepsilon)\int_0^t g(t-s)ds(g\star u)(t)\\ &\le \varepsilon E(t) + C(\varepsilon)(g\star u)(t).\end{aligned}$$

Hence, we obtain

$$I_2 \le \varepsilon\int_{t_1}^{t_2}\xi(t)E(t)dt + C(\varepsilon)\int_{t_1}^{t_2}\xi(t)(g\star u)(t)dt \le \varepsilon\int_{t_1}^{t_2}E(t)dt + C(\varepsilon)E(t_1). \tag{4.11}$$

Estimate for $I_3 = \int_{t_1}^{t_2}\xi(t)(g\star u)(t)\mathrm{d}t$.

By using (3.6), we have

$$I_3 = \int_{t_1}^{t_2}\xi(t)(g\star u)(t)\mathrm{d}t \lesssim E(t_1). \tag{4.12}$$

Estimate for $I_4 = \int_{t_1}^{t_2}\xi(t)\langle K(t)f(u(t)), u(t)\rangle\mathrm{d}t - 2\int_{t_1}^{t_2}\xi(t)\int_0^1 K(x,t)F(u(x,t))\mathrm{d}x\mathrm{d}t$

First, by $[K_2]$, we have

$$\begin{aligned}
\langle K(t)f(u(t)), u(t)\rangle - 2\int_0^1 K(x,t)F(u(x,t))\mathrm{d}x &\leqslant (d_2-2)\int_0^1 K(x,t)F(u(x,t))\mathrm{d}x \\
&\leqslant (d_2-2)\bar{d}_2K_2\left(\|u(t)\|_p^p + \sum_{i=1}^N \|u(t)\|_{q_i}^{q_i}\right) \leqslant (d_2-2)\bar{d}_2K_2\left(S_p^p\,\|u(t)\|_b^p + \sum_{i=1}^N S_{q_i}^{q_i}\,\|u(t)\|_b^{q_i}\right) \\
&= (d_2-2)\bar{d}_2K_2\left(S_p^p\,\|u(t)\|_b^{p-2} + \sum_{i=1}^N S_{q_i}^{q_i-2}\,\|u(t)\|_b^{q_i-2}\right)\|u(t)\|_b^2 \\
&\leqslant \frac{2p(d_2-2)\bar{d}_2K_2}{p-2}\left[S_p^p\ell^{-\frac{p}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2}{2}} + \sum_{i=1}^N S_{q_i}^{q_i}\ell^{-\frac{q_i}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{q_i-2}{2}}\right]E(t).
\end{aligned}$$

Consequently, we obtain

$$I_4 \leqslant \frac{2p(d_2-2)\bar{d}_2K_2}{p-2}\left[S_p^p\ell^{-\frac{p}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2}{2}} + \sum_{i=1}^N S_{q_i}^{q_i}\ell^{-\frac{q_i}{2}}\left(\frac{2pE(0)}{p-2}\right)^{\frac{q_i-2}{2}}\right]\int_{t_1}^{t_2}\xi(t)E(t)\mathrm{d}t. \tag{4.13}$$

Choose $\varepsilon > 0$ sufficiently small, from (4.9)–(4.13), we obtain

$$\int_{t_1}^{t_2}\xi(t)E(t)\mathrm{d}t \lesssim E(t_1), \quad \forall t_1, t_2 \in [0,\infty),\ t_1 < t_2.$$

By letting t_2 go to infinity in the left-hand side in the aforementioned inequality, one can easily deduce that

$$\int_{t_1}^{\infty}\xi(t)E(t)\mathrm{d}t \lesssim E(t_1), \quad \forall t_1[0,\infty).$$

Thus, by adopting Lemma 2.3, (4.6) holds. Theorem 4.2 is proved. □

Remark 4.1. We provide examples to illustrate our results. Based on the general assumption 4, we derive various decay rates, where exponential and polynomial rates are specific instances. Here, we consider examples of the function g with σ is a positive constant function.

(1) $g(t) = \alpha\exp(-\beta t)$, where $\alpha, \beta > 0$, then $\xi(t) = \beta$. Consequently, from (4.6), we obtain the following exponential decay

$$E(t) \lesssim \exp(-\gamma_* t),$$

where $\gamma_* > 0$.

(2) $g(t) = \alpha\exp(-\beta(1+t)^\gamma)$, where $\alpha, \beta > 0, \gamma \in (0,1]$, then $\xi(t) = \beta\gamma(1+t)^{\gamma-1}$. Consequently, from (4.6), we obtain the following exponential decay:

$$E(t) \lesssim \exp(-C_*(1+t)^\gamma),$$

where $C_* > 0$.

(3) $g(t) = \frac{\alpha}{(1+t)^{\gamma}}$, where $\alpha, \gamma > 0$, then $\xi(t) = \frac{\gamma}{t+1}$. Consequently, from (4.6), we obtain the following polynomial decay:

$$E(t) \lesssim \frac{1}{(1+t)^{\kappa}},$$

where $\kappa > 0$.

(4) $g(t) = \alpha \exp(-\beta \ln^{\gamma}(1+t))$, where $\alpha, \beta > 0, \gamma \in (0,1]$, then $\xi(t) = \frac{\beta\gamma \ln^{\gamma-1}(1+t)}{1+t}$. Consequently, from (4.6), we obtain the following decay estimate:

$$E(t) \lesssim \exp(-C_* \ln^{\gamma}(1+t)),$$

where $C_* > 0$.

(5) $g(t) = \frac{a}{(2+t)^{\gamma} \ln^{\beta}(2+t)}$, where $a > 0$ and $\gamma > 1$ and $\beta \in \mathbb{R}$ or $\gamma = 1$ and $\beta > 1$, then $\xi(t) = \frac{\gamma \ln(2+t) + \beta}{(2+t)\ln(2+t)}$. Consequently, from (4.6), we obtain the following decay estimate:

$$E(t) \leqslant [(2+t)^{\gamma} \ln(2+t)^{\beta}]^{-C_*}.$$

5 Conclusion

In this study, we investigate both finite-time blowup solutions and globally existing solutions of a class of nonlinear viscoelastic heat equations. Our primary results concentrate on the finite-time blowup and the long-term behavior of the global solution. Specifically, we demonstrate that, under certain conditions, the solution can still blow up in finite time even with nonnegative initial energy. Furthermore, for solutions that exist globally over time, we provide decay estimates for the weak solution. Notably, these decay estimates are heavily influenced by the behavior of the relaxation function.

Acknowledgements: Ngo Tran Vu is sincerely grateful to Do Thi Kim Phung for her ongoing support and encouragement throughout the preparation of this manuscript. Furthermore, the current version of this article has been significantly enriched by the insightful and invaluable feedback of anonymous referees, who carefully reviewed an initial draft and offered critical comments and perspectives.

Funding information: This research was funded by the University of Economics Ho Chi Minh City, Vietnam (UEH).

Author contributions: Each of the authors contributed to each part of this study equally. All authors read and proved the final vision of the manuscript.

Conflict of interest: The authors declare no potential conflict of interests.

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