Research Article

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Quantitative estimates for perturbed sampling Kantorovich operators in Orlicz spaces

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Abstract: In the present work, we establish a quantitative estimate for the perturbed sampling Kantorovich operators in Orlicz spaces, in terms of the modulus of smoothness, defined by means of its modular functional. From the obtained result, we also deduce the qualitative order of approximation, by considering functions in suitable Lipschitz classes. This allows us to apply the above results in certain Orlicz spaces of particular interest, such as the interpolation spaces, the exponential spaces and the L^p-spaces, $1 \leq p < +\infty$. In particular, in the latter case, we also provide an estimate established using a direct proof based on certain properties of the L^p-modulus of smoothness, which are not valid in the general case of Orlicz spaces. The possibility of using a direct approach allows us to improve the estimate that can be deduced as a consequence of the one achieved in Orlicz spaces. In the final part of the article, we furnish some estimates and the corresponding qualitative order of approximation in the space of uniformly continuous and bounded functions.

Keywords: perturbed sampling Kantorovich operators, quantitative estimates, order of approximation, modulus of smoothness, Lipschitz classes

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1 Introduction

Bardaro et al. [1] first introduced the sampling Kantorovich (SK) operators as a development of the generalized sampling operators (e.g., [2,3]), and later, a wide research has been conducted by some authors with the aim to create some generalizations of these operators, making them more suitable to different fields of applications (e.g., [4–8]). It is important to note that one of the main advantages in considering SK-type operators resides in the possibility to approximate/reconstruct signals which are not necessarily continuous. It is worth noting that all these sampling operators are different generalizations of the celebrated Wittaker-Kotelnikov-Shannon sampling theorem (e.g., [9]).

Currently, the study of these operators and their variants (e.g., [10–15]) continues to be an active area of research in the field of signal processing, since these operators are largely useful for the applications in image reconstruction and enhancement, as for instance, in medicine (e.g., [16,17]).

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Over the past few years, SK operators have been studied from many theoretical points of view: the convergence has been examined in [18] in the general context of modular spaces, while other approximation results have been investigated in [19,20].

The order of approximation for these operators has been widely examined in terms of the modulus of smoothness. Several works can be recalled in this context, as for instance, [5,21–24] for SK operators, [25,26] for Durrmeyer-type operators and [27] for Steklov sampling operators.

In real-world scenarios, signals are often affected by various types of noise. To adapt these operators to such situations, a version of the SK operators perturbed by multiplicative noise has been introduced in [28]. Convergence results for these perturbed operators have been established in Orlicz spaces in [28] and in modular spaces in [29].

However, the rate of convergence for these operators in Orlicz spaces remains an open problem and this is the main focus of the present work.

In this study, we establish quantitative estimates for the order of approximation of the above perturbed operators; to do this, we exploit the definition of the modulus of smoothness, defined by means of the modular which generates the space. This allows us to apply the above results in certain Orlicz spaces of particular interest, such as the interpolation spaces, the exponential spaces and the L^p-spaces, $1 \leq p < +\infty$. In particular, in the latter case, we also provide an estimate established using a direct proof based on certain properties of the L^p-modulus of smoothness, which are not valid in the general case of Orlicz spaces. Finally, we briefly mention some examples of kernels for which the theory established in this work holds.

2 Notations

From now on, we denote with $M(\mathbb{R})$ the space of all measurable functions and with $C(\mathbb{R})$ the space of all uniformly continuous and bounded functions, endowed with $\|\cdot\|_{C(\mathbb{R})}$, defined as $\|f\|_{C(\mathbb{R})} := \sup_{x\in\mathbb{R}}|f(x)|, f \in C(\mathbb{R})$.

We recall some basic useful notions as follows.

Definition 2.1. A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a *φ-function* if it satisfies the following conditions:
(φ1) φ is a non-decreasing and continuous function;
(φ2) $\varphi(0) = 0$, $\varphi(u) > 0$ if $u > 0$ and $\lim_{u\to+\infty}\varphi(u) = +\infty$.

Now, we consider the functional I^φ associated with a given φ-function φ and defined as follows:

$$I^\varphi[f] := \int_{\mathbb{R}} \varphi(|f(x)|)\mathrm{d}x,$$

for every $f \in M(\mathbb{R})$. The functional I^φ is modular and it can be used in order to define the Orlicz space. Indeed, the *Orlicz space* generated by φ is

$$L^\varphi(\mathbb{R}) := \{f \in M(\mathbb{R}) : I^\varphi[\lambda f] < +\infty, \text{ for some } \lambda > 0\}.$$

Now, we recall the notion of the *modular convergence* in Orlicz spaces [30–32].

Definition 2.2. A net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is *modularly convergent* to $f \in L^\varphi(\mathbb{R})$, if there exists $\lambda > 0$ such that

$$\lim_{w\to+\infty} I^\varphi[\lambda(f_w - f)] = \lim_{w\to+\infty} \int_{\mathbb{R}} \varphi(\lambda|f_w(x) - f(x)|)\mathrm{d}x = 0.$$

Further, in order to obtain some estimates, we introduce the following definition in Orlicz spaces [30].

Definition 2.3. For any fixed $f \in L^{\varphi}(\mathbb{R})$, we denote by

$$\omega(f,\delta)_{\varphi} := \sup_{|t|\le\delta} I^{\varphi}[f(\cdot+t)-f(\cdot)], \quad \delta>0,$$

the *modulus of smoothness* in the Orlicz space $L^{\varphi}(\mathbb{R})$ and with respect to the modular I^{φ}.

It is well known that, since $f \in L^{\varphi}(\mathbb{R})$, there exists $\lambda > 0$ such that $\omega(\lambda f,\delta)_{\varphi} \to 0$, as $\delta \to 0^{+}$. For further details regarding Orlicz spaces refer [30,33–35].

In order to define the operators considered in this work for any function $\chi : \mathbb{R} \to \mathbb{R}$, we can define the following useful tools. The discrete and the continuous absolute moments of order r ($r \ge 0$) of χ are defined in this way, respectively.

$$m_r(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^{r}$$

and

$$\widetilde{m}_r(\chi) := \int_{\mathbb{R}} |\chi(t)||t|^{r}\,\mathrm{d}t.$$

It is well known that if χ is bounded on $\mathbb{R}$ and χ is $O(|u|^{-r-1-\varepsilon})$, $|u| \to +\infty$, $\varepsilon > 0$, we obtain that $m_{\beta}(\chi) < +\infty$ and $\widetilde{m}_{\beta}(\chi) < +\infty$, for every $0 \le \beta \le r$.

From now on, the function χ will be called a *kernel* if it satisfies the following assumptions:

(χ1) $\chi \in L^{1}(\mathbb{R})$ is bounded in a neighbourhood of the origin;

(χ2) for some $\mu > 0$

$$\sum_{k\in\mathbb{Z}} \chi(wx-k) - 1 = O(w^{-\mu}), \quad \text{as } w \to +\infty$$

uniformly with respect to $x \in \mathbb{R}$;

(χ3) there exists $\beta > 0$ such that $m_{\beta}(\chi) < +\infty$.

From the above properties, for any given kernel χ, it is possible to prove the following condition [1]:

$$m_{\nu}(\chi) < +\infty, \quad 0 \le \nu < \beta. \tag{2.1}$$

Remark 2.4. We note that rather than assuming condition (χ2), we can directly suppose as true the following (stronger) property:

$$\sum_{k\in\mathbb{Z}} \chi(u-k) = 1, \quad \forall u \in \mathbb{R}. \tag{2.2}$$

Indeed, if condition (2.2) is satisfied, (χ2) is trivially fulfilled for every $\mu > 0$.

Now, we recall the definition of the family of generalized SK operators perturbed by multiplicative noise, $(K_w^{\chi,\mathcal{G}})_{w>0}$ [28], which will be considered in the following, as:

$$(K_w^{\chi,\mathcal{G}}f)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k) \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u)\,\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,\mathrm{d}u},$$

where $\mathcal{G} := (\mathcal{G}_w)_{w>0}$ is a family of noise sequences, with $\mathcal{G}_w = (g_{k,w})_{k\in\mathbb{Z}}$, $g_{k,w} : \mathbb{R} \to \mathbb{R}^{+}$ are locally integrable noise functions with $\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,\mathrm{d}u \ne 0$, for every $k \in \mathbb{Z}$ and $w > 0$, and $f : \mathbb{R} \to \mathbb{R}$ such that $g_{k,w} f$ are locally integrable and the above series is convergent for every $x \in \mathbb{R}$.

Remark 2.5. As we have already noted, there are many applications of the above theory in image reconstruction and signal processing across various fields, such as medicine and engineering (e.g., [16]). A real signal can be affected by multiplicative noise, such as that encountered in synthetic aperture radar remote sensing systems, where a “Speckle”-type noise is detected. The presence of the noise does not hinder the reconstruction process and the perturbed operator is capable of reconstructing the original signal, eliminating the effects provided by noise sources.

3 Quantitative estimate in Orlicz spaces

In this section, we prove a quantitative estimate for the perturbed SK operators, by using the modulus of smoothness of $L^\varphi(\mathbb{R})$.

From now on, we require that the φ-function is convex and that the following hypothesis on the boundedness of the noise functions $g_{k,w}$ is true:

There exist positive constants δ, σ such that

$$0 < \delta \le g_{k,w}(u) \le \sigma, \quad \text{for every } u \in \mathbb{R}, k \in \mathbb{Z}, w > 0.$$

In order to ensure that the operators are well-defined in $L^\varphi(\mathbb{R})$, we recall the following result.

Theorem 3.1. [28] *For every $f \in L^\varphi(\mathbb{R})$, there holds*:

$$I^\varphi[\lambda K_w^{\chi,\mathcal{G}} f] \le \frac{\sigma}{\delta}\frac{\|\chi\|_1}{m_0(\chi)} I^\varphi[\lambda m_0(\chi) f],$$

for every $w > 0$. In particular, if $f \in L^\varphi(\mathbb{R})$, it turns out that $K_w^{\chi,\mathcal{G}} f \in L^\varphi(\mathbb{R})$.

Now, we can establish the following estimate in Orlicz spaces.

Theorem 3.2. *Assume that for any fixed $0 < \alpha < 1$, the integral*

$$w \int_{|y|>1/w^\alpha} |\chi(wy)| \mathrm{d}y \le K w^{-\gamma}, \quad as\ w \to +\infty, \tag{3.1}$$

for suitable positive constants K, γ depending on α and χ. Then, for $f \in L^\varphi(\mathbb{R})$, there exist $\lambda, \lambda_0 > 0$ such that

$$\begin{aligned} I^\varphi[\lambda(K_w^{\chi,\mathcal{G}} f - f)] \le \frac{1}{3}\frac{\sigma}{\delta}\Bigg\{ & \frac{m_0(\tau)}{m_0(\chi)} \|\chi\|_1 \omega\left(3\lambda m_0(\chi) f, \frac{1}{w^\alpha}\right)_\varphi \\ & + \frac{m_0(\tau)}{m_0(\chi)} K I^\varphi[\lambda_0 f] w^{-\gamma} + \omega\left(3\lambda m_0(\chi) f, \frac{1}{w}\right)_\varphi \Bigg\} + \frac{1}{3} I^\varphi[\lambda_0 f] w^{-\mu}, \end{aligned}$$

for every sufficiently large $w > 0$, where τ is the characteristic function of the set $[0,1]$ and $\mu > 0$ is the constant of the condition ($\chi 2$). Choosing $\lambda > 0$ sufficiently small, the above inequality implies the modular convergence of the perturbed SK operators $K_w^{\chi,\mathcal{G}} f$ to f.

Proof. Let $\lambda_0 > 0$ such that $I^\varphi[\lambda_0 f] < +\infty$ and we choose $\lambda > 0$ such that

$$\lambda \le \min\left\{\frac{\lambda_0}{6 m_0(\chi)}, \frac{\lambda_0}{3M}\right\}.$$

Now, by using the properties of φ, we can write the following:

$$I^\varphi[\lambda(K_w^{\chi,\mathcal{G}} f - f)] = \int_{\mathbb{R}} \varphi\left(\lambda \left| \sum_{k\in\mathbb{Z}} \chi(wx-k) \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)[f(u) - f(u+x-k/w)]\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right.\right.$$

$$+\sum_{k\in\mathbb{Z}}\chi(wx-k)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)f(u+x-k/w)\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}-f(x)\sum_{k\in\mathbb{Z}}\chi(wx-k)+f(x)\sum_{k\in\mathbb{Z}}\chi(wx-k)-f(x)\Bigg|\Bigg)\mathrm{d}x$$

$$\leq \frac{1}{3}\int_{\mathbb{R}}\varphi\left(3\lambda\left|\sum_{k\in\mathbb{Z}}\chi(wx-k)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)[f(u)-f(u+x-k/w)]\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right|\right)\mathrm{d}x$$

$$+\frac{1}{3}\int_{\mathbb{R}}\varphi\left(3\lambda\left|\sum_{k\in\mathbb{Z}}\chi(wx-k)\left[\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)f(u+x-k/w)\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}-f(x)\right]\right|\right)\mathrm{d}x$$

$$+\frac{1}{3}\int_{\mathbb{R}}\varphi\left(3\lambda\left|f(x)\left[\sum_{k\in\mathbb{Z}}\chi(wx-k)-1\right]\right|\right)\mathrm{d}x$$

$$=: I_1+I_2+I_3,\ w>0.$$

Now, we estimate I_1. By using the Jensen inequality twice, the condition on the noise functions $g_{k,w}$ and the Fubini-Tonelli theorem, we have

$$3I_1\leq\int_{\mathbb{R}}\varphi\left(3\lambda\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(u+x-k/w)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right)\mathrm{d}x$$

$$\leq\frac{1}{m_0(\chi)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\varphi\left(3\lambda m_0(\chi)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(u+x-k/w)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right)\mathrm{d}x$$

$$\leq\frac{1}{m_0(\chi)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\varphi(3\lambda m_0(\chi)|f(u)-f(u+x-k/w)|)\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\mathrm{d}x$$

$$\leq\frac{\sigma w}{\delta m_0(\chi)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\left[\int_{k/w}^{(k+1)/w}\varphi(3\lambda m_0(\chi)|f(u)-f(u+x-k/w)|)\mathrm{d}u\right]\mathrm{d}x$$

$$=\frac{\sigma w}{\delta m_0(\chi)}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\chi(wx-k)|\left[\int_{k/w}^{(k+1)/w}\varphi(3\lambda m_0(\chi)|f(u)-f(u+x-k/w)|)\mathrm{d}u\right]\mathrm{d}x.$$

Now, denoting by $\tau(u)$ the characteristic function of the set $[0,1]$, with the change in variables $y=x-k/w$ and the Fubini-Tonelli theorem, we obtain

$$3I_1\leq\frac{\sigma w}{\delta m_0(\chi)}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\chi(wy)|\left[\int_{\mathbb{R}}\tau(wu-k)\varphi(3\lambda m_0(\chi)|f(u)-f(u+y)|)\mathrm{d}u\right]\mathrm{d}y$$

$$=\frac{\sigma w}{\delta m_0(\chi)}\int_{\mathbb{R}}|\chi(wy)|\int_{\mathbb{R}}\left[\sum_{k\in\mathbb{Z}}\tau(wu-k)\right]\varphi(3\lambda m_0(\chi)|f(u)-f(u+y)|)\mathrm{d}u\mathrm{d}y$$

$$\leq\frac{\sigma w}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\int_{\mathbb{R}}|\chi(wy)|I^{\varphi}[3\lambda m_0(\chi)(f(\cdot)-f(\cdot+y))]\mathrm{d}y=:J,\quad w>0.$$

Let now $0 < \alpha < 1$ be fixed. We split the integral J as follows:

$$J = \frac{\sigma w}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\left\{\int\limits_{|y|\le 1/w^\alpha} + \int\limits_{|y|>1/w^\alpha}\right\}|\chi(wy)|I^\varphi[3\lambda m_0(\chi)(f(\cdot) - f(\cdot+y))]\mathrm{d}y =: J_1 + J_2.$$

For J_1, we obtain

$$\begin{aligned} J_1 &\le \frac{\sigma w}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\int\limits_{|y|\le 1/w^\alpha}|\chi(wy)|\omega\left(3\lambda m_0(\chi)f, \frac{1}{w^\alpha}\right)_\varphi \mathrm{d}y \\ &= \frac{\sigma}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\omega\left(3\lambda m_0(\chi)f, \frac{1}{w^\alpha}\right)_\varphi\int\limits_{|u|\le w^{1-\alpha}}|\chi(u)|\mathrm{d}u \\ &\le \frac{\sigma}{\delta}\|\chi\|_1\frac{m_0(\tau)}{m_0(\chi)}\omega\left(3\lambda m_0(\chi)f, \frac{1}{w^\alpha}\right)_\varphi, \quad w > 0. \end{aligned}$$

Regarding J_2, we have

$$\begin{aligned} J_2 &= \frac{\sigma w}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\int\limits_{|y|>1/w^\alpha}|\chi(wy)|I^\varphi[3\lambda m_0(\chi)(f(\cdot) - f(\cdot+y))]\mathrm{d}y \\ &\le \frac{\sigma w}{\delta}\frac{m_0(\tau)}{m_0(\chi)}\int\limits_{|y|>1/w^\alpha}|\chi(wy)|\frac{1}{2}\{I^\varphi[6\lambda m_0(\chi)f] + I^\varphi[6\lambda m_0(\chi)f(\cdot+y)]\}\mathrm{d}y, \end{aligned}$$

for $w > 0$. It is easy to prove that

$$I^\varphi[6\lambda m_0(\chi)f(\cdot+y)] = I^\varphi[6\lambda m_0(\chi)f],$$

for every $y \in \mathbb{R}$. Now, by taking into account assumption (3.1), we can obtain

$$\begin{aligned} J_2 &\le \frac{\sigma}{\delta}\frac{m_0(\tau)}{m_0(\chi)}I^\varphi[6\lambda m_0(\chi)f]w\int\limits_{|y|>1/w^\alpha}|\chi(wy)|\mathrm{d}y \\ &\le \frac{\sigma}{\delta}\frac{m_0(\tau)}{m_0(\chi)}I^\varphi[6\lambda m_0(\chi)f]Kw^{-\gamma} \le \frac{\sigma}{\delta}\frac{m_0(\tau)}{m_0(\chi)}I^\varphi[\lambda_0 f]Kw^{-\gamma}, \end{aligned}$$

for sufficiently large $w > 0$.

Now, we can estimate I_2. We use Jensen's inequality twice, the upper and lower boundedness of the noise functions, the change in variables $y = u - k/w$ and Fubini-Tonelli theorem in order to have

$$\begin{aligned} 3I_2 &\le \int\limits_{\mathbb{R}}\varphi\left(3\lambda\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u+x-k/w) - f(x)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right)\mathrm{d}x \\ &\le \frac{1}{m_0(\chi)}\int\limits_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\varphi\left(3\lambda m_0(\chi)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u+x-k/w) - f(x)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right)\mathrm{d}x \\ &\le \frac{1}{m_0(\chi)}\int\limits_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\varphi(3\lambda m_0(\chi)|f(u+x-k/w) - f(x)|)\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\mathrm{d}x \\ &\le \frac{1}{m_0(\chi)}\int\limits_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\frac{\sigma w}{\delta}\int\limits_{k/w}^{(k+1)/w}\varphi(3\lambda m_0(\chi)|f(u+x-k/w) - f(x)|)\mathrm{d}u\mathrm{d}x \\ &\le \frac{1}{m_0(\chi)}\int\limits_{\mathbb{R}}\left[\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\right]\frac{\sigma w}{\delta}\int\limits_0^{1/w}\varphi(3\lambda m_0(\chi)|f(x+y) - f(x)|)\mathrm{d}y\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sigma}{\delta} w \int_{\mathbb{R}} \int_0^{1/w} \varphi(3\lambda m_0(\chi)|f(x+y)-f(x)|)\mathrm{d}y\mathrm{d}x \\
&= \frac{\sigma}{\delta} w \int_0^{1/w} \int_{\mathbb{R}} \varphi(3\lambda m_0(\chi)|f(x+y)-f(x)|)\mathrm{d}x\mathrm{d}y \\
&= \frac{\sigma}{\delta} w \int_0^{1/w} I^{\varphi}[3\lambda m_0(\chi)(f(\cdot+y)-f(\cdot))]\mathrm{d}y \le \frac{\sigma}{\delta}\omega\left(3\lambda m_0(\chi)f, \frac{1}{w}\right)_{\varphi},
\end{aligned}
$$

for $w > 0$. For I_3, we denote by $T_0 \subset \mathbb{R}$ the set of all points of $\mathbb{R}$ for which $f \neq 0$ almost everywhere, and by the convexity of φ and by the condition (χ2), we obtain

$$
\begin{aligned}
3I_3 &= \int_{T_0} \varphi\left(3\lambda \left| f(x)\left[\sum_{k\in\mathbb{Z}} \chi(wx-k) - 1\right]\right|\right)\mathrm{d}x \\
&\le \int_{T_0} \varphi(3\lambda M w^{-\mu}|f(x)|)\mathrm{d}x \le w^{-\mu}\int_{T_0}\varphi(3\lambda M|f(x)|)\mathrm{d}x \\
&= w^{-\mu}\int_{\mathbb{R}}\varphi(3\lambda M|f(x)|)\mathrm{d}x = w^{-\mu} I^{\varphi}[3\lambda M f] \le w^{-\mu} I^{\varphi}[\lambda_0 f] < +\infty,
\end{aligned}
$$

for positive constants M and μ and for sufficiently large $w > 0$. This completes the proof. □

In case the kernel χ satisfies the strong condition (2.2) instead of (χ2), we can deduce the following corollary.

Corollary 3.3. *Let χ be a kernel satisfying* (2.2) *and assume that for any fixed* $0 < \alpha < 1$, *the integral*

$$
w \int_{|y|>1/w^{\alpha}} |\chi(wy)|\mathrm{d}y \le K w^{-\gamma}, \quad \text{as } w \to +\infty,
$$

for suitable positive constants K, γ depending on α and χ. Then, for $f \in L^{\varphi}(\mathbb{R})$, there exist $\lambda, \lambda_0 > 0$ such that

$$
\begin{aligned}
I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}} f - f)] \le \frac{1}{2}\frac{\sigma}{\delta}\Bigg\{ &\frac{m_0(\tau)}{m_0(\chi)}\|\chi\|_1 \omega\left(2\lambda m_0(\chi) f, \frac{1}{w^{\alpha}}\right)_{\varphi} \\
&+ \frac{m_0(\tau)}{m_0(\chi)} K I^{\varphi}[\lambda_0 f] w^{-\gamma} + \omega\left(2\lambda m_0(\chi) f, \frac{1}{w}\right)_{\varphi}\Bigg\},
\end{aligned}
$$

for every sufficiently large $w > 0$ and where τ is the characteristic function of the set $[0, 1]$.

Proof. Here we can split the term $I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}} f - f)]$ in this way

$$
\begin{aligned}
I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}} f - f)] &\le \frac{1}{2}\int_{\mathbb{R}} \varphi\left(2\lambda\left|\sum_{k\in\mathbb{Z}}\chi(wx-k)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)[f(u)-f(u+x-k/w)]\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u}\right|\right)\mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}} \varphi\left(2\lambda\left|\sum_{k\in\mathbb{Z}}\chi(wx-k)\frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u) f(u+x-k/w)\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u} - f(x)\right|\right)\mathrm{d}x \\
&=: I_1 + I_2, \quad w > 0.
\end{aligned}
$$

With the same steps of the proof of Theorem 3.2, we can estimate I_1 and I_2 in order to obtain the thesis. □

We note that assumption (3.1) is satisfied, for instance, by kernels having a sufficiently rapid decay, as kernels χ with compact support. Indeed, if we suppose that $\mathrm{supp}\chi \subset [-B, B]$, $B > 0$, we can write

$$w \int_{|y|>1/w^{\alpha}} |\chi(wy)|\mathrm{d}y = \int_{|u|>w^{1-\alpha}} |\chi(u)|\mathrm{d}u = 0,$$

for sufficiently large $w > B^{1/(1-\alpha)}$ and $0 < \alpha < 1$. So, Theorem 3.2 can be reformulated in this way.

Corollary 3.4. *Let χ be a kernel with compact support and let $f \in L^{\varphi}(\mathbb{R})$ be fixed. Then, for every $0 < \alpha < 1$, there exist positive λ and λ_0 such that*

$$I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}}f - f)] \leq \frac{\sigma}{3\delta}\frac{m_0(\tau)}{m_0(\chi)}\|\chi\|_1\,\omega\left(3\lambda m_0(\chi)f, \frac{1}{w^{\alpha}}\right)_{\varphi} + \frac{\sigma}{3\delta}\omega\left(3\lambda m_0(\chi)f, \frac{1}{w}\right)_{\varphi} + \frac{1}{3}I^{\varphi}[\lambda_0 f]w^{-\mu},$$

for sufficiently large $w > 0$, where τ is the characteristic function of the set $[0, 1]$ and $\mu > 0$ is the constant of the condition $(\chi 2)$.

Remark 3.5. If the kernel also satisfies condition (2.2) instead of $(\chi 2)$, the thesis of the Corollary 3.4 can be rewritten as:

For every $0 < \alpha < 1$, there exists a positive λ such that

$$I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}}f - f)] \leq \frac{\sigma}{2\delta}\frac{m_0(\tau)}{m_0(\chi)}\|\chi\|_1\,\omega\left(2\lambda m_0(\chi)f, \frac{1}{w^{\alpha}}\right)_{\varphi} + \frac{\sigma}{2\delta}\omega\left(2\lambda m_0(\chi)f, \frac{1}{w}\right)_{\varphi},$$

for sufficiently large $w > 0$ and where τ is the characteristic function of the set $[0, 1]$.

If the kernel χ does not have compact support, we may require this assumption on the continuous absolute moment

$$\widetilde{m}_q(\chi) < +\infty, \quad \text{for } q > 0.$$

Under this assumption, we obtain that condition (3.1) is true with $\gamma = (1 - \alpha)q$ and $K = \widetilde{m}_q(\chi)$ and so the thesis of Theorem 3.2 holds. For more details and for examples of kernels χ with unbounded support satisfying (3.1), refer, e.g., [24,36].

In order to deduce also the quantitative order of approximation of the function f through the operators $(K_w^{\chi,\mathcal{G}}f)_{w>0}$, we recall the definition of the Lipschitz class in Orlicz spaces, denoted by $\mathrm{Lip}_{\varphi}(\nu)$, $0 < \nu \leq 1$ and defined as follows:

$$\mathrm{Lip}_{\varphi}(\nu) := \left\{f \in L^{\varphi}(\mathbb{R}) | \exists\lambda > 0 : I^{\varphi}[\lambda(f(\cdot) - f(\cdot + t))] = \int_{\mathbb{R}} \varphi(\lambda|f(x) - f(x + t)|)\mathrm{d}x = O(|t|^{\nu}),\ t \to 0\right\}.$$

From the previous result, we can immediately deduce the following:

Corollary 3.6. *Under the assumptions of Theorem 3.2 with $0 < \alpha < 1$ and for any $f \in \mathrm{Lip}_{\varphi}(\nu)$, $0 < \nu \leq 1$, there exists a positive λ such that*

$$I^{\varphi}[\lambda(K_w^{\chi,\mathcal{G}}f - f)] = O(w^{-\varepsilon}), \quad w \to +\infty,$$

with $\varepsilon := \min\{\alpha\nu, \gamma, \mu\}$.

Remark 3.7. If (2.2) is fulfilled instead of $(\chi 2)$, we can obtain that ε in the thesis of Corollary 3.6 is minimum only between $\alpha\nu$ and γ.

In the literature, there are many examples of remarkable Orlicz spaces. Indeed, the φ-function defined as $\varphi(u) \coloneqq u^p, 1 \leq p < +\infty$ generates the well-known L^p-spaces, or with a φ-function in the form $\varphi(u) \coloneqq u^\alpha \log^\beta(u+e)$, for $\alpha \geq 1, \beta > 0$, we obtain the $L^\alpha \log^\beta L$-spaces or Zygmund spaces largely used, for instance, in the theory of partial differential equations. If instead we consider the φ-function as follows: $\varphi(u) \coloneqq e^{u^\gamma} - 1, \gamma > 0, u \geq 0$, we have the exponential spaces used in embedding theorems between Sobolev spaces. In all the above cases, the results developed here are valid.

4 Quantitative estimate in L^p-spaces

We have just noted that the theory established in Section 3 continues to be satisfied in the specific case of L^p-spaces. In this setting, if we consider the problem of the order of approximation by a direct approach, we can obtain a sharper estimate than the one deduced by what was achieved in Section 3. In order to do this, we will use the following well-known inequality:

$$\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f,\delta), \quad \lambda, \delta > 0, \tag{4.1}$$

that is true in L^p-spaces (but not in Orlicz spaces). In order to reach this goal, we consider the definition of the modulus of smoothness in $L^p(\mathbb{R})$:

$$\omega(f,\delta)_p \coloneqq \sup_{|h|\leq\delta} \|f(\cdot+h) - f(\cdot)\|_p = \sup_{|h|\leq\delta} \left(\int_{\mathbb{R}} |f(t+h) - f(t)|^p \mathrm{d}t \right)^{1/p},$$

with $\delta > 0$ and $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$.

With this definition we can prove the following quantitative estimate.

Theorem 4.1. *Let χ be a kernel such that $\widetilde{m}_p(\chi) < +\infty, 1 \leq p < +\infty$, then for every $f \in L^p(\mathbb{R})$, it turns out:*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_p \leq \frac{\sigma^{1/p}}{\delta^{1/p}} (2m_0(\chi))^{(p-1)/p} (m_0(\tau))^{1/p} (\|\chi\|_1 + \widetilde{m}_p(\chi))^{1/p} \omega\left(f, \frac{1}{w}\right)_p + m_0(\chi)\frac{\sigma}{\delta}\omega\left(f, \frac{1}{w}\right)_p + M\|f\|_p w^{-\mu},$$

for every sufficiently large $w > 0$, where τ is the characteristic function of the set $[0,1]$ and $M, \mu > 0$ are the constants of the condition $(\chi 2)$.

Proof. At first, we can proceed as in the first part of the proof of Theorem 3.2 and by using Minkowski inequality and the sub-additivity of the function $|\cdot|^{1/p}$, $p \geq 1$, we obtain

$$\begin{aligned}\|K_w^{\chi,\mathcal{G}} f - f\|_p \leq& \left(\int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u) - f(u+x-k/w)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u} \right]^p \mathrm{d}x \right)^{1/p} \\ &+ \left(\int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u+x-k/w) - f(x)|\mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\mathrm{d}u} \right]^p \mathrm{d}x \right)^{1/p} \\ &+ \left(\int_{\mathbb{R}} |f(x)|^p \left| \sum_{k\in\mathbb{Z}} \chi(wx-k) - 1 \right|^p \mathrm{d}x \right)^{1/p} \eqqcolon I_1 + I_2 + I_3.\end{aligned}$$

Now, we estimate I_1, by exploiting Jensen's inequality twice, the boundedness of the noise functions and Fubini-Tonelli theorem

$$I_1^p = \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(u+x-k/w)|du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)du} \right]^p dx$$

$$\leq \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left[m_0(\chi) \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(u+x-k/w)|du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)du} \right]^p dx$$

$$\leq (m_0(\chi))^{p-1} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(u+x-k/w)|^p du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)du} dx$$

$$\leq \frac{\sigma w}{\delta}(m_0(\chi))^{p-1} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left[\int_{k/w}^{(k+1)/w} |f(u)-f(u+x-k/w)|^p du \right] dx$$

$$= \frac{\sigma w}{\delta}(m_0(\chi))^{p-1} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\chi(wx-k)| \left[\int_{k/w}^{(k+1)/w} |f(u)-f(u+x-k/w)|^p du \right] dx$$

$$= \frac{\sigma w}{\delta}(m_0(\chi))^{p-1} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\chi(wx-k)| \left[\int_{\mathbb{R}} \tau(wu-k)|f(u)-f(u+x-k/w)|^p du \right] dx,$$

where τ is the characteristic function on the interval $[0,1]$.

Now, we use the change in variables $y = x - k/w$, Fubini-Tonelli theorem, property (4.1) with $\lambda = w|y|$ and $\delta = \frac{1}{w}$, the convexity of the function $|\cdot|^p$, $p \geq 1$ and the change in variables $wy = z$ in order to have

$$I_1^p = \frac{\sigma w}{\delta}(m_0(\chi))^{p-1} \int_{\mathbb{R}} |\chi(wy)| \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} \tau(wu-k) \right] |f(u)-f(u+y)|^p du dy$$

$$\leq \frac{\sigma}{\delta}(m_0(\chi))^{p-1} m_0(\tau) \int_{\mathbb{R}} w|\chi(wy)|(\omega(f,|y|)_p)^p dy$$

$$\leq \frac{\sigma}{\delta}(m_0(\chi))^{p-1} m_0(\tau) \left(\omega\left(f, \frac{1}{w}\right)_p \right)^p \int_{\mathbb{R}} w|\chi(wy)|(1+w|y|)^p dy$$

$$\leq \frac{\sigma}{\delta}(m_0(\chi))^{p-1} m_0(\tau) \left(\omega\left(f, \frac{1}{w}\right)_p \right)^p \int_{\mathbb{R}} w|\chi(wy)| 2^{p-1}(1+(w|y|)^p) dy$$

$$= \frac{\sigma}{\delta}(m_0(\chi))^{p-1} m_0(\tau) 2^{p-1} \left(\omega\left(f, \frac{1}{w}\right)_p \right)^p \int_{\mathbb{R}} |\chi(z)|(1+|z|^p) dz$$

$$= \frac{\sigma}{\delta}(m_0(\chi))^{p-1} m_0(\tau) 2^{p-1} \left(\omega\left(f, \frac{1}{w}\right)_p \right)^p (\|\chi\|_1 + \widetilde{m}_p(\chi)),$$

for every $w > 0$, where $\|\chi\|_1$ and $\widetilde{m}_p(\chi)$ are both finite.

Then, we estimate the integral I_2, where we use the boundedness of the functions $g_{k,w}$, the change in variables $y = u - k/w$, Jensen's inequality twice and Fubini-Tonelli's theorem:

$$I_2^p = \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u+x-k/w)-f(x)|du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)du} \right]^p dx$$

$$\leq \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\sigma}{\delta} w \int_0^{1/w} |f(x+y)-f(x)| \mathrm{d}y \right]^p \mathrm{d}x$$
$$\leq \frac{1}{m_0(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left[w \int_0^{1/w} m_0(\chi) \frac{\sigma}{\delta} |f(x+y)-f(x)| \mathrm{d}y \right]^p \mathrm{d}x$$
$$\leq (m_0(\chi))^{p-1} \int_{\mathbb{R}} \left[\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \right] w \frac{\sigma^p}{\delta^p} \left[\int_0^{1/w} |f(x+y)-f(x)|^p \mathrm{d}y \right] \mathrm{d}x$$
$$\leq (m_0(\chi))^p \frac{\sigma^p}{\delta^p} \int_{\mathbb{R}} w \left[\int_0^{1/w} |f(x+y)-f(x)|^p \mathrm{d}y \right] \mathrm{d}x$$
$$= (m_0(\chi))^p \frac{\sigma^p}{\delta^p} \int_0^{1/w} w \left[\int_{\mathbb{R}} |f(x+y)-f(x)|^p \mathrm{d}x \right] \mathrm{d}y$$
$$\leq (m_0(\chi))^p \frac{\sigma^p}{\delta^p} \left(\omega\left(f, \frac{1}{w}\right)_p \right)^p.$$

Finally, we also estimate I_3 by using the property $(\chi 2)$:

$$I_3^p = \int_{\mathbb{R}} |f(x)|^p \left| \sum_{k\in\mathbb{Z}} \chi(wx-k) - 1 \right|^p \mathrm{d}x \leq \|f\|_p^p M^p w^{-\mu p},$$

with $M > 0$ and for sufficiently large $w > 0$. By combining the estimates on I_1, I_2 and I_3, we obtain the thesis. □

If we assume that the kernel χ satisfies condition (2.2), as noted in Section 3, we obtain the following corollary.

Corollary 4.2. *Let χ be a kernel satisfying* (2.2) *such that $\widetilde{m}_p(\chi) < +\infty$, $1 \leq p < +\infty$, then for every $f \in L^p(\mathbb{R})$*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_p \leq \frac{\sigma^{1/p}}{\delta^{1/p}} (2m_0(\chi))^{(p-1)/p} (m_0(\tau))^{1/p} (\|\chi\|_1 + \widetilde{m}_p(\chi))^{1/p} \omega\left(f, \frac{1}{w}\right)_p + m_0(\chi) \frac{\sigma}{\delta} \omega\left(f, \frac{1}{w}\right)_p,$$

for every $w > 0$ and where τ is the characteristic function of the set $[0, 1]$.

Now, if we consider the Lipschitz class in $L^p(\mathbb{R})$, defined as follows:

$$\mathrm{Lip}(\nu, p) := \{f \in L^p(\mathbb{R}) : \|f(\cdot + h) - f(\cdot)\|_p = O(|h|^\nu),\ h \to 0\},$$

with $0 < \nu \leq 1$, $p \geq 1$, we can establish this result as a consequence of the previous theorem.

Corollary 4.3. *Let χ be a kernel such that $\widetilde{m}_p(\chi) < +\infty$, $1 \leq p < +\infty$, then for every $f \in \mathrm{Lip}(\nu, p)$, with $0 < \nu \leq 1$ and $1 \leq p < +\infty$:*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_p = O(w^{-\varepsilon}), \quad w \to +\infty,$$

where $\varepsilon := \min\{\nu, \mu\}$.

In case the kernel satisfies the strong condition (2.2), it turns out that $\|K_w^{\chi,\mathcal{G}} f - f\|_p = O(w^{-\nu})$, $w \to +\infty$, where ν is the constant arising from the class $\mathrm{Lip}(\nu, p)$.

5 Quantitative estimate in $C(\mathbb{R})$

In order to provide a quantitative estimate for the operators $(K_w^{\chi,\mathcal{G}})_{w>0}$ in $C(\mathbb{R})$, we recall the definition of the modulus of continuity in this space:

$$\omega(f,\delta) := \sup\{|f(x)-f(y)| : x,y \in \mathbb{R}, |x-y| \le \delta\}, \quad \delta > 0.$$

It is well-known that also in $C(\mathbb{R})$, the following property $\omega(f,\lambda\delta) \le (1+\lambda)\omega(f,\delta)$, $\lambda, \delta > 0$, mentioned before, is true.

We start with the following.

Theorem 5.1. *Let χ be a kernel satisfying condition $(\chi 3)$ with $\beta \ge 1$, then for $f \in C(\mathbb{R})$, it turns out*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} \le \omega\left(f, \frac{1}{w}\right)\left\{\left(1 + \frac{\sigma}{\delta}\right) m_0(\chi) + \frac{\sigma}{\delta} m_1(\chi)\right\} + \|f\|_{C(\mathbb{R})} M w^{-\mu},$$

for every sufficiently large $w > 0$ and where $M, \mu > 0$ are the constants of condition $(\chi 2)$.

Proof. Let $x \in \mathbb{R}$ be fixed. We have

$$|(K_w^{\chi,\mathcal{G}} f)(x) - f(x)| \le \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)|f(u)-f(x)|\,du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du} + |f(x)| \left| \sum_{k\in\mathbb{Z}} \chi(wx-k) - 1 \right| := I_1 + I_2.$$

I_1 turns out to be

$$\begin{aligned} I_1 &\le \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\omega(f,|u-x|)\,du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du} \\ &\le \omega\left(f,\frac{1}{w}\right) \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)(1+w|u-x|)\,du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du} \\ &= \omega\left(f,\frac{1}{w}\right) \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left\{1 + \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u) w|u-x|\,du}{\int_{k/w}^{(k+1)/w} g_{k,w}(u)\,du}\right\} \\ &\le \omega\left(f,\frac{1}{w}\right) \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left\{1 + \frac{\sigma}{\delta} w \int_{k/w}^{(k+1)/w} w|u-x|\,du\right\}, \end{aligned}$$

for every $w > 0$, where the previous inequalities are consequences of both the property on the modulus of continuity $\omega(f,\lambda\delta) \le (1+\lambda)\omega(f,\delta)$, with $\lambda = w|u-x|$ and $\delta = \frac{1}{w}$ and of the boundedness of the noise functions.

Now, we observe that for every $u \in [\frac{k}{w}, \frac{k+1}{w}]$ and $x \in \mathbb{R}$:

$$|u-x| \le \left|u - \frac{k}{w}\right| + \left|\frac{k}{w} - x\right| \le \frac{1}{w} + \frac{|wx-k|}{w},$$

so, we obtain the following estimate:

$$w^2 \int_{k/w}^{(k+1)/w} |u - x| \mathrm{d}u \leq 1 + |wx - k|,$$

and by using this inequality, we finally conclude

$$\begin{aligned} I_1 &\leq \omega\left(f, \frac{1}{w}\right) m_0(\chi) + \omega\left(f, \frac{1}{w}\right) \frac{\sigma}{\delta} \sum_{k \in \mathbb{Z}} |\chi(wx - k)|(1 + |wx - k|) \\ &\leq \omega\left(f, \frac{1}{w}\right) \left\{\left(1 + \frac{\sigma}{\delta}\right) m_0(\chi) + \frac{\sigma}{\delta} m_1(\chi)\right\}, \end{aligned}$$

for $w > 0$. For I_2, by using condition $(\chi 2)$, we can obtain the following inequalities:

$$I_2 \leq |f(x)| M w^{-\mu} \leq \|f\|_{C(\mathbb{R})} M w^{-\mu},$$

for $M > 0$ and for every sufficiently large $w > 0$ and this completes the proof. □

Since Theorem 5.1 holds if $m_\beta(\chi) < +\infty$, with $\beta \geq 1$, it is quite natural also what happens if $0 < \beta < 1$. In case of $0 < \beta < 1$ in condition $(\chi 3)$, we can state the following result.

Theorem 5.2. *Let χ be a kernel satisfying condition $(\chi 3)$ with $0 < \beta < 1$. Then, for $f \in C(\mathbb{R})$, it turns out:*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} \leq \omega\left(f, \frac{1}{w^\beta}\right) \frac{\sigma}{\delta} (2 m_0(\chi) + m_\beta(\chi)) + 2^{\beta+1} \|f\|_{C(\mathbb{R})} \frac{\sigma}{\delta} m_\beta(\chi) w^{-\beta} + \|f\|_{C(\mathbb{R})} M w^{-\mu},$$

for every sufficiently large $w > 0$ and where $M, \mu > 0$ are the constants of condition $(\chi 2)$.

Proof. Let $x \in \mathbb{R}$ be fixed. We can write

$$\begin{aligned} &|(K_w^{\chi,\mathcal{G}} f)(x) - f(x)| \\ &= \left| \sum_{k \in \mathbb{Z}} \chi(wx - k) \frac{\int_{k/w}^{(k+1)/w} g_{k,w}(u)[f(u) - f(x)] \mathrm{d}u}{\int_{k/w}^{(k+1)/w} g_{k,w}(u) \mathrm{d}u} + f(x) \left[\sum_{k \in \mathbb{Z}} \chi(wx - k) - 1 \right] \right| \\ &\leq \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \frac{\sigma}{\delta} w \int_{k/w}^{(k+1)/w} |f(u) - f(x)| \mathrm{d}u + |f(x)| \left| \sum_{k \in \mathbb{Z}} \chi(wx - k) - 1 \right| \\ &= \left\{ \sum_{|wx-k| \leq w/2} + \sum_{|wx-k| > w/2} \right\} |\chi(wx - k)| \frac{\sigma}{\delta} \left[w \int_{k/w}^{(k+1)/w} |f(u) - f(x)| \mathrm{d}u \right] \\ &\quad + |f(x)| \left| \sum_{k \in \mathbb{Z}} \chi(wx - k) - 1 \right| =: I_1 + I_2 + I_3, \end{aligned}$$

for $w > 0$. Regarding the estimate on I_1, we observe that, for every $u \in [\frac{k}{w}, \frac{k+1}{w}]$ and if $|wx - k| \leq w/2$, one has

$$|u - x| \leq \left| u - \frac{k}{w} \right| + \left| \frac{k}{w} - x \right| \leq \frac{1}{w} + \frac{|wx - k|}{w} \leq \frac{1}{w} + \frac{1}{2} \leq 1, \tag{5.1}$$

for sufficiently large $w > 0$ and so, for $0 < \beta < 1$, the following inequality is true:

$$\omega(f, |u - x|) \leq \omega(f, |u - x|^\beta).$$

Now, thanks to the property on the modulus of continuity $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$ with $\lambda = w^\beta |u - x|^\beta$ and $\delta = w^{-\beta}$, we obtain

$$I_1 \leq \sum_{|wx-k| \leq w/2} |\chi(wx - k)| \frac{\sigma}{\delta} w \int_{k/w}^{(k+1)/w} \omega(f, |u - x|^\beta) \mathrm{d}u$$

$$\leq \sum_{|wx-k|\leq w/2} |\chi(wx-k)| \frac{\sigma}{\delta} w \int_{k/w}^{(k+1)/w} (1 + w^\beta\, |u-x|^\beta)\omega\left(f, \frac{1}{w^\beta}\right) du$$
$$= \omega\left(f, \frac{1}{w^\beta}\right)\frac{\sigma}{\delta}\left[\sum_{|wx-k|\leq w/2} |\chi(wx-k)| w \int_{k/w}^{(k+1)/w} w^\beta\, |u-x|^\beta du + \sum_{|wx-k|\leq w/2} |\chi(wx-k)|\right]$$
$$= \omega\left(f, \frac{1}{w^\beta}\right)\frac{\sigma}{\delta}[I_{1,1} + I_{1,2}].$$

For the estimate of the term $I_{1,1}$, we use (5.1) and the sub-additivity of $|\cdot|^\beta$, with $0 < \beta < 1$, in order to obtain

$$I_{1,1} \leq \sum_{|wx-k|\leq w/2} |\chi(wx-k)| w \int_{k/w}^{(k+1)/w} w^\beta\left[\frac{1}{w^\beta} + \frac{|wx-k|^\beta}{w^\beta}\right] du$$
$$= \sum_{|wx-k|\leq w/2} |\chi(wx-k)| + \sum_{|wx-k|\leq w/2} |\chi(wx-k)||wx-k|^\beta$$
$$\leq m_0(\chi) + m_\beta(\chi),$$

which is a finite quantity thanks to conditions (χ3) and (2.1) of kernels. On the other hand, for $I_{1,2}$, it is trivially proved that $I_{1,2} \leq m_0(\chi)$. Moreover, for what concerns I_2, by using that $f \in C(\mathbb{R})$, we conclude that

$$I_2 \leq 2\|f\|_{C(\mathbb{R})}\frac{\sigma}{\delta}\sum_{|wx-k|>w/2} |\chi(wx-k)|$$
$$\leq 2\|f\|_{C(\mathbb{R})}\frac{\sigma}{\delta}\sum_{|wx-k|>w/2} \frac{|wx-k|^\beta}{|wx-k|^\beta}|\chi(wx-k)|$$
$$\leq 2^{\beta+1}\|f\|_{C(\mathbb{R})}\frac{\sigma}{\delta}\frac{1}{w^\beta} m_\beta(\chi) < +\infty.$$

Finally, by condition (χ2), we obtain

$$I_3 \leq \|f\|_{C(\mathbb{R})} M w^{-\mu},$$

for $M > 0$ and for sufficiently large $w > 0$. By combining previous estimates and by passing to the supremum, we have the thesis. □

If the kernel satisfies hypothesis (2.2), we obtain the following results for $\beta \geq 1$ and for $0 < \beta < 1$, respectively.

Corollary 5.3. *Let χ be a kernel satisfying* (2.2) *and condition* (χ3) *with* $\beta \geq 1$, *then for* $f \in C(\mathbb{R})$, *it turns out*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} \leq \omega\left(f, \frac{1}{w}\right)\left\{\left(1 + \frac{\sigma}{\delta}\right)m_0(\chi) + \frac{\sigma}{\delta}m_1(\chi)\right\},$$

for every $w > 0$.

Corollary 5.4. *Let χ be a kernel satisfying* (2.2) *and condition* (χ3) *with* $0 < \beta < 1$, *then for* $f \in C(\mathbb{R})$, *it turns out*:

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} \leq \omega\left(f, \frac{1}{w^\beta}\right)\frac{\sigma}{\delta}(2m_0(\chi) + m_\beta(\chi)) + 2^{\beta+1}\|f\|_{C(\mathbb{R})}\frac{\sigma}{\delta}m_\beta(\chi)\frac{1}{w^\beta},$$

for every sufficiently large $w > 0$.

In order to study the rate of approximation for the operators in $C(\mathbb{R})$, we define the Lipschitz class $\mathrm{Lip}_\infty(\nu)$, $0 < \nu \leq 1$ as follows:

$$\mathrm{Lip}_\infty(\nu) := \{f \in C(\mathbb{R}) : \|f(\cdot) - f(\cdot + t)\|_{C(\mathbb{R})} = O(|t|^\nu),\, t \to 0\},$$

and we establish these results as a direct consequence of previous theorems. The first corollary is for $\beta \geq 1$.

Corollary 5.5. *Let χ be a kernel satisfying condition (χ3) with $\beta \geq 1$ and $f \in \mathrm{Lip}_\infty(\nu)$, $0 < \nu \leq 1$, then*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} = O(w^{-\theta}), \quad w \to +\infty,$$

where $\theta \coloneqq \min\{\nu, \mu\}$.

Remark 5.6. If the kernel satisfies condition (2.2), we obtain that the qualitative order of approximation is $w^{-\nu}$, as $w \to +\infty$.

The second one is for $0 < \beta < 1$.

Corollary 5.7. *Let χ be a kernel satisfying condition (χ3) with $0 < \beta < 1$ and $f \in \mathrm{Lip}_\infty(\nu)$, $0 < \nu \leq 1$, then*

$$\|K_w^{\chi,\mathcal{G}} f - f\|_{C(\mathbb{R})} = O(w^{-l}), \quad w \to +\infty,$$

where $l \coloneqq \min\{\beta\nu, \mu\}$.

Remark 5.8. If property (2.2) is fulfilled, the order is $w^{-\beta\nu}$, as $w \to +\infty$.

As examples of kernels satisfying assumptions (χ1)–(χ3), we recall the well-known central B-splines of order $n \in \mathbb{N}$, which have compact support and so, for those kernels, it is possible to apply Corollary 3.4 instead of the general Theorem 3.2. Moreover, we mention Fejer kernel and de la Vallée Poussin kernel, for which condition (χ3) is satisfied for every $\beta < 1$ (e.g., [1,2]) and therefore, Theorem 5.1 cannot be applied, while Theorem 5.2 holds. For other examples of kernels, the reader can refer [37].

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