Research Article

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The asymptotic behavior for the Navier-Stokes-Voigt-Brinkman-Forchheimer equations with memory and Tresca friction in a thin domain

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Abstract: In this article, we investigate the behavior of weak solutions for the three-dimensional Navier-Stokes-Voigt-Brinkman-Forchheimer fluid model with memory and Tresca friction law within a thin domain. We analyze the asymptotic behavior as one dimension of the fluid domain approaches zero. We derive the limit problem and obtain the specific Reynolds equation, while also establishing the uniqueness of the limit velocity and pressure distributions.

Keywords: asymptotic behavior, Navier-Stokes-Voigt-Brinkman-Forchheimer equations, Reynolds equation, mathematical model, nonlinear systems, Tresca friction law

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1 Introduction

The Navier-Stokes equations are well known in the fields of applied mathematics and engineering, particularly for their effectiveness in modeling turbulence in fluid phenomena, which has been intensively studied for the past 90 years. In 1973, the mathematician Oskolkov [1] modified the Navier-Stokes equation by adding the pseudoparabolic regularization $-\nu\Delta\partial_t u$ for velocity field u, and this model is called Navier-Stokes-Voigt. In recent years, this model has attracted significant attention from mathematicians, resulting in numerous research papers focused on studying the properties of its solutions. This surge in interest is partly due to the model yielding interesting results in various applications, including its recent use in image processing [2]. Let us briefly review the available literature on the mathematical analysis of Navier-Stokes-Voigt-type models. Anh and Thanh [3] demonstrated the existence and uniqueness of solutions for the Navier-Stokes-Voigt model, followed by an examination of the mean square exponential stability and the almost sure exponential stability of the stationary solutions. Zelati and Gal [4] investigated the long-term behavior of the three-dimensional

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Navier-Stokes-Voigt model. They established the existence of global and exponential attractors of optimal regularity. The existence and time decay rates of solutions for the Navier-Stokes-Voigt model have been studied in [5]. Layton and Rebholz [6] studied the analytically and numerically the relaxation time of flow evolution governed by the Navier-Stokes-Voigt model.

The Brinkman-Forchheimer equation [7,8] describes the motion of fluid flow in a saturated porous medium and has been studied by many researchers. Djoko and Razafimandimby [9] studied the Brinkman-Forchheimer equations driven under slip boundary conditions of the friction type. They proved the existence and uniqueness of weak solutions. Le [10] proved the existence of global weak solutions, the exponential stability of a stationary solution, and the existence of a global attractor for the three-dimensional convective Brinkman-Forchheimer equations with finite delay and fast growing nonlinearity in bounded domains with homogeneous Dirichlet boundary conditions. Cai and Jiu [11] incorporated the damping term $|u|^{p-2}u$ into the classical 3D Navier-Stokes equations and explored its impact on the well-posedness of the resulting equations. This damping term widely used in geophysical hydrodynamics arises from the resistance to the motion of the flow or by friction effects; for more details, see [12].

Regarding the memory term, its significance extends to various phenomena like non-Newtonian flows, soil mechanics, and heat conduction theory, as documented in [13]. Several researchers have delved into the behavior of solutions for the Navier-Stokes model incorporating memory effects. Notable studies include those referenced in [14,15].

On the other hand, the study of models placed in thin domains appears in many applications such as (thin elastic bodies, thin rods, plates, or shells) for solid mechanics, and (lubrication, meteorology, ocean dynamics) for fluid mechanics. During the past few decades, the lubrication theory was successfully applied to the modelling of dewetting processes in microscopic and nanoscopic liquid films on solid substrates. The previous works [16,17] addressed the study of three-dimensional Navier-Stokes equations in thin domains without friction forces. In studying the asymptotic behavior of Stokes equations in a thin domain with Tresca friction law, please refer to [18–20]. The reader can also explore related research studies concerning the asymptotic behavior elastic and viscosity materials in a thin domain with the Tresca friction law, as presented in [21–23].

Based on the principles of lubrication theory, which typically involves very thin flow domains, this article investigates the asymptotic behavior of solutions to the initial-boundary value problem governing the 3D Navier-Stokes-Voigt-Brinkman-Forchheimer equations. Incorporating a memory term and Tresca friction law, we aim to know the behavior of the solution as one dimension of the domain approaches zero and to establish the specific Reynolds equation.

This article is organized as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on function spaces and some of the inequalities frequently used in this article. In Section 3, we give the mechanical problem governed by the Navier-Stokes-Voigt-Brinkman-Forchheimer model with a memory term and Tresca friction law. In Section 4, we derive the variational formulation of the problem and introduce the theorem of the existence and uniqueness of the weak solution. In Section 5, we use the change of variable $z = x_3/\varepsilon$ to transform the initial problem posed in the domain Ω^ε, to a new problem posed on a fixed domain Ω independent of parameter ε. Then we establish some a priori estimates for the velocity and pressure fields, independently of ε. Finally, we derive and study the limit problem when ε tends to zero. Moreover, we show that the limit pressure is given as the unique solution of a Reynolds equation.

2 Mathematical setting

In this section, we describe the necessary function spaces needed to obtain the asymptotic behavior results of Navier-Stokes-Voigt-Brinkman-Forchheimer equations.

Let ω be a bounded domain in $\mathbb{R}^2$ situated in the plane of equation $x_3 = 0$. We suppose that ω represents the lower surface of the domain occupied by the substance. The upper surface $\partial\Omega_1^\varepsilon$ is defined by

$$\partial\Omega_1^\varepsilon = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x_3 = \varepsilon h(x') \text{ and } x' = (x_1, x_2) \in \omega\},$$

where h is a function defined and bounded on $\bar{\omega}$, such that:

$$0 < \underline{h} \le h(x') \le \bar{h}, \quad \forall (x', 0) \in \omega.$$

We study the velocity of a substance in

$$\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \omega,\ 0 < x_3 < \varepsilon h(x')\},$$

with Lipschitz boundary $\partial\Omega^\varepsilon = \partial\Omega_1^\varepsilon \cup \partial\Omega_L^\varepsilon \cup \bar{\omega}$, where $\partial\Omega_L^\varepsilon$ is the lateral surface of Ω.

Throughout this work, we denote by:

- $L^p(\Omega^\varepsilon)$ is the Lebesgue space for the norm

$$\|u^\varepsilon\|_{L^p(\Omega^\varepsilon)} = \left[\int_{\Omega^\varepsilon} |u^\varepsilon|^p \mathrm{d}x\right]^{\frac{1}{p}}, \quad p \in [1, \infty[.$$

For $p = 2$, we use $(\cdot,\cdot)$ to denote the inner product, i.e.,

$$(u^\varepsilon, v^\varepsilon) \coloneqq \sum_{i=1}^{3} \int_{\Omega^\varepsilon} u_i^\varepsilon . v_i^\varepsilon \mathrm{d}x, \quad u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon), v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon) \in L^2(\Omega^\varepsilon)^3.$$

- $H^1(\Omega^\varepsilon)^3$ is the Sobolev space that gives

$$H^1(\Omega^\varepsilon) = \{u^\varepsilon \in L^2(\Omega^\varepsilon)^3 : \nabla u^\varepsilon \in L^2(\Omega^\varepsilon)^{3\times 3}\},$$

with inner product and associated norm

$$\langle u, v\rangle_{H^1(\Omega^\varepsilon)} = \int_{\Omega^\varepsilon} u.v \mathrm{d}x + \int_{\Omega^\varepsilon} \nabla u.\nabla v \mathrm{d}x, \quad \|u\|_{H^1(\Omega^\varepsilon)} = [\|u\|_{L^2(\Omega^\varepsilon)^3} + \|\nabla u\|_{L^2(\Omega^\varepsilon)^{3\times 3}}]^{\frac{1}{2}}.$$

- $H_0^1(\Omega^\varepsilon)^3$ is the closure of $\mathcal{D}(\Omega^\varepsilon)^3$ in $H^1(\Omega^\varepsilon)^3$, and $H^{-1}(\Omega^\varepsilon)^3$ is the dual space of $H_0^1(\Omega^\varepsilon)^3$.

For any separable Banach space E equipped with the norm $\|.\|_E$, we denote by $L^p(0, T; E)$ the Banach space consisting of (classes of) functions $t \longrightarrow f(t)$ measurable from $[0, T] \longmapsto E$ (for the measure dt) such that

$$\|u^\varepsilon\|_{L^p(0,T;\,E)} = \left[\int_0^T \|u^\varepsilon(s)\|_E^p \mathrm{d}s\right]^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

Let us introduce the following spaces

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon)^3 : v = 0 \text{ on } \partial\Omega_1^\varepsilon \cup \partial\Omega_L^\varepsilon,\ v.n = 0 \text{ on } \omega\},$$

where $n = (n_1, n_2, n_3)$ is the normal vector outward to $\partial\Omega^\varepsilon$.

$$V_{\mathrm{div}}^\varepsilon = \{v \in V^\varepsilon :\ \mathrm{div}(v) = 0\}$$

and

$$L_0^2(\Omega^\varepsilon) = \left\{\varphi \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} \varphi \mathrm{d}x = 0\right\}.$$

The following inequalities are frequently used in the article

- Cauchy-Schwarz inequality

$$\int_{\Omega^\varepsilon} |u^\varepsilon| . |v^\varepsilon| \mathrm{d}x \le \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} . \|v^\varepsilon\|_{L^2(\Omega^\varepsilon)^3}.$$

- Poincaré's inequality

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \le \varepsilon\bar{h}\ \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}}. \tag{1}$$

- Young's inequality

$$a.b \leq \frac{a^2}{2} + \frac{b^2}{2}, \quad \forall a, b > 0. \tag{2}$$

- Sobolev's inequality

$$\|u^\varepsilon\|_{L^p(\Omega^\varepsilon)^3} \leq c_s \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}}, \quad \text{for } p \in [2, 6]\cdot. \tag{3}$$

3 The dynamical system

In the domain Ω^ε, we study the asymptotic behavior of weak solutions to the following Navier-Stokes-Voight-Brinkman-Forchheimer equations with a memory:

Find the velocity field $u^\varepsilon : \Omega^\varepsilon \times]0, T[\to \mathbb{R}^3$ and the pressure $\pi^\varepsilon : \Omega^\varepsilon \times]0, T[\to \mathbb{R}$, such that

$$\rho \partial_t u^\varepsilon - \operatorname{div}(\sigma^\varepsilon(u^\varepsilon, \pi^\varepsilon)) + \sqrt{\varepsilon}(u^\varepsilon.\nabla u^\varepsilon) + r^\varepsilon u^\varepsilon + \alpha^\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon = f^\varepsilon \quad \text{in } \Omega^\varepsilon \times]0, T[, \tag{4}$$

$$\sigma^\varepsilon(u^\varepsilon, \pi^\varepsilon) = \mu \nabla u^\varepsilon + \nu \nabla(\partial_t u^\varepsilon) + \phi \star \nabla u^\varepsilon - \pi^\varepsilon I_3 \quad \text{in } \Omega^\varepsilon \times]0, T[, \tag{5}$$

$$\operatorname{div}(u^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon \times]0, T[, \tag{6}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega_1^\varepsilon \times]0, T[, \tag{7}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega_L^\varepsilon \times]0, T[, \tag{8}$$

$$u^\varepsilon \cdot n = 0 \quad \text{on } \omega \times]0, T[, \tag{9}$$

$$\left.\begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \Rightarrow u_\tau^\varepsilon = 0, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \Rightarrow \exists \beta > 0, \ \text{such that } u_\tau^\varepsilon = -\beta \sigma_\tau^\varepsilon, \end{array}\right\} \quad \text{on } \omega \times]0, T[. \tag{10}$$

In equation (4) representing the equilibrium of the fluid, $f^\varepsilon : \Omega^\varepsilon \times]0, T[\to \mathbb{R}^3$ represents an external body force, ρ is the mass density per unit of reference volum, $\mu > 0$ is the Brinkman coefficient, $\nu > 0$ is the length-scale parameter characterizing the elasticity of the fluid, $r^\varepsilon > 0$ is the Darcy coefficient, α^ε is the Forchheimer coefficient, and $p \in [2, 6]$ is a constant. Equation (5) represents the law of behavior of the fluid for the Navier-Stokes-Voigt model with a memory

$$(\phi \star \nabla u)(t) := \int_0^t \phi(t - s) \nabla u(s) \mathrm{d}s,$$

where the function $\phi : [0; T] \to \mathbb{R}$, is usually called the memory kernel and satisfies certain conditions specified later. Equation (6) represents the incompressibility equation. The boundary conditions (7) and (8) are the nonslip boundary condition on $\partial\Omega_1^\varepsilon \times]0, T[$ and $\partial\Omega_L^\varepsilon \times]0, T[$. Since there is a no-flux condition across ω, we have equation (9). Condition (10) represents a Tresca friction law on ω with a friction coefficient k^ε, where $|.|$ is the $\mathbb{R}^2$ Euclidean norm and

$$u_n^\varepsilon = u^\varepsilon.n, \quad u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon.n, \quad \sigma_n^\varepsilon = (\sigma_n^\varepsilon).n, \quad \sigma_\tau^\varepsilon = \sigma^\varepsilon.n - (\sigma_n^\varepsilon).n$$

are the normal and the tangential velocity u^ε, and the components of the normal and the tangential stress tensor σ^ε, respectively.

Equations (4)–(10) are supplemented with the initial condition

$$u^\varepsilon(x, 0) = u_0^\varepsilon, \quad \forall x \in \Omega^\varepsilon. \tag{11}$$

Remark 1. On $\omega \times]0, T[$ the third component of velocity equals zero. Indeed, according to condition (9), we have

$$u^\varepsilon.n = u_1^\varepsilon.n_1 + u_2^\varepsilon.n_2 + u_3^\varepsilon.n_3 = 0 \quad \text{on } \omega \times]0, T[,$$

where $n = (n_1, n_2, n_3) = (0, 0, -1)$ is the unit normal to ω. So $u_3^\varepsilon = 0$, on $\omega \times]0, T[$.

Properties of the kernel $\phi(\cdot)$

Let $\phi(\cdot)$ satisfies the following conditions:

(1) $\phi(t) \in C^2[0, T]$,

(2) $(-1)^k \frac{d^k}{dt^k}(\phi(t)) \geq 0$, for $t \in [0, T]$, $k = 0; 1; 2$.

(3) $\phi(\cdot)$ is positive kernel, i.e.,

$$\int_0^T ((\phi \star u^\varepsilon)(t), u^\varepsilon(t))dt = \int_0^T\int_0^t \phi(t-s)(u^\varepsilon(s), u^\varepsilon(t))dsdt \geq 0,$$

for all $u^\varepsilon \in L^2(0, T; L^2(\Omega^\varepsilon))$ and every $T > 0$.

By the Lemma 2.6, [24], we have

$$\begin{aligned}\int_0^T \|(\phi \star u^\varepsilon)(t)\|^2_{L^2(\Omega^\varepsilon)}dt &\leq \int_0^T\left(\int_0^t \phi(t-s)\cdot\|u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)}ds\right)^2 dt \\ &\leq \left(\int_0^T \phi(t)dt\right)^2\left(\int_0^T \|u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}dt\right).\end{aligned} \tag{12}$$

By using the Cauchy-Schwarz inequality and (12), we can obtain the following estimate:

$$\begin{aligned}\int_0^t ((\phi \star u^\varepsilon)(s), \psi(s))ds &= \left(\int_0^t \|(\phi \star u^\varepsilon)(s)\|^2_{L^2(\Omega^\varepsilon)}\right)^{\frac{1}{2}} \cdot \left(\int_0^t \|\psi(s)\|^2_{L^2(\Omega^\varepsilon)}ds\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^t \phi(s)ds\right)\left(\int_0^t \|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t \|\psi(s)\|^2_{L^2(\Omega^\varepsilon)}ds\right)^{\frac{1}{2}},\end{aligned}$$

for $u^\varepsilon, \psi \in L^2(0, T; L^2(\Omega^\varepsilon))$.

We can take $\phi(t) = c\exp(-\delta t)$ as an example of a kernel, using Lemma 4.1, [25], it is easy to see that this function is a positive kernel.

4 Weak formulation

To derive the weak formulation of the problem, we will need the following lemma.

Lemma 1. [26] *The condition of Tresca* (10) *is equivalent to*

$$\sigma^\varepsilon_\tau . u^\varepsilon_\tau + k^\varepsilon |u^\varepsilon_\tau| = 0 \quad on\ \omega \times]0, T[.$$

By multiplying equation (4) by test-functions $(\varphi - u^\varepsilon)$ and integrating over Ω^ε, then applying Green's formula and the aforementioned lemma, we obtain the following weak formulation of problems (4)–(11):

Find $(u^\varepsilon(t), \pi^\varepsilon(t)) \in L^2(0, T; V^\varepsilon_{\text{div}}) \times L^2(0, T; L^2_0(\Omega^\varepsilon))$, such that for almost all t

$$\begin{aligned}&(\rho\partial_t u^\varepsilon(t), \varphi - u^\varepsilon(t)) + \mu\mathcal{A}(u^\varepsilon(t), \varphi - u^\varepsilon(t)) + \nu\mathcal{A}(\partial_t u^\varepsilon(t), \varphi - u^\varepsilon(t)) \\ &\qquad + \mathcal{A}(\phi\star u^\varepsilon(t), \varphi - u^\varepsilon(t)) + \sqrt{\varepsilon}\mathcal{B}(u^\varepsilon(t), u^\varepsilon(t), \varphi - u^\varepsilon(t)) \\ &\qquad + ((r^\varepsilon + \alpha^\varepsilon |u^\varepsilon(t)|^{p-2})u^\varepsilon(t), \varphi - u^\varepsilon(t)) - (\pi^\varepsilon(t), \operatorname{div}(\varphi)) + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon(t)) \\ &\quad \geq (f^\varepsilon(t), \varphi - u^\varepsilon(t)), \quad \forall\varphi \in V^\varepsilon, \\ &u^\varepsilon(x, 0) = u^\varepsilon_0, \quad \forall x \in \Omega^\varepsilon.\end{aligned} \tag{13}$$

With

$$\mathcal{A}(u,v)=\int_{\Omega^\varepsilon}\nabla u.\nabla v\mathrm{d}x,\quad j^\varepsilon(v)=\int_\omega k^\varepsilon|v|\mathrm{d}x',\quad (f,v)=\int_{\Omega^\varepsilon}f.v\mathrm{d}x,$$
$$\mathcal{B}(u,v,w)=\int_{\Omega^\varepsilon}u.\nabla v.w\mathrm{d}x=\sum_{i,j=1}^{3}\int_{\Omega^\varepsilon}u_i.\partial_{x_j}v_i.w_j\mathrm{d}x.$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous and coercive on $V^\varepsilon_{\mathrm{div}}\times V^\varepsilon_{\mathrm{div}}$ and $j^\varepsilon(\cdot)$ is a functional continuous convex, but nondifferentiable on $V^\varepsilon_{\mathrm{div}}$. The trilinear form $\mathcal{B}(\cdot,\cdot,\cdot)$ satisfies

$$\mathcal{B}(u,v,v)=0,\quad \forall u,v\in V^\varepsilon_{\mathrm{div}},$$
$$|\mathcal{B}(u,v,w)|\le\|u\|_{L^3(\Omega^\varepsilon)^3}\|\nabla v\|_{L^2(\Omega^\varepsilon)^3}\|w\|_{L^6(\Omega^\varepsilon)^3},\quad \forall u,v,w\in V^\varepsilon_{\mathrm{div}}.$$

Theorem 1. *Under the assumptions*

$$\begin{aligned}&f^\varepsilon(t),\partial_t f^\varepsilon(t)\in L^2(0,T;\,L^2(\Omega^\varepsilon)^3),\\&u_0\in H^1(\Omega^\varepsilon)^3,\quad (u_0)_\tau=0,\\&k^\varepsilon\in L^\infty(\omega),\quad k^\varepsilon>0.\end{aligned}\tag{14}$$

Problem (13) *admits a unique solution* $u^\varepsilon(t)\in L^2(0,T;\,V^\varepsilon_{\mathrm{div}})$, *such that*

$$\partial_t u^\varepsilon(t)\in L^2(0,T;\,V^\varepsilon_{\mathrm{div}}).$$

Proof. The proof is based on the regularization method, which is based on an approximation of nondifferentaible term $j^\varepsilon(\cdot)$ by a family of differentaible once $j^\varepsilon_\xi(\cdot)$, where

$$j^\varepsilon_\xi(v)=\int_\omega k^\varepsilon(x')\phi_\xi(|v|^2)\mathrm{d}x',\ \text{ with }\ \phi_\xi(v)=\frac{1}{1+\xi}\,|v|^{1+\xi},\quad \xi>0,$$

clearly $j^\varepsilon_\xi(\cdot)$ is convex and Gateaux differentiable with Gateaux derivative $(j_\xi)'$ defined on $V^\varepsilon_{\mathrm{div}}$ and given by

$$((j^\varepsilon_\xi)'(v),\varphi)=\int_\omega k^\varepsilon(x')\frac{|v|^{1+\xi}v.\varphi}{1+\xi}\mathrm{d}x'.$$

Since $j^\varepsilon_\xi(\cdot)$ is differentiable, adopting the classical arguments in [26], one can state that (13) is equivalent to
Find $u^\varepsilon_\xi(t)\in L^2(0,T;\,V^\varepsilon_{\mathrm{div}})$ with $\partial_t u^\varepsilon_\xi(t)\in L^2(0,T;\,V^\varepsilon_{\mathrm{div}})$, such that for almost all t

$$\begin{aligned}&(\rho\partial_t u^\varepsilon_\xi(t),\varphi)+\mu\mathcal{A}(u^\varepsilon_\xi(t),\varphi)+\nu\mathcal{A}(\partial_t u^\varepsilon_\xi(t),\varphi)+\mathcal{A}(\phi\star u^\varepsilon_\xi(t),\varphi)\\&\quad+\sqrt{\varepsilon}\mathcal{B}(u^\varepsilon_\xi(t),u^\varepsilon_\xi(t),\varphi)+((r^\varepsilon+\alpha^\varepsilon\,|u^\varepsilon_\xi(t)|^{p-2})u^\varepsilon_\xi(t),\varphi)+((j^\varepsilon_\xi)'(u^\varepsilon(t)),\varphi)\\&=(f^\varepsilon(t),\varphi),\quad \forall\varphi\in V^\varepsilon_{\mathrm{div}}.\end{aligned}\tag{15}$$

By using Galerkin's method, we show that there exists a unique solution $u^\varepsilon_\xi(t)$ of this last approximate problem (see the works of Duvaut and Lions [26]. Then, the limit of $u^\varepsilon_\xi(t)$ when ξ tends to zero is a solution of (13). □

5 Asymptotic analysis of the weak solution

For the asymptotic analysis of the problem (13), we introduce a small change of the variable $z=x_3/\varepsilon$. For (x',x_3) in Ω^ε, we have (x',z) in

$$\Omega=\{(x',z)\in\mathbb{R}^3:(x',0)\in\omega,\ 0<z<h(x')\},$$

and we denote by $\partial\bar{\Omega}=\partial\bar{\Omega}_1\cup\partial\bar{\Omega}_L\cup\bar{\omega}$ its boundary.

Following this scaling, we define the following unknown functions in Ω

$$\left.\begin{aligned}&\hat{u}_i^\varepsilon(x',z,t)=u_i^\varepsilon(x',x_3,t),\quad i=1,2,\\&\hat{u}_3^\varepsilon(x',z,t)=\varepsilon^{-1}u_3^\varepsilon(x',x_3,t),\\&\hat{\pi}^\varepsilon(x',z,t)=\varepsilon^2\pi^\varepsilon(x',x_3,t).\end{aligned}\right\}\tag{16}$$

For the data of problem (13), it is assumed that they depend on ε as follows:

$$\left.\begin{aligned}&\hat{f}(x',z,t)=\varepsilon^2 f^\varepsilon(x',x_3,t),\\&\hat{k}=\varepsilon k^\varepsilon,\\&\hat{a}=\varepsilon^2 a^\varepsilon,\quad \hat{r}=\varepsilon^2 r^\varepsilon,\end{aligned}\right\}\tag{17}$$

and $\hat{f}$, $\hat{k}$, $\hat{a}$, and $\hat{r}$ do not depend on ε.

We now introduce the functional framework on Ω as follows:

$$\begin{aligned}&V=\{\varphi\in H^1(\Omega)^3:\varphi=0\ \text{on}\ \partial\Omega_1\cup\partial\Omega_L\ \text{and}\ \varphi.n=0\ \text{on}\ \omega\},\\&V_{\mathrm{div}}=\{\varphi\in V:\ \mathrm{div}(\varphi)=0\},\\&\Pi(V)=\{\varphi\in H^1(\Omega)^2:\varphi=(\varphi_1,\varphi_2),\ \varphi_i=0\ \text{on}\ \partial\Omega_1\cup\partial\Omega_L,\ i=1,2\},\\&L_0^2(\Omega)=\left\{\varphi\in L^2(\Omega):\int_\Omega\varphi\,\mathrm{d}x'\mathrm{d}z=0\right\}.\end{aligned}$$

Now, let

$$V_z=\{v=(v_1,v_2)\in L^2(\Omega)^2:\partial_z v_i\in L^2(\Omega),\ i=1,2\ \text{and}\ v=0\ \text{on}\ \partial\Omega_1\},$$

be the Banach space with the norm

$$\|v\|_{V_z}=\left(\sum_{i=1}^{2}(\|v_i\|^2_{L^2(\Omega)}+\|\partial_z v_i\|^2_{L^2(\Omega)})\right)^{\frac12}.$$

From (16) and (17), we deduce that problem (13) is equivalent to the following variational inequality:

Find $(\hat{u}^\varepsilon,\hat{\pi}^\varepsilon)\in L^2(0,T;\,V_{\mathrm{div}})\times L^2(0,T;\,L_0^2(\Omega))$, such that

$$\begin{aligned}&\varepsilon^2\sum_{i=1}^{2}(\rho\partial_t\hat{u}_i^\varepsilon,\hat{\varphi}_i-\hat{u}_i^\varepsilon)+\varepsilon^4(\rho\partial_t\hat{u}_3^\varepsilon,\hat{\varphi}_3-\hat{u}_3^\varepsilon)+\mu\breve{\mathcal{A}}(u^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)+\nu\breve{\mathcal{A}}(\partial_t\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)\\&\quad\breve{\mathcal{A}}(\phi\star\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)+\sqrt{\varepsilon}\breve{\mathcal{B}}(\hat{u}^\varepsilon,\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)+\sum_{i=1}^{2}((\hat{r}+\hat{a}\,|\hat{u}_i^\varepsilon|^{p-2})\hat{u}_i^\varepsilon,\hat{\varphi}_i-\hat{u}_i^\varepsilon)\\&\quad+\varepsilon^2((\hat{r}+\hat{a}\varepsilon^{p-2}\,|\hat{u}_3^\varepsilon|^{p-2})\hat{u}_3^\varepsilon,\hat{\varphi}_3-\hat{u}_3^\varepsilon)-(\hat{\pi}^\varepsilon,\mathrm{div}(\hat{\varphi}))+\hat{J}(\hat{\varphi})-\hat{J}(\hat{u}^\varepsilon)\\&\quad\geq\sum_{i=1}^{2}(\hat{f}_i,\hat{\varphi}_i-\hat{u}_i^\varepsilon)+\varepsilon(\hat{f}_3,\hat{\varphi}_3-\hat{u}_3^\varepsilon),\quad\forall\hat{\varphi}\in V,\\&\hat{u}^\varepsilon(x',z,0)=\hat{u}_0,\quad\forall(x',z)\in\Omega,\end{aligned}\tag{18}$$

where

$$\begin{aligned}\hat{J}(\hat{\varphi})&=\int_\omega\hat{k}|\hat{\varphi}|\mathrm{d}x',\\\breve{\mathcal{A}}(\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)&=\varepsilon^2\sum_{i,j=1}^{2}\int_\Omega(\partial_{x_j}\hat{u}_i^\varepsilon).\partial_{x_j}(\hat{\varphi}_i-\hat{u}_i^\varepsilon)\mathrm{d}x'\mathrm{d}z+\sum_{i=1}^{2}\int_\Omega[\partial_z\hat{u}_i^\varepsilon.\partial_z(\hat{\varphi}_i-\hat{u}_i^\varepsilon)\\&\quad+\varepsilon^4\partial_{x_i}\hat{u}_3^\varepsilon.\partial_{x_i}(\hat{\varphi}_3-\hat{u}_3^\varepsilon)]\mathrm{d}x'\mathrm{d}z+\varepsilon^2\int_\Omega\partial_z\hat{u}_3^\varepsilon\partial_z(\hat{\varphi}_3-\hat{u}_3^\varepsilon)\mathrm{d}x'\mathrm{d}z,\\\breve{\mathcal{B}}(\hat{u}^\varepsilon,\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)&=\varepsilon^2\sum_{i,j=1}^{2}\int_\Omega\hat{u}_i^\varepsilon.\partial_{x_i}\hat{u}_j^\varepsilon.(\hat{\varphi}_j-\hat{u}_j^\varepsilon)\mathrm{d}x'\mathrm{d}z+\varepsilon^4\sum_{i=1}^{2}\int_\Omega\hat{u}_i^\varepsilon.\partial_{x_i}\hat{u}_3^\varepsilon.(\hat{\varphi}_3-\hat{u}_3^\varepsilon)\mathrm{d}x'\mathrm{d}z\\&\quad+\varepsilon^2\sum_{i=1}^{2}\int_\Omega\hat{u}_3^\varepsilon.\partial_z\hat{u}_i^\varepsilon.(\hat{\varphi}_i-\hat{u}_i^\varepsilon)\mathrm{d}x'\mathrm{d}z+\varepsilon^4\sum_{i=1}^{2}\int_\Omega\hat{u}_3^\varepsilon.\partial_z\hat{u}_3^\varepsilon.(\hat{\varphi}_3-\hat{u}_3^\varepsilon)\mathrm{d}x'\mathrm{d}z,\end{aligned}$$

and

$$(\hat{\pi}^\varepsilon, \operatorname{div}(\hat{\varphi})) = \sum_{j=1}^{2} \int_\Omega \hat{\pi}^\varepsilon.\partial_{x_j}\hat{\varphi}_j \mathrm{d}x'\mathrm{d}z + \int_\Omega \hat{\pi}^\varepsilon.\partial_z\hat{\varphi}_3 \mathrm{d}x'\mathrm{d}z.$$

5.1 Uniform estimates

First, we will obtain estimates on $\hat{u}^\varepsilon$ and $\partial_t\hat{u}^\varepsilon$.

Theorem 2. *Let $(u^\varepsilon, \pi^\varepsilon) \in L^2(0,T;\, V^\varepsilon_{\mathrm{div}}) \times L^2(0,T;\, L^2_0(\Omega^\varepsilon))$ be a solution to the problem* (13), *where $k^\varepsilon \in L^\infty_+(\omega)$ and $u_0 \in L^{2(p-1)}(\Omega)^3 \cap H^1(\Omega)^3$. There exists a constant c independent of ε such that*

$$\begin{aligned}&\sum_{i=1}^{2}(\|\partial_z\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)} + \|\varepsilon^2\partial_{x_i}\hat{u}_3^\varepsilon\|^2_{L^2(\Omega)}) + \sum_{i,j=1}^{2}\|\varepsilon\partial_{x_j}\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)} + \|\varepsilon\partial_z\hat{u}_3^\varepsilon\|^2_{L^2(\Omega)}\\ &\quad + \sum_{i=1}^{2}\|\hat{u}_i^\varepsilon\|^p_{L^p(0,T;\,L^p(\Omega))} + \|\varepsilon\hat{u}_3^\varepsilon\|^p_{L^p(0,T;\,L^p(\Omega))} \le c,\end{aligned} \tag{19}$$

$$\begin{aligned}&\sum_{i=1}^{2}(\|\partial_z(\partial_t\hat{u}_i^\varepsilon)\|^2_{L^2(\Omega)} + \|\varepsilon^2\partial_{x_i}(\partial_t\hat{u}_3^\varepsilon)\|^2_{L^2(\Omega)}) + \sum_{i,j=1}^{2}\|\varepsilon\partial_{x_j}(\partial_t\hat{u}_i^\varepsilon)\|^2_{L^2(\Omega)}\\ &\quad + \|\varepsilon\partial_z(\partial_t\hat{u}_3^\varepsilon)\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\|\varepsilon\partial_t\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)} + \|\varepsilon^2\partial_t\hat{u}_3^\varepsilon\|^2_{L^2(\Omega)} \le c.\end{aligned} \tag{20}$$

Proof. Choosing $\varphi = (0,0,0)$ in (13), we have

$$\begin{aligned}&(\rho\partial_t u^\varepsilon, u^\varepsilon) + \mu\mathcal{A}(u^\varepsilon, u^\varepsilon) + \nu\mathcal{A}(\partial_t u^\varepsilon, u^\varepsilon) + \sqrt{\varepsilon}\mathcal{B}(u^\varepsilon, u^\varepsilon, u^\varepsilon) + \mathcal{A}(\phi\star u^\varepsilon, u^\varepsilon) + ((r^\varepsilon + \alpha^\varepsilon\,|\hat{u}^\varepsilon|^{p-2})u^\varepsilon, u^\varepsilon) + j^\varepsilon(u^\varepsilon)\\ &\quad \le (f^\varepsilon, u^\varepsilon),\end{aligned}$$

by using the identity $\mathcal{B}(u^\varepsilon, u^\varepsilon, u^\varepsilon) = 0$, and the fact that $j^\varepsilon(u^\varepsilon) \ge 0$, we deduce that

$$\begin{aligned}&\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\mathcal{A}(u^\varepsilon(t), u^\varepsilon(t))) + \mu\mathcal{A}(u^\varepsilon(t), u^\varepsilon(t)) + \mathcal{A}(\phi\star u^\varepsilon(t), u^\varepsilon(t)) + ((r^\varepsilon + \alpha^\varepsilon\,|\hat{u}^\varepsilon(t)|^{p-2})u^\varepsilon(t), u^\varepsilon(t))\\ &\quad \le \|f^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}\|u^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}.\end{aligned}$$

Now, integration from 0 to t, we obtain

$$\begin{aligned}&\frac{\rho}{2}\,\|u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + \mu\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s + \nu\,\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\\ &\quad + \int_0^t\mathcal{A}((\phi * u^\varepsilon)(s), u^\varepsilon(s))\mathrm{d}s + \alpha^\varepsilon\int_0^t\|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s\\ &\le \frac{\rho}{2}\,\|u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\,\|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \int_0^t\|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\|u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s,\end{aligned}$$

since $\phi(t)$ is a positive kernel, we obtain

$$\begin{aligned}&\frac{\rho}{2}\,\|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \mu\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s + \nu\,\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \alpha^\varepsilon\int_0^t\|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s\\ &\le \frac{\rho}{2}\,\|u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\,\|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \int_0^t\|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\|u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s.\end{aligned} \tag{21}$$

On the other hand, we have by Poincaré's inequality (1) and Young's inequality (2)

$$\int_0^t \|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\|u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s \leq \frac{1}{\mu}(\varepsilon\bar{h})^2\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s + \mu\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\mathrm{d}s.$$

Thus, the inequality (21) becomes

$$\frac{\nu}{2}\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \alpha^\varepsilon\int_0^t \|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s \leq \frac{1}{\mu}(\varepsilon\bar{h})^2\|f^\varepsilon\|^2_{L^2(0,T;L^2(\Omega^\varepsilon)^3)} + \frac{1}{2}\|u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{1}{2}\nu\|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}. \tag{22}$$

Now, since

$$\begin{aligned}\varepsilon^2\|f^\varepsilon\|^2_{L^2(0,T;\,L^2(\Omega^\varepsilon)^3)} &= \varepsilon^{-1}\|\hat{f}\|^2_{L^2(0,T;\,L^2(\Omega)^3)},\\ \|\partial_{x_3}u_i^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} &= \varepsilon^{-1}\|\partial_z\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)}, \quad i = 1,2,\end{aligned}$$

we multiply the inequality (22) by ε, and we deduce that

$$\frac{\nu}{2}[\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}] + \hat{\alpha}\left[\frac{1}{\varepsilon}\|u^\varepsilon\|^p_{L^p(0,T;\,L^p(\Omega^\varepsilon)^3)}\right] \leq A, \tag{23}$$

where

$$A = \frac{1}{\mu}\bar{h}^2\|\hat{f}\|^2_{L^2(0,T;\,L^2(\Omega)^3)} + \frac{1}{2}\hat{\alpha}\|\hat{u}_0\|^2_{L^2(\Omega)^3} + \frac{1}{2}\nu\|\nabla\hat{u}_0\|^2_{L^2(\Omega)^{3\times3}}.$$

From (23), we find (19) after the change of variables.

To find the estimate (20), we differentiate (15), respect to t and we take $\varphi = \partial_t u_\xi^\varepsilon$, we obtain

$$\begin{aligned}&\rho(\partial_t^2u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) + \mu\mathcal{A}(\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) + \nu\mathcal{A}(\partial_t^2u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) + \mathcal{A}(\phi' * u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t))\\ &\quad + \phi(0)(\nabla u_\xi^\varepsilon(t),\nabla\partial_tu_\xi^\varepsilon(t)) + \sqrt{\varepsilon}\mathcal{B}(\partial_tu_\xi^\varepsilon(t),u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) + \sqrt{\varepsilon}\mathcal{B}(u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t))\\ &\quad + ((r^\varepsilon + (p-1)\alpha^\varepsilon|u_\xi^\varepsilon(t)|^{p-2})\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) + (\partial_t(j_\xi^\varepsilon)'(u_\xi^\varepsilon),\partial_tu_\xi^\varepsilon(t))\\ &= (\partial_tf^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)),\end{aligned}$$

as $\mathcal{B}(u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t)) = 0$, and $\mathcal{A}(\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t))$, $((r^\varepsilon + (p-1)\alpha^\varepsilon|u_\xi^\varepsilon(t)|^{p-2})\partial_tu_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t))$, and $((j_\xi^\varepsilon)'(u_\xi^\varepsilon(t)),\partial_tu_\xi^\varepsilon(t))$ are positive terms, we have

$$\begin{aligned}&\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_tu_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\|\nabla\partial_tu_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}) + \int_0^t(\phi'(t-s)\nabla u_\xi^\varepsilon(s),\partial_tu_\xi^\varepsilon(t))\mathrm{d}s\\ &\quad + \phi(0)(\nabla u_\xi^\varepsilon(t),\nabla(\partial_tu_\xi^\varepsilon(t))) + \sqrt{\varepsilon}\mathcal{B}(\partial_tu_\xi^\varepsilon(t),u_\xi^\varepsilon(t),\partial_tu_\xi^\varepsilon(t))\\ &\leq \|\partial_tf^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}\|\partial_tu_\xi^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}.\end{aligned} \tag{24}$$

By simple calculation, we have

$$\begin{aligned}&\int_0^t\phi'(t-s)\int_{\Omega^\varepsilon}\nabla u^\varepsilon(s).\nabla\partial_tu^\varepsilon(t)\mathrm{d}x\mathrm{d}s\\ &\quad = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\phi'(t-s)\|\nabla u^\varepsilon(s)-\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\right] + \int_0^t\phi''(t-s)\|\nabla u^\varepsilon(s)-\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\\ &\qquad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\phi'(s)\mathrm{d}s.\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\right] - \frac{1}{2}\phi'(t)\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}.\end{aligned} \tag{25}$$

By combining (21) and (25), we obtain

$$\begin{aligned}&\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\|\partial_t u_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\mathcal{A}(\partial_t u_\xi^\varepsilon(t), \partial_t u_\xi^\varepsilon(t)) + \frac{1}{2}\int_0^t \phi'(s)\mathrm{d}s.\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \frac{\phi(0)}{2}\|\nabla\partial_t u_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3}\\
&\quad - \frac{1}{2}\int_0^t \phi'(t-s)\|\nabla u^\varepsilon(s) - \nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\Bigg] + \int_0^t \phi''(t-s)\|\nabla u^\varepsilon(s) - \nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\\
&\quad - \frac{1}{2}\phi'(t)\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \sqrt{\varepsilon}\mathcal{B}(\partial_t u_\xi^\varepsilon(t), u_\xi^\varepsilon(t), \partial_t u_\xi^\varepsilon(t))\\
&\le \|\partial_t f^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}.\|\partial_t u_\xi^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3},\end{aligned}$$

and using the fact that $\phi'(t) < 0$ and $\phi''(t) > 0$ leads to

$$\begin{aligned}&\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\|\partial_t u_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0))\|\nabla\partial_t u_\xi^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{1}{2}\phi(t).\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\\
&\quad - \frac{1}{2}\phi(0).\|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \frac{1}{2}\int_0^t \phi'(t-s)\|\nabla u^\varepsilon(s) - \nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\Bigg] + \sqrt{\varepsilon}\mathcal{B}(\partial_t u_\xi^\varepsilon(t), u_\xi^\varepsilon(t), \partial_t u_\xi^\varepsilon(t))\\
&\le \|\partial_t f^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}.\|\partial_t u_\xi^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)^3}.\end{aligned}$$

Then, by integrating this inequality over $(0, t)$, we find

$$\begin{aligned}&\frac{\rho}{2}\|\partial_t u_\xi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0))\|\nabla\partial_t u_\xi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} - \frac{1}{2}\int_0^t \phi'(t-s)\|\nabla u^\varepsilon(s) - \nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\mathrm{d}s\\
&\le \frac{(\varepsilon\bar{h})^2}{\nu}\int_0^t \|\partial_t f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s + \nu\int_0^t \|\nabla\partial_t u_\xi^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\mathrm{d}s + \frac{1}{2}\phi(0).\|\nabla u_0\|^2_{L^2(\Omega^\varepsilon)}\\
&\quad + \frac{\rho}{2}\|\partial_t u_\xi^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0))\|\nabla\partial_t u_\xi^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \int_0^t \sqrt{\varepsilon}|\mathcal{B}(\partial_t u_\xi^\varepsilon(s), u_\xi^\varepsilon(s), \partial_t u_\xi^\varepsilon(s))|\mathrm{d}s.\end{aligned}\tag{26}$$

Next, we estimate $|\mathcal{B}(\partial_t u_\xi^\varepsilon(s), u_\xi^\varepsilon(s), \partial_t u_\xi^\varepsilon(s))|$ using Cauchy-Schwartz and Sobolev's inequalities, and we obtain

$$\begin{aligned}&\sqrt{\varepsilon}\int_0^t |\mathcal{B}(\partial_t u_\xi^\varepsilon(s), u_\xi^\varepsilon(s), \partial_t u_\xi^\varepsilon(s))|\mathrm{d}s\\
&\le \sqrt{\varepsilon}\int_0^t \|\partial_t u_\xi^\varepsilon(s)\|_{L^3(\Omega^\varepsilon)^3}\|\nabla u_\xi^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\|\partial_t u_\xi^\varepsilon(s)\|_{L^6(\Omega^\varepsilon)^3}\mathrm{d}s\\
&\le c_s\int_0^t \sqrt{\varepsilon}\,\|\nabla u_\xi^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^{3\times 3}}\|\nabla\partial_t u_\xi^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\mathrm{d}s,\end{aligned}$$

By the aforementioned estimates, the inequality (26) leads to

$$\begin{aligned}&\frac{\rho}{2}\|\partial_t u_\xi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0))\|\nabla\partial_t u_\xi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\\
&\le \frac{(\varepsilon\bar{h})^2}{\nu}\int_0^t \|\partial_t f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s + \int_0^t (\nu + c_s\sqrt{\varepsilon}\,\|\nabla u_\xi^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^{3\times 3}})\|\nabla\partial_t u_\xi^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\mathrm{d}s\\
&\quad + \left(\frac{\rho}{2} + \nu + \phi(0)\right)[\|\partial_t u_\xi^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)^3} + \|\nabla\partial_t u_\xi^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}].\end{aligned}\tag{27}$$

It remains for us to estimate $\partial_t u_\xi^\varepsilon(0)$. From (14) and (15), we deduce

$$\begin{aligned}&\rho(\partial_t u_\xi^\varepsilon(0), \varphi) + \nu\mathcal{A}(\partial_t u_\xi^\varepsilon(0), \varphi) + \mu\mathcal{A}(u_\xi^\varepsilon(0), \varphi) + \sqrt{\varepsilon}\mathcal{B}(u_\xi^\varepsilon(0), u_\xi^\varepsilon(0), \varphi) + \alpha^\varepsilon(|u_\xi^\varepsilon(0)|^{p-2}u_\xi^\varepsilon(0), \varphi)\\
&= (f^\varepsilon(0), \varphi), \quad \text{for all } \varphi \in H_0^1(\Omega^\varepsilon)^3.\end{aligned}$$

By using Cauchy-Schwarz's and Poincaré's inequalities (1), we obtain

$$\begin{aligned}
&|\rho(\partial_t u_\xi^\varepsilon(0),\varphi) + \nu\mathcal{A}(\partial_t u_\xi^\varepsilon(0),\varphi)| \\
&\le \mu \|\nabla u_0^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}} + c_s\sqrt{\varepsilon}\ \|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}} \\
&\quad + \frac{\hat{a}\bar{h}}{\sqrt{\varepsilon}}\left(\int_\Omega |\hat{u}_0|^{2(p-1)}\mathrm{d}x'\mathrm{d}z\right)^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^3} + \varepsilon\bar{h}\ \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)^3}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle\partial_t u_\xi^\varepsilon(0),\varphi\rangle_{H^1(\Omega^\varepsilon)^3}| &\le (\mu \|\nabla u_0^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}} + \hat{a}\bar{h}^2\ \|\nabla u_0^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}} + c_s\sqrt{\varepsilon}\ \|\nabla u_0^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}})\|\varphi\|_{H^1(\Omega^\varepsilon)^3} \\
&\quad + \left(\frac{\hat{a}\bar{h}}{\sqrt{\varepsilon}}\left(\int_\Omega |\hat{u}_0|^{2(p-1)}\mathrm{d}x'\mathrm{d}z\right)^{\frac{1}{2}} + \varepsilon\bar{h}\ \|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)^3}\right)\|\varphi\|_{H^1(\Omega^\varepsilon)^3}.
\end{aligned}$$

We multiply this inequality by $\sqrt{\varepsilon}$, we obtain

$$\sqrt{\varepsilon}\ \|\partial_t u_\xi^\varepsilon(0)\|_{H^1(\Omega^\varepsilon)^3} \le c_0,$$

where $c_0 = (\mu + \hat{a}\bar{h}^2)\|\hat{u}_0\|_{H^1(\Omega)^3} + c_s(\hat{a}\bar{h}+1)\|\hat{u}_0\|^2_{H^1(\Omega)^3} + \bar{h}\ \|\hat{f}(0)\|_{L^2(\Omega)^3}$ is a constant independent of ε.

Now, passing to the limit in (27) when ξ tends to zero, we deduce that

$$\begin{aligned}
&\frac{\rho}{2}\ \|\partial_t u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0))\|\nabla\partial_t u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} \\
&\le \int_0^t \left[\frac{1}{2}\ \|\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0) + c_s\sqrt{\varepsilon}\ \|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^{3\times3}})\|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]\mathrm{d}s \\
&\quad + \frac{(\varepsilon\bar{h})^2}{\nu}\ \|\partial_t f^\varepsilon\|^2_{L^2(0,T;L^2(\Omega^\varepsilon)^3)} + \rho\left(\frac{1}{2} + \nu + \phi(0)\right)(c_0)^2.
\end{aligned}$$

We multiply this inequality by ε and use the fact that $\sqrt{\varepsilon}\ \|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^{3\times3}} \le c_1$, and we find

$$\begin{aligned}
&\varepsilon\left[\frac{1}{2}\ \|\partial_t u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \nu\ \|\nabla\partial_t u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right] \\
&\le \int_0^t \varepsilon\left[\frac{1}{2}\ \|\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3} + (\nu + \phi(0) + c_1.c_s)\|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]\mathrm{d}s + B,
\end{aligned}$$

where $B = \frac{\bar{h}^2}{\nu}\ \|\partial_t\hat{f}\|^2_{L^2(0,T;L^2(\Omega)^3)} + \rho\left(\frac{1}{2} + \nu + \phi(0)\right)(c_0')^2$ is a constant independent of ε.

By using the Gronwall lemma, we complete the proof. □

Now, we are looking for *a priori* estimates on the pressure $\hat{\pi}^\varepsilon$. For this, we need to establish the following result.

Theorem 3. *Under the hypotheses of Theorem 2, there exists a constant c independent of ε such that*

$$\|\partial_{x_1}\hat{\pi}^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le c, \tag{28}$$

$$\|\partial_{x_2}\hat{\pi}^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le c, \tag{29}$$

$$\|\partial_z\hat{\pi}^\varepsilon\|_{L^2(0,T;\ H^{-1}(\Omega))} \le \varepsilon.\ c. \tag{30}$$

Proof. Let ψ in $L^2(0,T;\ H_0^1(\Omega^\varepsilon)^3)$, putting in (13), $\varphi = u^\varepsilon + \psi$, we deduce

$$\begin{aligned}
(\pi^\varepsilon,\ \mathrm{div}(\psi)) &= \rho(\partial_t u^\varepsilon,\psi) + \mu\mathcal{A}(u^\varepsilon,\psi) + \nu\mathcal{A}(\partial_t u^\varepsilon,\psi) + \mathcal{A}(\phi\star u^\varepsilon,\psi) \\
&\quad + \sqrt{\varepsilon}\mathcal{B}(u^\varepsilon,u^\varepsilon,\psi) + (r^\varepsilon + \alpha^\varepsilon\ |u^\varepsilon|^{p-2}u^\varepsilon,\psi) + (f^\varepsilon,\psi).
\end{aligned}$$

After using Cauchy-Schwarz, Poincaré, Sobolev's, and Young's inequalities, one obtains

$$\begin{aligned}
&\left|\int_0^t (\pi^\varepsilon, \operatorname{div}(\psi))\mathrm{d}s\right| \\
&\quad \leq \left(\mu + \hat{r}\bar{h}^2 + \left(\int_0^t \phi(s)\mathrm{d}s\right)^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + ((\rho\bar{h})^2 + \nu)\int_0^t \|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s \\
&\qquad + \frac{(\varepsilon\bar{h})^2}{\mu+\nu}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s + (1 + 2\mu + 2\nu + \hat{r}\bar{h}^2)\int_0^t \|\nabla\psi(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s \\
&\qquad + c_s\int_0^t \sqrt{\varepsilon}\ \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\psi(s)\|_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s + \alpha^\varepsilon\int_0^t (|u^\varepsilon(s)|^{p-2}u^\varepsilon(s), \psi(s))\mathrm{d}s.
\end{aligned} \tag{31}$$

The last term of (31) can be estimated by

$$\begin{aligned}
&\alpha^\varepsilon\int_0^t\int_{\Omega^\varepsilon} ||u^\varepsilon(s)|^{p-2}u^\varepsilon(s)\psi|\mathrm{d}x'\mathrm{d}x_3\mathrm{d}s \\
&\quad \leq \left|\frac{\hat{\alpha}}{\varepsilon^2}\int_0^t\left(\int_{\Omega^\varepsilon}|u^\varepsilon(s)|^p\mathrm{d}x'\mathrm{d}x_3\right)^{\frac{1}{q}}\left(\int_{\Omega^\varepsilon}|\psi|^p\mathrm{d}x'\mathrm{d}x_3\right)^{\frac{1}{p}}\mathrm{d}s\right|, \qquad \frac{1}{p}+\frac{1}{q} = 1 \\
&\quad \leq \frac{\hat{\alpha}}{\varepsilon}\int_0^t \frac{1}{\varepsilon}\ \|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s \\
&\quad \leq \frac{1}{\varepsilon}c'' + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\left|\int_0^t (\pi^\varepsilon, \operatorname{div}(\psi))\mathrm{d}s\right| \\
&\quad \leq \left(\mu + \hat{r}\bar{h}^2 + \left(\int_0^t \phi(s)\mathrm{d}s\right)^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + ((\rho\bar{h})^2 + \nu)\int_0^t \|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s \\
&\qquad + \frac{(\varepsilon\bar{h})^2}{\mu+\nu}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\mathrm{d}s + (1 + 2\mu + 2\nu + c_s)\int_0^t \|\nabla\psi(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s \\
&\qquad + c_s\int_0^t \varepsilon\ \|\nabla u^\varepsilon(s)\|^4_{L^2(\Omega^\varepsilon)^{3\times3}}\mathrm{d}s + \frac{1}{\varepsilon}c'' + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^p_{L^p(\Omega^\varepsilon)^3}\mathrm{d}s.
\end{aligned} \tag{32}$$

Now, we choose $\psi = (\psi_1, 0, 0)$, $\psi = (0, \psi_2, 0)$, and $\psi = (0, 0, \psi_3)$, respectively, in (32), then change variables, to obtain (28)–(30). □

5.2 The limit problem

The objective of this paragraph is to find the limit behavior of the sequences $(\hat{u}^\varepsilon)$ and $(\hat{\pi}^\varepsilon)$ as $\varepsilon \to 0$. We will demonstrate the weak convergence of these sequences and give the limit problem that characterizes these limits.

Lemma 2. *Under the hypotheses of Theorems* 2 *and* 3, *there exists* $u_i^* \in L^2(0,T;\ V_z), i = 1,2$ *and* $\pi^* \in L^2(0,T;\ L_0^2(\Omega))$ *such that*

$$\begin{aligned} &\hat{u}_i^\varepsilon \rightharpoonup u_i^*, \quad i = 1,2, \ \text{weakly in } L^2(0,T;\ V_z) \\ &\partial_t \hat{u}_i^\varepsilon \rightharpoonup \partial_t u_i^*, \quad i = 1,2, \ \textit{weakly in } L^2(0,T;\ V_z), \end{aligned} \tag{33}$$

$$|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon \rightharpoonup |u_i^*|^{p-2}u_i^*, \quad i = 1,2, \ \textit{weakly in } L^q(0,T;\ L^q(\Omega)), \tag{34}$$

$$\varepsilon\partial_{x_j}\hat{u}_i^\varepsilon \rightharpoonup 0, \ \varepsilon\partial_{x_j}(\partial_t\hat{u}_i^\varepsilon) \rightharpoonup 0, \quad i,j = 1,2, \ \textit{weakly in } L^2(0,T;\ L^2(\Omega)), \tag{35}$$

$$\varepsilon^2\partial_{x_i}\hat{u}_3^\varepsilon \rightharpoonup 0, \ \varepsilon\partial_{x_i}(\partial_t\hat{u}_3^\varepsilon) \rightharpoonup 0, \quad i = 1,2, \ \textit{weakly in } L^2(0,T;\ L^2(\Omega)), \tag{36}$$

$$\varepsilon\partial_z\hat{u}_3^\varepsilon \rightharpoonup 0, \varepsilon\partial_z(\partial_t\hat{u}_3^\varepsilon) \rightharpoonup 0, \quad \textit{weakly in } L^2(0,T;\ L^2(\Omega)), \tag{37}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0, \ \varepsilon\partial_t\hat{u}_3^\varepsilon \rightharpoonup 0, \quad \textit{weakly in } L^2(0,T;\ L^2(\Omega)), \tag{38}$$

$$\hat{\pi}^\varepsilon \rightharpoonup \pi^*, \quad \text{weakly } \textit{in } L^2(0,T;\ L_0^2(\Omega)). \tag{39}$$

Proof. By Theorem 2, there exists a constant C_T independent of ε such that

$$\|\partial_z\hat{u}_i^\varepsilon\|^2_{L^2(0,T;\ L^2(\Omega))} \le C_T, \quad \|\partial_z(\partial_t\hat{u}_i^\varepsilon)\|^2_{L^2(0,T;L^2(\Omega))} \le C_T, \quad i = 1,2.$$

Thus, by using these estimates with Poincaré's inequality in the domain Ω ($\|\hat{u}_i^\varepsilon\|_{L^2(0,T;\ L^2(\Omega))} \le \bar{h}\ \|\partial_z\hat{u}_i^\varepsilon\|_{L^2(0,T;L^2(\Omega))}$), we obtain (33). We obtain (35)–(36) according to (19), (20), and (33). For (34), we have

$$\int_0^T\int_\Omega (||\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon|)^q \mathrm{d}x'\mathrm{d}z\mathrm{d}s \le \int_0^T \|\hat{u}_i^\varepsilon(s)\|^p_{L^p(\Omega)}\mathrm{d}s \le C,$$

so $|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon \rightharpoonup |u_i^*|^{p-2}u_i^*, i = 1,2$ a.e in $L^q(0,T;\ L^q(\Omega))$. Because $\operatorname{div}(\hat{u}^\varepsilon) = 0$, by (20) and with a particular choice of the test function, we obtain (37) and (38) [19]. To find (39), we use (28)–(30) and the following inequality:

$$\|\hat{\pi}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le c'\ \|\nabla\hat{\pi}^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega)^3)}. \qquad \square$$

Theorem 4. *The weak limit* (u^*, π^*) *satisfies the following properties*:

$$\pi^*(x', z, t) = \pi^*(x', t), \quad \textit{a.e in } \Omega, \tag{40}$$

$$\int_\Omega \pi^*(\partial_{x_1}u_1^* + \partial_{x_2}u_2^*)\mathrm{d}x'\mathrm{d}z = 0. \tag{41}$$

Also (u^*, π^*) *is characterized as solution of the variational inequality and the limit problem*

$$\begin{aligned} &\sum_{i=1}^2\int_\Omega \mu\partial_z u_i^*.\partial_z(\hat{\varphi}_i - u_i^*)\mathrm{d}x'\mathrm{d}z + \sum_{i=1}^2\int_\Omega \nu\partial_z(\partial_t u_i^*).\partial_z(\hat{\varphi}_i - u_i^*)\mathrm{d}x'\mathrm{d}z \\ &\quad + \sum_{i=1}^2\int_\Omega\int_0^t \phi(t-s)\partial_z u_i^*(s)\mathrm{d}s.\partial_z(\hat{\varphi}_i - u_i^*)\mathrm{d}x'\mathrm{d}z + \sum_{i=1}^2\int_\Omega (\hat{r} + \hat{\alpha}\ |u_i^*|^{p-2})u_i^*.(\hat{\varphi}_i - u_i^*)\mathrm{d}x'\mathrm{d}z \\ &\quad - \int_\Omega \pi^*(\partial_{x_1}\hat{\varphi}_1 + \partial_{x_2}\hat{\varphi}_2)\mathrm{d}x'\mathrm{d}z + \int_\omega \hat{k}(|\hat{\varphi}| - |u^*|)\mathrm{d}x' \\ &\ge \sum_{i=1}^2\int_\Omega \hat{f}_i.(\hat{\varphi}_i - u_i^*)\mathrm{d}x'\mathrm{d}z, \quad \forall\hat{\varphi} \in \Pi(V), \end{aligned} \tag{42}$$

$$\begin{aligned} &-\partial_z^2[\mu u_i^* + \nu\partial_t u_i^* + \phi\star u_i^*](t) + (\hat{r} + \hat{\alpha}\ |u_i^*|^{p-2})u_i^*(t) + \partial_{x_i}\pi^*(t) \\ &= \hat{f}_i(t), \quad i = 1,2 \ \textit{a.e on } \Omega \times (0,T). \end{aligned} \tag{43}$$

Proof. By choosing $\varphi_i = \hat{u}_i^\varepsilon$ $(i = 1, 2)$, $\varphi_3 = \hat{u}_3^\varepsilon \pm \psi$, where $\psi \in L^2(0, T; H_0^1(\Omega))$ in (18), and using the convergence results (33)–(39), we deduce

$$\int_0^t (\pi^*(s), \partial_z \psi(s)) \mathrm{d}s = -\int_0^t \int_{\Omega^\varepsilon} \frac{\partial \pi^*(s)}{\partial z} \psi(s) \mathrm{d}x' \mathrm{d}z \mathrm{d}s = 0, \quad \forall \psi \in L^2(0, T; H_0^1(\Omega)).$$

Therefore π^* depends only on x'.

Let $\theta \in \mathcal{D}(]0, T[\times\omega)$, and by using the fact $\operatorname{div}(\hat{u}^\varepsilon) = 0$ in Ω, we have

$$\begin{aligned} &\int_0^t \int_\Omega \theta(x', s)(\partial_{x_1} \hat{u}_1^\varepsilon + \partial_{x_2} \hat{u}_2^\varepsilon + \partial_z \hat{u}_3^\varepsilon) \mathrm{d}x' \mathrm{d}z \mathrm{d}s \\ &\quad = \int_0^t \int_\Omega \theta(x', s)(\partial_{x_1} \hat{u}_1^\varepsilon + \partial_{x_2} \hat{u}_2^\varepsilon) \mathrm{d}x' \mathrm{d}z \mathrm{d}s \\ &\quad = 0. \end{aligned}$$

As $\hat{u}_i^\varepsilon \rightharpoonup u_i^*$, $i = 1, 2$ in $L^2(0, T; V_z)$, we obtain

$$\int_0^t \int_\Omega \theta(x', s)(\partial_{x_1} u_1^* + \partial_{x_2} u_2^*) \mathrm{d}x' \mathrm{d}z \mathrm{d}s = 0. \tag{44}$$

Now, using (40), π^* is now in $L^2(0, T; L^2(w))$, and this means that there exists (θ_m) in $\mathcal{D}(]0, T[\times\omega)$ such that $\theta_m \to \pi^*$ in $L^2(0, T; L^2(w))$, and then from (44), we obtain (41) when $m \to \infty$.

By passing the limit in (18), using the result of Lemma 2, and the fact that $\hat{J}(\cdot)$ is convex and lower semi-continuous, we obtain

$$\begin{aligned} &\sum_{i=1}^2 \int_\Omega \mu \partial_z u_i^* . \partial_z(\hat{\varphi}_i - u_i^*) \mathrm{d}x' \mathrm{d}z + \sum_{i=1}^2 \int_\Omega \nu \partial_z(\partial_t u_i^*) . \partial_z(\hat{\varphi}_i - u_i^*) \mathrm{d}x' \mathrm{d}z \\ &\sum_{i=1}^2 \int_\Omega \int_0^t \phi(t-s) \star \partial_z u_i^*(s) \mathrm{d}s . \partial_z(\hat{\varphi}_i - u_i^*) \mathrm{d}x' \mathrm{d}z + \sum_{i=1}^2 \int_\Omega (\hat{r} + \hat{\alpha}\, |u_i^*|^{p-2}) u_i^* (\hat{\varphi}_i - u_i^*) \mathrm{d}x' \mathrm{d}z \\ &\qquad - \int_\Omega \pi^* (\partial_{x_1} \hat{\varphi}_1 + \partial_{x_2} \hat{\varphi}_2) \mathrm{d}x' \mathrm{d}z + \int_\omega \hat{k}(|\hat{\varphi}| - |u^*|) \mathrm{d}x' \\ &\qquad \geq \sum_{i=1}^2 \int_\Omega \hat{f}_i . (\hat{\varphi}_i - u_i^*) \mathrm{d}x' \mathrm{d}z, \quad \forall \hat{\varphi} \in \Pi(V). \end{aligned}$$

Now, by choosing the aforementioned variational inequality,

$$\hat{\varphi}_i = u_i^* \pm w_i, \quad \text{where } w_i \in H_0^1(\Omega), i = 1, 2,$$

We find

$$\begin{aligned} &\sum_{i=1}^2 \int_\Omega [\mu \partial_z u_i^* + \nu \partial_z(\partial_t u_i^*) + \phi \star \partial_z u_i^*] . \partial_z w_i \mathrm{d}x' \mathrm{d}z + \sum_{i=1}^2 \int_\Omega (\hat{r} + \hat{\alpha}\, |u_i^*|^{p-2}) u_i^* . w_i \mathrm{d}x' \mathrm{d}z - \sum_{i=1}^2 \int_\Omega \pi^* \partial_{x_i} w_i \mathrm{d}x' \mathrm{d}z \\ &\quad = \sum_{i=1}^2 \int_\Omega \hat{f}_i w_i \mathrm{d}x' \mathrm{d}z. \end{aligned}$$

Using the Green's formula and choosing $w_1 = 0$ and $w_2 \in H_0^1(\Omega)$, then $w_2 = 0$ and $w_1 \in H_0^1(\Omega)$, we obtain

$$\begin{aligned} &- \int_\Omega \partial_z [\mu \partial_z u_i^*(t) + \nu \partial_z(\partial_t u_i^*(t)) + \phi \star \partial_z u_i^*(t)] w_i \mathrm{d}x' \mathrm{d}z \\ &\quad + \sum_{i=1}^2 \int_\Omega (\hat{r} + \hat{\alpha}\, |u_i^*|^{p-2}) u_i^*(t) w_i \mathrm{d}x' \mathrm{d}z + \int_\Omega \partial_{x_i} \pi^*(t) w_i \mathrm{d}x' \mathrm{d}z \\ &\quad = \int_\Omega \hat{f}_i(t) w_i \mathrm{d}x' \mathrm{d}z, \quad \forall w_i \in H_0^1(\Omega),\ i = 1, 2. \end{aligned}$$

Thus

$$\begin{aligned}&-\partial_z^2[\mu u_i^*(t)+\nu\partial_t u_i^*(t)+\phi\star u_i^*(t)]+(\hat r+\hat\alpha\,|u_i^*|^{p-2})u_i^*(t)+\partial_{x_i}\pi^*(t)\\&=\hat f_i(t),\quad \text{in } H^{-1}(\Omega),\ \text{for } i=1,2.\end{aligned}\tag{45}$$

As $\hat f_i(t)\in L^2(\Omega)$, then (45) is valid in $L^2(\Omega)$. □

5.3 A specific Reynolds equation and the limit boundary conditions

Theorem 5. *Let*

$$\tau^*(x',t)=\partial_z u^*(x',0,t)\quad \textit{and}\quad \ell^*(x',t)=u^*(x',0,t),$$

with the traces of the velocity u^* on ω. Under the assumptions of Lemma 3.5, the solution u^* and π^* satisfy the weak generalized of Reynolds equation:

$$\begin{aligned}&\int_\omega\left(\frac{h^3(x')}{12}\nabla\pi^*-\frac{h(x')}{2}(\mu\ell^*+\nu\partial_t\ell^*+\phi\star\ell^*)+\int_0^h(\mu u^*+\nu\partial_t u^*+\phi\star u^*)(x',\zeta,t)\mathrm{d}\zeta+\tilde F+\tilde U\right)\nabla\psi\mathrm{d}x'\\&=0,\quad \forall\psi\in H^1(\omega)^2,\end{aligned}\tag{46}$$

with

$$\tilde F(x',h,t)=\int_0^h F(x',z,t)\mathrm{d}z-\frac{h}{2}F(x',h,t),\quad \tilde U(x,h,t)=-\int_0^h U(x,z,t)\mathrm{d}z+\frac{\hat\alpha h}{2}U(x,h,t),$$
$$F(x',z,t)=\int_0^z\int_0^\zeta\hat f(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta,\quad U(x,z,t)=\int_0^z\int_0^\zeta(\hat r+\hat\alpha\,|u^*|^{p-2})u^*(x,\eta,t)\mathrm{d}\eta\mathrm{d}\zeta.$$

Moreover, the traces τ^* and ℓ^* satisfy the following inequality:

$$\int_\omega\hat k(|\psi+\ell^*(t)|-|\ell^*(t)|)\mathrm{d}x'-\int_\omega(\mu\tau^*(t)+\nu\partial_t\tau^*(t)+\phi\star\tau^*(t))\psi\mathrm{d}x'\geq 0,\quad \forall\psi\in L^2(\omega)^2,\tag{47}$$

and the following limit form of the Tresca boundary conditions

$$\left.\begin{aligned}&\left|\begin{matrix}\mu\tau^*(t)+\nu\partial_t\tau^*(t)\\+\phi\star\tau^*(t)\end{matrix}\right|<\hat k\Rightarrow\ell^*(t)=0,\\&\left|\begin{matrix}\mu\tau^*(t)+\nu\partial_t\tau^*(t)\\+\phi\star\tau^*(t)\end{matrix}\right|=\hat k\Rightarrow\exists\lambda>0:\left[\begin{matrix}\ell^*(t)=\\\lambda\left(\begin{matrix}\mu\tau^*(t)+\nu\partial_t\tau^*(t)\\+\phi\star\tau^*(t)\end{matrix}\right)\end{matrix}\right]\end{aligned}\right\}\quad a.e\ on\ \omega\times]0,T[.\tag{48}$$

Proof. To prove (46), we integrate (43) over $(0,z)$, and we note that

$$\begin{aligned}&-\mu\partial_z u_i^*-\nu\partial_z(\partial_t u_i^*)-\phi\star\partial_z u_i^*+\mu\partial_z u_i^*(x',0,t)+\nu\partial_z(\partial_t u_i^*)(x',0,t)+\phi\star\partial_z u_i^*(x',0,t)+z\partial_{x_i}\pi^*\\&=-\int_0^z(\hat r+\hat\alpha\,|u_i^*|^{p-2})u_i^*(x',\eta,t)\mathrm{d}\eta+\int_0^z\hat f_i(x',\eta,t)\mathrm{d}\eta.\end{aligned}$$

By integrating for the second time between 0 and z, we obtain

$$\begin{aligned}&[\mu u_i^*+\nu\partial_t u_i^*+\phi\star u_i^*](x',z,t)\\&=\mu z\tau_i^*+\nu z\partial_t\tau_i^*+\phi\star\tau_i^*+\mu\ell_i^*+\nu\partial_t\ell_i^*+\phi\star\ell_i^*\\&\quad+\frac{z^2}{2}\partial_{x_i}\pi^*+\int_0^z\int_0^\zeta(\hat r+\hat\alpha\,|u_i^*|^{p-2})u_i^*(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta-\int_0^z\int_0^\zeta\hat f_i(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta.\end{aligned}\tag{49}$$

We replace z by $h(x')$, and hence,

$$h(x')[\mu\tau_i^* + \nu\partial_t\tau_i^* + \phi\star\tau_i^*] + \mu\ell_i^* + \nu\partial_t\ell_i^* + \phi\star\ell_i^* + \frac{h(x')^2}{2}\partial_{x_i}\pi^* + \int_0^z\int_0^\zeta(\hat{r} + \hat{\alpha}\,|u_i^*|^{p-2})u_i^*(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta = \int_0^h\int_0^\zeta f_i(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta. \tag{50}$$

By integrating (49) from 0 to $h(x')$, we obtain

$$\begin{aligned}\mu\int_0^h u_i^*(x',\zeta,t)\mathrm{d}\zeta + \nu\int_0^h\partial_t u_i^*(x',\zeta,t)\mathrm{d}\zeta + \int_0^h\phi\star u_i^*(x',\zeta,t)\mathrm{d}\zeta \\ = h(x')[\mu\ell_i^* + \nu\partial_t\ell_i^* + \phi\star\ell_i^*] + \frac{h(x')^2}{2}[\mu\tau_i^* + \nu\partial_t\tau_i^* + \phi\star\tau_i^*] + \frac{h(x')^3}{6}\partial_{x_i}\pi^* \\ + \int_0^h\int_0^z\int_0^\zeta(\hat{r} + \hat{\alpha}\,|u_i^*|^{p-2})u_i^*(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}z - \int_0^h\int_0^z\int_0^\zeta\hat{f}_i(x',\eta,t)\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}z.\end{aligned} \tag{51}$$

From (50) and (51), we deduce (46).

By choosing in (42) $\varphi = (u_1^* + \psi_1, u_2^* + \psi_2)$, where $\psi \in \Pi(V)^2$, we obtain

$$\begin{aligned}\int_\omega\hat{k}(|\psi + \ell^*| - |\ell^*|)\mathrm{d}x' \geq -\sum_{i=1}^2\int_\Omega[\mu\partial_z u_i^*(t) + \nu\partial_z(\partial_t u_i^*(t)) + \phi\star\partial_z u_i^*(t)]\partial_z\psi_i\mathrm{d}x'\mathrm{d}z \\ -\sum_{i=1}^2\int_\Omega(\hat{r} + \hat{\alpha}\,|u_i^*|^{p-2})u_i^*.\psi_i\mathrm{d}x'\mathrm{d}z + \sum_{i=1}^2\int_\Omega\pi^*(\partial_{x_i}\psi_i)\mathrm{d}x'\mathrm{d}z + \sum_{i=1}^2\int_\Omega\hat{f}_i.\psi_i\mathrm{d}x'\mathrm{d}z.\end{aligned}$$

By using now the Green formula, equality (43), and the fact that $\psi_i = 0$ on $\partial\Omega_1 \cup \partial\Omega_L$, we obtain

$$\int_\omega\hat{k}(|\psi + \ell^*| - |\ell^*|)\mathrm{d}x' - \int_\omega(\mu\tau^*(t) + \nu\partial_t\tau^*(t) + \phi\star\tau^*(t))\psi \geq 0, \quad \forall\psi \in \Pi(V)^2.$$

This inequality remain valid for any $\psi \in \mathcal{D}(\omega)^2$, and by density of $\mathcal{D}(\omega)^2$ in $L^2(\omega)^2$, it also remain valid for any $\psi \in L^2(\omega)^2$.

For (48), we choose $\psi = \pm\ell^*$ in (47), and then we follow the same techniques of [19]. □

5.4 Uniqueness

Theorem 6. *The solution* (u^*, π^*) *of the limit problems* (42) *and* (43) *is unique in* $L^2(0,T;\,V_z) \times L^2(0,T;\,L_0^2(\omega))$.

Proof. Let us suppose that there exist two solutions (u^{*1}, π^{*1}) and (u^{*2}, π^{*2}) of the limit problems (42) and (43), and we take $\hat{\varphi} = u^{*2}$ and $\hat{\varphi} = u^{*1}$ from (42), and by summing the two inequalities, we obtain

$$\begin{aligned}\sum_{i=1}^2\mu\int_\Omega(\partial_z u_i^{*1} - \partial_z u_i^{*2}).(\partial_z u_i^{*1} - \partial_z u_i^{*2})\mathrm{d}x'\mathrm{d}z \\ + \nu\int_\Omega(\partial_z(\partial_t u_i^{*1}) - \partial_z(\partial_t u_i^{*2})).(\partial_z u_i^{*1} - \partial_z u_i^{*2})\mathrm{d}x'\mathrm{d}z \\ + \int_\Omega\phi\star(\partial_z u_i^{*1} - \partial_z u_i^{*2}).(\partial_z u_i^{*1} - \partial_z u_i^{*2})\mathrm{d}x'\mathrm{d}z \\ + \hat{\alpha}\sum_{i=1}^2\int_\Omega[(\hat{r} + \hat{\alpha}\,|u_i^{*1}|^{p-2})u_i^{*1} - (\hat{r} + \hat{\alpha}\,|u_i^{*2}|^{p-2})u_i^{*2}](u_i^{*1} - u_i^{*2})\mathrm{d}x'\mathrm{d}z \\ \leq 0.\end{aligned}$$

The last term is monotone, and $\phi(\cdot)$ is positive kernel, and thus, we find

$$\|\partial_z(u^{*1} - u^{*2})\|^2_{L^2(\Omega)^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_z(u^{*1} - u^{*2})\|^2_{L^2(\Omega)^2} \leq 0.$$

By integrating this inequality over $(0, t)$, we obtain

$$\|\partial_z(u^{*1} - u^{*2})\|^2_{L^2(0,T;L^2(\Omega))^2} \leq 0.$$

Now, by Poincaré's inequality, we find

$$\|u^{*1} - u^{*2}\|_{L^2(0,T;V_z)} = 0.$$

The uniqueness of the π^* in $L^2(0, T; L^2_0(\omega))$ is inferred from (46). First, we obtain

$$\int_0^t\int_\omega \frac{h^3}{12}\nabla(\pi^{*1}(s) - \pi^{*2}(s))\nabla\psi\mathrm{d}x'\mathrm{d}s = 0.$$

Then we take $\psi = \pi^{*1} - \pi^{*2}$, and using Poincaré's inequality, we obtain

$$\pi^{*1} = \pi^{*2}, \quad \text{a.e in } \omega \times]0, T[.$$

This ends the proof of the uniqueness. □

6 Conclusion

This article investigates the asymptotic behavior of solutions to the Navier-Stokes-Voigt-Brinkman-Forchheimer equations in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$, incorporating a memory term and the Tresca friction law. By using the functional theory, we analyze how the solution evolves as one dimension of the domain tends to zero. We first introduce the problem statement and derive the associated variational formulation, and we mentioned the existence and uniqueness of weak solutions for the governing mechanical system. Then, we obtained a priori estimates for the velocity field and pressure, which hold independently of the parameter ε. Furthermore, we derived the specific Reynolds equation linked to variational inequalities and prove the uniqueness of the limit problem. These results contribute to a deeper understanding of the fluid flow behavior in thin domains, with significant applications in fluid mechanics and material science.

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