Research Article

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Fixed point results for generalized convex orbital Lipschitz operators

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Abstract: Krasnoselskii's iteration is a classical and important method for approximating the fixed point of an operator that satisfies certain conditions. Many authors have used this approach to obtain several famous fixed point theorems for different types of operators. It is well known that Kirk's iteration can be seen as a generalization of Krasnoselskii's iteration, in which the iterates are generated by a certain generalized averaged mapping. This approximation method is of great practical significance because the iterative formula contains more information related to the operator in question. The purpose of this study is to define weak (α_n, β_i)-convex orbital Lipschitz operators. These concepts not only extend the previously introduced Popescu-type convex orbital (λ, β)-Lipschitz operators in *Fixed-point results for convex orbital operators*, (Demonstr. Math. **56** (2023), 20220184), but also encompass many classical contractive operators. Popescu also proved a fixed point result for his proposed operator using the graphic contraction principle and obtained an approximation of the fixed point with Krasnoselskii's iterates. To extend Popescu's main results from Krasnoselskii's iterative scheme to Kirk's iterative scheme, several fixed point theorems are established, in which an appropriate Kirk's iterative algorithm can be used to approximate the fixed point of a k-fold averaged mapping associated with our presented convex orbital Lipschitz operators. These results not only generalize, but also complement the existing results documented in the previous literature.

Keywords: fixed point, convex orbital Lipschitz operator, weak (α_n, β_i)-convex orbital Lipschitz operator, weakly Picard operator, Ulam-Hyers stability, well-posedness

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1 Introduction and preliminaries

Let (X, d) be a metric space and $T : X \to X$ be an operator. We define the nth iterate of T as $T^n = T^{n-1} \circ T, n \in \{0\} \cup \mathbb{N}$ where $T^0 = I$ (identity operator). An element $x^* \in X$ is said to be a fixed point of T if $Tx^* = x^*$, and we denote the set of all fixed points of T by $F(T)$. The symbol $Gr(T) := \{(x, Tx) \in X \times X, x \in X\}$ denotes the graph of T.

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We recall some basic and important concepts in the fixed point theory.

Definition 1.1. Let (X, d) be a metric space. Then, $T : X \to X$ is said to be a Picard operator if:
(i) $F(T) = \{x^*\}$;
(ii) the sequence $\{T^n x_0\}_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for every $x_0 \in X$.

Definition 1.2. Let (X, d) be a metric space. Then, $T : X \to X$ is said to be a weak Picard operator if for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges and the limit is a fixed point of T.

If $T : X \to X$ is a weak Picard operator, the operator $T^\infty : X \to F(T)$ given by $T^\infty x \coloneqq \lim_{n\to\infty} T^n(x)$ is a retraction mapping (set retraction).

Definition 1.3. [1] Let (X, d) be a metric space, $T : X \to X$ be an operator such that $F(T) \neq \emptyset$. Let $r : X \to F(T)$ be a set retraction. We say that T satisfies the retraction-displacement condition if there exists $c > 0$ such that for every $x \in X$,

$$d(x, r(x)) \leq cd(x, Tx).$$

Definition 1.4. Let (X, d) be a metric space, $T : X \to X$ be an operator such that $F(T)$ is nonempty, and $r : X \to F(T)$ be a set retraction. Then,
(i) the fixed point equation $x = Tx$ is called well posed in the sense of Reich and Zaslavski [2,3] if for each $x^* \in F(T)$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which $d(x_n, Tx_n) \to 0$ as $n \to \infty$, we have $x_n \to x^*$ as $n \to \infty$;
(ii) the operator T has the Ostrowski property [4,5] if for each $x^* \in F(T)$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which $d(x_{n+1}, Tx_n) \to 0$ as $n \to \infty$, we have $x_n \to x^*$ as $n \to \infty$;
(iii) the fixed point equation $x = Tx$ is Ulam-Hyers stable [6,7] if there exists a constant $K > 0$ such that for each $\varepsilon > 0$ and each $v^* \in X$ with $d(v^*, Tv^*) \leq \varepsilon$, there exists $x^* \in X$ with $Tx^* = x^*$ such that $d(x^*, v^*) \leq K\varepsilon$.

Remark 1.1. In particular, if $F(T) = \{x^*\}$, then we obtain the classical notions of well-posedness [2], Ostrowski stability property [4,5] and Ulam-Hyers stability [6,7].

In a recent publication, Popescu [8] proposed two novel types of operators, namely, weak convex orbital Lipschitz operator and convex orbital (λ, β)-Lipschitz operator. These operators are comparatively weaker than the ones introduced earlier by Petrusel et al. [9].

Definition 1.5. [9] Let $(X, ||\cdot||)$ be a normed space and A be a nonempty and convex subset of X. Let $T : A \to A$ be an operator. We say that T is a convex orbital λ-Lipschitz operator if there exists $\beta > 0$ such that for every $\lambda \in (0, 1]$ and every $x \in A$,

$$||Tx - TT_\lambda x|| \leq \beta\lambda||x - Tx||,$$

where $T_\lambda x \coloneqq (1 - \lambda)x + \lambda Tx$.

Definition 1.6. [8] Let $(X, ||\cdot||)$ be a normed space and A be a nonempty and convex subset of X. Let $T : A \to A$ be an operator. We say that T is a weak convex orbital Lipschitz operator if for every $\lambda \in (0, 1]$, there exists $\beta > 0$ such that for every $x \in A$,

$$||Tx - TT_\lambda x|| \leq \beta\lambda||x - Tx||,$$

where $T_\lambda x \coloneqq (1 - \lambda)x + \lambda Tx$.

Definition 1.7. [8] Let $(X, \|\cdot\|)$ be a normed space and A be a nonempty and convex subset of X. Let $T : A \to A$ be an operator. We say that T is a convex orbital (λ, β)-Lipschitz operator if there exist $\lambda \in (0, 1]$ and $\beta > 0$ such that for every $x \in A$,

$$\|Tx - TT_\lambda x\| \leq \beta\lambda\|x - Tx\|,$$

where $T_\lambda x \coloneqq (1 - \lambda)x + \lambda Tx$.

It has been observed that each convex orbital λ-Lipschitz operator can be classified as a weak convex orbital Lipschitz operator. Similarly, every weak convex orbital Lipschitz operator can be categorized as a convex orbital (λ, β)-Lipschitz operator. Various types of operators fall under this category, such as the Banach contractions, Kannan contractions, Ćirć-Reich-Rus contractions, Berinde contractions, non-expansive operators, enriched (b, θ)-contractions, and Lipschitz operators.

Popescu [8] demonstrated the existence of a unique fixed point for a convex orbital (λ, β)-Lipschitz operator T using the graphic contraction principle [10]. This fixed point can be approximated through the Krasnoselskii's iterates, which is defined by $x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n$ for every initial point x_0 and $\lambda \in [0, 1)$. In the sequential approximation is essentially the Picard's iteration of the averaged mapping T_λ (refer to [11]). Additionally, Popescu also investigated the stability properties and well-posedness of the fixed point equation $x = Tx$.

Theorem 1.1. (Graphic contraction principle, [10]) *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic k-contraction, i.e., there exists $k \in (0, 1)$ such that*

$$d(f(x), f^2(x)) \leq kd(x, f(x)),$$

for all $x \in X$.

If f has a closed graph, then:
(1) *for each $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges in (X, d) to a fixed point $x^*(x)$ of f;*
(2) *$F(f) = F(f^n) \neq \emptyset$ for all $n \in \mathbb{N}$;*
(3) *f is a weakly Picard operator;*
(4) *$d(x, f^\infty(x)) \leq \frac{1}{1-k}d(x, f(x))$, for all $x \in X$, i.e., f is a $\frac{1}{1-k}$-weakly Picard operator;*
(5) *the fixed point equation $x = f(x)$ is well-posed in the sense of Reich and Zaslavski;*
(6) *the fixed point equation $x = f(x)$ is Ulam-Hyers stable;*
(7) *if $k < \frac{1}{3}$, then $d(f(x), f^\infty(x)) \leq \frac{k}{1-2k}d(x, f^\infty(x))$, for all $x \in X$, i.e., f is a $\frac{k}{1-2k}$-quasicontraction;*
(8) *if $k < \frac{1}{3}$, then f has the Ostrowski stability property.*

Theorem 1.2. [8] *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, A be a nonempty closed and convex subset of X, and $T : A \to A$ be an operator with a closed graph. We suppose that*
(*i*) *T is a convex orbital (λ, β)-Lipschitz operator with $\beta \geq 1$;*
(*ii*) $\mathrm{Re}\langle Tu - Tv, u - v\rangle \leq \mu\|u - v\|^2$ *for any $u, v \in A$, where $\mu < \frac{2-\lambda(1+\beta^2)}{2(1-\lambda)}$.*

Then, for every $x_0 \in A$, the sequence $\{x_m\} \subset A$, defined by

$$x_m = (1 - \lambda)x_{m-1} + \lambda Tx_{m-1}, \quad m \in \mathbb{N},$$

converges to the unique fixed point $x^* \in A$ of T.

Theorem 1.3. [8] *Let $(X, \|\cdot\|)$ be a Banach space and A be a nonempty closed and convex subset of X. Let $T : A \to A$ be a convex orbital (λ, β)-Lipschitz operator with $\beta < 1$. Then, the following conclusions hold:*
(*i*) *T satisfies the following retraction-displacement condition:*

$$\|x - x^*(x)\| \leq \frac{1}{1-\beta}\|x - Tx\|,$$

for every $x \in A$;
(*ii*) *the fixed point equation $x = Tx$ is Ulam-Hyers stable;*
(*iii*) *if $\beta < \frac{1}{3}$ and $\lambda > \frac{2}{3(1-\beta)}$, then T has the Ostrowski stability property.*

In recent times, Nithiarayaphaks and Sintunavarat [12] introduced a novel notion of weak enriched contraction mappings along with a new variant of averaged mapping, known as a double-averaged mapping. This mapping is defined as $T_{\alpha_1,\alpha_2} := (1-\alpha_1-\alpha_2)I + \alpha_1 T + \alpha_2 T^2$, where $\alpha_1 > 0, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 \leq 1$. It can be observed that T_{α_1,α_2} is an extension of T_λ, where $T_\lambda = T_{\alpha_1,0}$, $\alpha_1 = \lambda$. Nithiarayaphaks and Sintunavarat [12] proved the existence and uniqueness of the fixed point of a double averaged mapping associated with a weak enriched contraction mapping. This can be achieved by using an appropriate Kirk's iterative algorithm of order $k = 2$ described in [14]. Relevant results can be found in the following statement.

Theorem 1.4. [12] *Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a weak enriched contraction mapping, i.e., there exist nonnegative real numbers a, b and $w \in [0, a+b+1)$ such that, for any $x, y \in C$,*

$$\|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| \leq w\|x-y\|.$$

Then, there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that the following assertions hold:

(*i*) $|F(T_{\alpha_1,\alpha_2})| = 1$; $\mathrm{F}(\mathrm{T}_{\alpha_1,\alpha_2})$ is a singleton set.

(*ii*) *for every $x_0 \in C$, the iteration $\{x_n\} \subset C$ given by*

$$x_n = (1-\alpha_1-\alpha_2)x_{n-1} + \alpha_1 T x_{n-1} + \alpha_2 T^2 x_{n-1},$$

for all $n \in \mathbb{N}$ converges to the unique fixed point of T_{α_1,α_2}.

Recently, Zhou et al. [13] introduced the notion of a k-fold averaged mappings, which can be viewed as a generalization of the averaged mapping [11] and double averaged mappings [12]. They then prove the existence of unique fixed point of the k-fold averaged mapping associated with certain generalized weak enriched contractions introduced herein.

Let K be a nonempty subset of a Banach space X and T be a self-mapping defined on X. A mapping $\bar{T}$ defined on K is called a k-fold averaged mapping associated with T defined by

$$\bar{T} := (1-\bar{\alpha}_1-\bar{\alpha}_2-\ldots-\bar{\alpha}_k)I + \bar{\alpha}_1 T + \bar{\alpha}_2 T^2 + \ldots + \bar{\alpha}_k T^k,$$

where $\bar{\alpha}_i > 0$, $\sum_{i=1}^{k}\bar{\alpha}_i \in (0,1]$, $k \geq 3$, $k \in \mathbb{N}$.

It seems useful to unify the fixed point results mentioned earlier by using Kirk's iteration scheme with higher-order k generated by a generalized convex orbital Lipschitz operators. This is a twofold unification: (1) generalization of convex orbital Lipschitz operators in a way that the several existing contraction mappings are deduced as special cases, and (2) a consideration of a Kirk's iteration scheme with the order greater than 2.

The important contributions from this work are highlighted as follows:

(1) The concepts of a generalized convex orbital Lipschitz operators, namely, (α_n, β_i)-convex orbital Lipschitz operator and weak (α_n, β_i)-convex orbital Lipschitz operators, which cover the Popescu-type operators and other existing generalized contractive operators.

(2) Existence of unique fixed point of the k-fold averaged mapping associated with weak (α_n, β_i)-convex orbital Lipschitz operators is proved in the framework of Banach spaces.

(3) The Ulam-Hyers stability and Ostrowski stability property of the k-fold averaged mapping are studied.

2 Main results

First, let us introduce the two generalizations of convex orbital Lipschitz operators, called (α_n, β_i)-convex orbital Lipschitz operator and weak (α_n, β_i)-convex orbital Lipschitz operators.

Definition 2.1. Let $(X, \|\cdot\|)$ be a normed space, A be a nonempty and convex subset of X and i, n be any positive integers with $i \leq n$, $2 \leq n$. Let $T : A \to A$ be an operator. We say that T is an (α_n, β_i)-convex orbital Lipschitz operator of type I if there exist $\beta_k > 0$, $k = 1, 2, \ldots, i$, such that for any $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$, with $\sum_{j=1}^{n}\alpha_j \in (0,1]$

and every $x \in A$, one has

$$||T^k x - T^k T_{\alpha_n} x|| \leq \beta_k \sum_{j=1}^{n} \alpha_j ||x - T^j x||, \tag{1}$$

where $T_{\alpha_n} x \coloneqq \left(1 - \sum_{j=1}^{n} \alpha_j\right) x + \sum_{j=1}^{n} \alpha_j T^j x$.

Definition 2.2. Let $(X, ||\cdot||)$ be a normed space, A be a nonempty and convex subset of X and i, n be any fixed positive integers with $i \leq n, 2 \leq n$. Let $T : A \to A$ be an operator. We say that T is an (α_n, β_i)-convex orbital Lipschitz operator of type II if there exist $\beta_k > 0, k = 1, 2, \ldots, i$, and $\alpha_1 > 0, \alpha_j \geq 0, 2 \leq j \leq n$ with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$ such that for every $x \in A$, one has

$$||T^k x - T^k T_{\alpha_n} x|| \leq \beta_k \sum_{j=1}^{n} \alpha_j ||x - T^j x||, \tag{2}$$

where $T_{\alpha_n} x \coloneqq \left(1 - \sum_{j=1}^{n} \alpha_j\right) x + \sum_{j=1}^{n} \alpha_j T^j x$.

Definition 2.3. Let $(X, ||\cdot||)$ be a normed space, A be a nonempty and convex subset of X and i, n be any fixed positive integers with $i \leq n, 2 \leq n$. Let $T : A \to A$ be an operator. We say that T is a weak (α_n, β_i)-convex orbital Lipschitz operator of type I if there exist $\beta_k > 0, k = 1, 2, \ldots, i$, such that for any $\alpha_1 > 0, \alpha_j \geq 0, 2 \leq j \leq n$, with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$, and for every $x \in A$, one has

$$||T^k x - T^k T_{\alpha_n} x|| \leq \beta_k \left\| \sum_{j=1}^{n} \alpha_j (x - T^j x) \right\|, \tag{3}$$

where $T_{\alpha_n} x \coloneqq \left(1 - \sum_{j=1}^{n} \alpha_j\right) x + \sum_{j=1}^{n} \alpha_j T^j x$.

Definition 2.4. Let $(X, ||\cdot||)$ be a normed space, A be a nonempty and convex subset of X and i, n be any fixed positive integer with $i \leq n, 2 \leq n$. Let $T : A \to A$ be an operator. We say that T is a weak (α_n, β_i)-convex orbital Lipschitz operator of type II if there exist $\beta_k > 0, k = 1, 2, \ldots, i$, and $\alpha_1 > 0, \alpha_j \geq 0, 2 \leq j \leq n$, with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$ such that for every $x \in A$, one has

$$||T^k x - T^k T_{\alpha_n} x|| \leq \beta_k \left\| \sum_{j=1}^{n} \alpha_j (x - T^j x) \right\|, \tag{4}$$

where $T_{\alpha_n} x \coloneqq \left(1 - \sum_{j=1}^{n} \alpha_j\right) x + \sum_{j=1}^{n} \alpha_j T^j x$.

It is evident that any weak (α_n, β_i)-convex orbital Lipschitz operator is also a (α_n, β_i)-convex orbital Lipschitz operator. However, the converse of the previous statement does not hold. It is important to note that the aforementioned definitions exhibit an implication relation where a weak (α_n, β_i)-convex orbital Lipschitz operator of type I implies a weak (α_n, β_i)-convex orbital Lipschitz operator of type II.

Remark 2.1. The (α_n, β_i)-convex orbital Lipschitz operators mentioned earlier are an extension of the convex orbital Lipschitz operators presented by Popescu [8]. The Popescu-type operators can be viewed as (α_1, β_1)-convex orbital Lipschitz operators, where T_λ can be considered as a specific case of T_{α_n} for $n = 1$, $\alpha_1 = \lambda \in (0, 1]$, and $\beta_1 = \beta > 0$. It is known that the k-fold averaged mappings can be viewed as a generalization of the averaged mappings T_λ.

Remark 2.2. By the definition of T_{α_n}, we can rewrite (3) as

$$||T^k x - T^k T_{\alpha_n} x|| \leq \beta_k ||x - T_{\alpha_n} x||. \tag{5}$$

This concept is weaker than the definition of uniform k-Lipschitz mapping provided by K. Goebel and W.A. Kirk in 1973 in their seminal work [15], defined in a normed space.

Given a normed space $(X, \|\cdot\|)$, a mapping $T : X \to X$ is said to be uniformly k-Lipschitzian if there exists some $k > 0$ such that

$$\|T^n x - T^n y\| \le k\|x - y\|, \quad \text{for all } x, y \in X \text{ and } n \in \mathbb{N}. \tag{6}$$

It is clear that every uniform k-Lipschitz mapping produced by Geobel et al. is also uniformly continuous, and therefore continuous. However, our definition is weaker than the definition of uniform k-Lipschitz. In Geobel et al.'s definition, $x \in X$ is selected and a corresponding $y \in X$ is chosen to satisfy condition (6). In contrast, our definition selects $x \in X$ and recursively generates the other element y as $y = Tx$. If we substitute $y = T_{\alpha_n}x$ and $\beta = \max\{\beta_k\}$, for all k into (5), we obtain the definition of k-Lipschitz by Geobel et al. Therefore, the concept of (3) is broader than the concept of uniform k-Lipschitz. This definition can be seen as a potential generalization of Geobel et al.'s definition, with weaker conditions.

On the other hand, the motivation of providing the aforementioned definitions is to generalize the notions provided by Popescu [8], in which it contains the convex combination of $T^j x (j = 0, 1, \ldots, n)$. In algorithm design to solve practical engineering or physics problems, it is necessary to obtain more information about the iterative function, such as its various order function values, which may represent certain specific observations. Therefore, considering such operators and approximation algorithms of its fixed points is of great practical significance.

Example 2.1. Let $(X, \|\cdot\|)$ be a normed space, A be a nonempty and convex subset of X, and $T : A \to A$ be an L-Lipschitz operator, i.e., $L > 0$, and for any $x, y \in A$,

$$\|Tx - Ty\| \le L\|x - y\|.$$

Then, T is a weak (α_n, β_1)-convex orbital Lipschitz operator of type I, (also a (α_n, β_1)-convex orbital Lipschitz operator of type I), where $\beta_1 = L$. Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, we have

$$\|Tx - TT_{\alpha_n}x\| \le L\|x - T_{\alpha_n}x\| = L\left\|\sum_{j=1}^{n}\alpha_j(x - T^j x)\right\| \le L\left[\sum_{j=1}^{n}\alpha_j\|x - T^j x\|\right],$$

for any $\alpha_1 > 0$, $\alpha_j \ge 0$, $j = 2, 3, \ldots, n$, with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$ and for all $x \in A$.

Example 2.2. Let $(X, \|\cdot\|)$ be a normed space and A be a nonempty and convex subset of X, and $T : A \to A$ be an Kannan contraction, i.e., there exists $\gamma \in [0, \frac{1}{2})$ such that for any $x, y \in A$,

$$\|Tx - Ty\| \le \gamma[\|x - Tx\| + \|y - Ty\|].$$

Then, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, provided that $\gamma < \frac{\alpha_1}{2} \in \left(0, \frac{1}{2}\right]$. Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, we obtain

$$\begin{aligned}\|Tx - TT_{\alpha_n}x\| &\le \gamma[\|x - Tx\| + \|T_{\alpha_n}x - TT_{\alpha_n}x\|]\\ &= \gamma\left[\|x - Tx\| + \left\|\left(1 - \sum_{j=1}^{n}\alpha_j\right)x + \sum_{j=1}^{n}\alpha_i T^j x - TT_{\alpha_n}x\right\|\right]\\ &\le \gamma\left[2\|x - Tx\| + \sum_{j=1}^{n}\alpha_j\|x - T^j x\| + \|Tx - TT_{\alpha_n}x\|\right].\end{aligned}$$

Therefore

$$(1 - \gamma)\|Tx - TT_{\alpha_n}x\| \le (2\gamma + \gamma\alpha_1)\|x - Tx\| + \gamma\sum_{j=2}^{n}\alpha_j\|x - T^j x\| \le (1 + \gamma)\sum_{j=1}^{n}\alpha_j\|x - T^j x\|,$$

for any $\alpha_1 > 0, \alpha_j \ge 0, 2 \le j \le n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$ and $x \in A$. Hence,

$$\|Tx - TT_{\alpha_n}x\| \le \frac{1 + \gamma}{1 - \gamma}\sum_{j=1}^{n}\alpha_j\|x - T^j x\|.$$

Consequently, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, provided that $\gamma < \frac{\alpha_1}{2} \leq \frac{1}{2}$, where $\beta_1 = \frac{1+\gamma}{1-\gamma}$.

Example 2.3. Let $(X, \|\cdot\|)$ be a normed space and A be a nonempty and convex subset of X, and $T : A \to A$ be a Berinde (α, L)-contraction, i.e., there exist $\alpha, L \in [0, \infty)$ with $\alpha < 1$ such that for any $x, y \in A$,

$$\|Tx - Ty\| \leq \alpha\|x - Tx\| + L\|y - Tx\|.$$

Then, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I. Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, we obtain

$$\begin{aligned}\|Tx - TT_{\alpha_n}x\| &\leq \alpha\|x - T_{\alpha_n}x\| + L\|T_{\alpha_n}x - Tx\|\\ &\leq \alpha\|x - T_{\alpha_n}x\| + L(\|T_{\alpha_n}x - x\| + \|x - Tx\|)\\ &= [(\alpha + L)\alpha_1 + L]\|x - Tx\| + (\alpha + L)\sum_{j=2}^{n}\alpha_j\|x - T^jx\|\\ &\leq (\alpha + 2L)\sum_{j=1}^{n}\alpha_j\|x - T^jx\|,\end{aligned}$$

for any $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$ and $x \in A$.

Therefore, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, where $\beta_1 = \alpha + 2L$.

Example 2.4. Let $(X, \|\cdot\|)$ be a normed space, A be a nonempty and convex subset of X, and $T : A \to A$ be an enriched (b, θ)-contraction, i.e., there exist $b \geq 0$, $\theta \in (0, b + 1]$ such that for any $x, y \in A$,

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|.$$

Then, T is a weak (α_n, β_1)-convex orbital Lipschitz operator of type I. Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, we obtain

$$\left\|b\left(\sum_{j=1}^{n}\alpha_j(x - T^jx)\right) + Tx - TT_{\alpha_n}x\right\| \leq \theta\left\|\sum_{j=1}^{n}\alpha_j(x - T^jx)\right\|,$$

so

$$\|Tx - TT_{\alpha_n}x\| - b\left\|\sum_{j=1}^{n}\alpha_j(x - T^jx)\right\| \leq \theta\left\|\sum_{j=1}^{n}\alpha_j(x - T^jx)\right\|,$$

for any $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$ and $x \in A$. Hence, we obtain

$$\|Tx - TT_{\alpha_n}x\| \leq (b + \theta)\left\|\sum_{j=1}^{n}\alpha_j(x - T^jx)\right\| \leq (b + \theta)\sum_{j=1}^{n}\alpha_j\|x - T^jx\|,$$

showing that T is a weak (α_n, β_1)-convex orbital Lipschitz operator of type I with the coefficient $\beta_1 = b + \theta$.

Example 2.5. Let $(X, \|\cdot\|)$ be a normed space, A be a nonempty and convex subset of X, and $T : A \to A$ be an enriched Kannan contraction, i.e., there exist $b \geq 0$, $\gamma \in [0, \frac{1}{2})$ such that for any $x, y \in A$,

$$\|b(x - y) + Tx - Ty\| \leq \gamma(\|x - Tx\| + \|y - Ty\|).$$

Then, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, provided that $\gamma < \frac{\alpha_1}{2} \in \left(0, \frac{1}{2}\right]$.

Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, we obtain that for any $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$,

$$\left\|b\left[\sum_{j=1}^{n}\alpha_i(x - T^jx)\right] + Tx - TT_{\alpha_n}x\right\|$$

$$\le \gamma\left(\|x - Tx\| + \left\|\sum_{j=1}^{n} \alpha_i(T^j x - x) + (x - TT_{\alpha_n}x)\right\|\right)$$
$$\le \gamma\left(\|x - Tx\| + \sum_{j=1}^{n} \alpha_i\|x - T^j x\| + \|x - Tx\| + \|Tx - TT_{\alpha_n}x\|\right).$$

Therefore

$$(1-\gamma)\|Tx - TT_{\alpha_n}x\| \le (2\gamma + \gamma\alpha_1 + b\alpha_1)\|x - Tx\| + (\gamma + b)\sum_{j=2}^{n} \alpha_j\|x - T^j x\| \le (1 + \gamma + b)\sum_{j=1}^{n} \alpha_j\|x - T^j x\|.$$

provided that $\gamma < \frac{\alpha_1}{2} \le \frac{1}{2}$. We thus obtain

$$\|Tx - TT_{\alpha_n}x\| \le \frac{1 + \gamma + b}{1 - \gamma}\sum_{j=1}^{n} \alpha_j\|x - T^j x\|.$$

Consequently, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, where $\beta_1 = \frac{1+\gamma+b}{1-\gamma}$, provided that $\gamma < \frac{\alpha_1}{2} \in \left[0, \frac{1}{2}\right]$.

Example 2.6. Let $(X, \|\cdot\|)$ be a normed space, A be a nonempty and convex subset of X, and $T : A \to A$ be an enriched Ćirić-Reich-Rus contraction, i.e., there exist $b \ge 0$, $k, l \ge 0$ with $k + 2l < 1$ such that for any $x, y \in A$,

$$\|b(x - y) + Tx - Ty\| \le k\|x - y\| + l(\|x - Tx\| + \|y - Ty\|).$$

Then, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I.

Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, then for any $\alpha_1 > 0, \alpha_j \ge 0, 2 \le j \le n$ with $\sum_{j=1}^{n}\alpha_i \in (0, 1]$,

$$\left\|b\left[\sum_{j=1}^{n} \alpha_j(x - T^j x)\right] + Tx - TT_{\alpha_n}x\right\| \le k\left\|\sum_{j=1}^{n} \alpha_j(x - T^j x)\right\| + l\left(\|x - Tx\| + \left\|\sum_{j=1}^{n} \alpha_j(T^j x - x) + (x - TT_{\alpha_n}x)\right\|\right)$$
$$\le k\left\|\sum_{j=1}^{n} \alpha_j(x - T^j x)\right\| + l\left(\|x - Tx\| + \sum_{j=1}^{n} \alpha_j\|x - T^j x\| + \|x - Tx\| + \|Tx - TT_{\alpha_n}x\|\right).$$

Hence,

$$(1 - l)\|Tx - TT_{\alpha_n}x\| \le (k\alpha_1 + 2l + l\alpha_1 + b\alpha_1)\|x - Tx\| + (k + l + b)\sum_{j=2}^{n} \alpha_j\|x - T^j x\| \le (2 + b)\sum_{j=1}^{n} \alpha_j\|x - T^j x\|.$$

Thus, we obtain

$$\|Tx - TT_{\alpha_n}x\| \le \frac{2 + b}{1 - l}\sum_{j=1}^{n} \alpha_j\|x - T^j x\|.$$

Therefore, T is an (α_n, β_1)-convex orbital Lipschitz operator of type I, where $\beta_1 = \frac{2+b}{1-l}$.

Example 2.7. Every weak enriched contraction produced in [12] is also a weak (α_n, β_1)-convex orbital Lipschitz operator of type I provided by the assumption (W) being fulfilled.

(W) For any $x, y \in A$, there exists $K > 0$ such that

$$\|T^2 x - T^2 y\| \le K\|x - y\|. \tag{7}$$

Let A be a closed convex subset of a normed space $(X, \|\cdot\|)$ and $T: A \to A$ be a strong enriched contraction mapping, i.e., there exist nonnegative real numbers a, b, and $w \in [0, a + b + 1)$ such that for any $x, y \in A$,

$$\|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| \le w\|x-y\|.$$

Indeed, if we choose $y = T_{\alpha_n}x$ in the aforementioned inequality, then for any $\alpha_1 > 0, \alpha_j \ge 0, 2 \le j \le n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$,

$$\begin{aligned}
&\|a(x - T_{\alpha_n}x) + Tx - TT_{\alpha_n}x + b(T^2x - T^2T_{\alpha_n}x)\| \\
&= \left\| a\sum_{j=1}^{n}\alpha_j(x - T^jx) + Tx - TT_{\alpha_n}x + b(T^2x - T^2T_{\alpha_n}x)\right\| \\
&\le w\left\|\sum_{j=1}^{n}\alpha_j(x - T^jx)\right\|.
\end{aligned}$$

From (7), there exists $K > 0$ such that $\|T^2x - T^2T_{\alpha_n}x\| \le K\|x - T_{\alpha_n}x\|$. Then,

$$\|Tx - TT_{\alpha_n}x\| \le \left\|\sum_{j=1}^{n}a\alpha_j(x - T^jx)\right\| + \left\|\sum_{j=1}^{n}bK\alpha_j(x - T^jx)\right\| + \sum_{j=1}^{n}w\alpha_j\|(x - T^jx)\| \le (a + bK + w)\sum_{j=1}^{n}\alpha_j\|x - T^jx\|.$$

Hence, T is a weak (α_n, β_1)-convex orbital Lipschitz operator of type I with the coefficient $\beta_1 = (a + bK + w)$.

Our main results will now be presented in the following theorems.

In what follows, for a nonempty closed and convex subset A of a normed space and some fixed positive integers $2 \le n, i \le n$, we denote by T_{α_n} the operator $T_{\alpha_n}: A \to A$ given by

$$T_{\alpha_n}x = \left(1 - \sum_{j=1}^{n}\alpha_j\right)x + \sum_{j=1}^{n}\alpha_jT^jx,$$

where $\alpha_1 > 0, \alpha_j \ge 0, 2 \le j \le n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X and n be any positive integer with $2 \le n$. Let $T: A \to A$ be a weak (α_n, β_n)-convex orbital Lipschitz operator of type II with closed graph, where $0 < \beta_k < 1, k = 1, 2, \ldots, n$. Then, for every $x_0 \in A$, the sequence $\{x_m\} \subset A$ defined by*

$$x_m = \left(1 - \sum_{j=1}^{n}\alpha_j\right)x_{m-1} + \sum_{j=1}^{n}\alpha_jT^jx_{m-1}, \quad m \in \mathbb{N},$$

where $\alpha_1 > 0, \alpha_j \ge 0, 2 \le j \le n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$, converges to a fixed point of T_{α_n}.

Proof. For $x, y \in A$, we have

$$\|T_{\alpha_n}x - T_{\alpha_n}y\| = \left\|\left(1 - \sum_{j=1}^{n}\alpha_j\right)(x - y) + \sum_{j=1}^{n}\alpha_j(T^jx - T^jy)\right\| \le \left(1 - \sum_{j=1}^{n}\alpha_j\right)\|x - y\| + \sum_{j=1}^{n}\alpha_j\|T^jx - T^jy\|.$$

Taking $y = T_{\alpha_n}x$ in the aforementioned inequality, it follows from (2) that

$$\begin{aligned}
\|T_{\alpha_n}x - T_{\alpha_n}^2x\| &\le \left(1 - \sum_{j=1}^{n}\alpha_j\right)\|x - T_{\alpha_n}x\| + \sum_{j=1}^{n}\alpha_j\|T^jx - T^jT_{\alpha_n}x\| \\
&\le \left(1 - \sum_{j=1}^{n}\alpha_j\right)\|x - T_{\alpha_n}x\| + \sum_{j=1}^{n}\alpha_j\beta_j\|x - T_{\alpha_n}x\| \\
&= \left(\left(1 - \sum_{j=1}^{n}\alpha_j\right) + \sum_{j=1}^{n}\beta_j\alpha_j\right)\|x - T_{\alpha_n}x\|.
\end{aligned}$$

Since $\beta_j < 1$ for $j = 1, 2, \ldots, n$, if we denote $\gamma \coloneqq (1 - \sum_{j=1}^{n}\alpha_j) + \sum_{j=1}^{n}\alpha_j\beta_j$, then $\gamma < 1$ and

$$\|T_{\alpha_n}x - T_{\alpha_n}^2 x\| \leq \gamma\|x - T_{\alpha_n}x\|,$$

for all $x \in A$. This implies that T_{α_n} is a graphic γ-contraction. Hence, it follows from the graphic contraction principle (Theorem 1.1) that T_{α_n} is a weakly Picard operator, i.e., the sequence $\{x_m\}$ defined earlier converges to a fixed point of T_{α_n}. □

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X, and i, n be any fixed positive integers with $i < n$, $2 \leq n$. Let $T : A \to A$ be a weak (α_n, β_i)-convex orbital Lipschitz operator of type II with closed graph, where $0 < \beta_k < 1$, $k = 1, 2, \ldots, i$. Suppose that the following assumption is satisfied*:

(W') *for any $x, y \in A$ and $r \in \{i + 1, \ldots, n\}$ there exists $0 < L_r < 1$ such that*

$$\|T^r x - T^r y\| \leq L_r\|x - y\|. \tag{8}$$

Then, for every $x_0 \in A$, the sequence $\{x_m\} \subset A$ defined by

$$x_m = \left(1 - \sum_{j=1}^{n}\alpha_j\right)x_{m-1} + \sum_{j=1}^{n}\alpha_j T^j x_{m-1}, \quad m \in \mathbb{N},$$

where $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n}\alpha_j \in (0, 1]$, converges to a fixed point of T_{α_n}.

Proof. For $x, y \in A$ we have

$$\|T_{\alpha_n}x - T_{\alpha_n}y\| = \left\|\left(1 - \sum_{j=1}^{n}\alpha_j\right)(x - y) + \sum_{j=1}^{n}\alpha_j(T^j x - T^j y)\right\| \leq \left(1 - \sum_{j=1}^{n}\alpha_j\right)\|x - y\| + \sum_{j=1}^{n}\alpha_j\|T^j x - T^j y\|.$$

Taking $y = T_{\alpha_n}x$ in the aforementioned inequality, it follows from (2) and (8) that

$$\begin{aligned}\|T_{\alpha_n}x - T_{\alpha_n}^2 x\| &\leq \left(1 - \sum_{j=1}^{n}\alpha_j\right)\|x - T_{\alpha_n}x\| + \sum_{j=1}^{n}\alpha_j\|T^j x - T^j T_{\alpha_n}x\| \\ &\leq \left(1 - \sum_{j=1}^{n}\alpha_j\right)\left\|\sum_{j=1}^{n}\alpha_j(x - T^j x)\right\| + \sum_{k=1}^{i}\alpha_k\beta_k\left(\left\|\sum_{j=1}^{n}\alpha_j(x - T^j x)\right\|\right) + \sum_{r=i+1}^{n}\alpha_r L_r\|x - T_{\alpha_n}x\| \\ &= \left(\left(1 - \sum_{j=1}^{n}\alpha_j\right) + \sum_{k=1}^{i}\alpha_k\beta_k + \sum_{r=i+1}^{n}\alpha_r L_r\right)\|x - T_{\alpha_n}x\|.\end{aligned}$$

Since $\beta_k < 1$ for $k = 1, 2, \ldots, i$, if we denote $\gamma \coloneqq (1 - \sum_{j=1}^{n}\alpha_j) + \sum_{k=1}^{i}\alpha_k\beta_k + \sum_{r=i+1}^{n}\alpha_r L_r$, then $\gamma < 1$ and

$$\|T_{\alpha_n}x - T_{\alpha_n}^2 x\| \leq \gamma\|x - T_{\alpha_n}x\|,$$

for all $x \in A$. This implies that T_{α_n} is a graphic γ-contraction. Hence, it follows from the graphic contraction principle that T_{α_n} is a weakly Picard operator so the sequence $\{x_m\}$ defined earlier converges to a fixed point of T_{α_n}. □

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X, and n be any fixed integer with $n \geq 2$. Let $T : A \to A$ be a weak (α_n, β)-convex orbital Lipschitz operator of type II with closed graph and $0 < \beta < 1$. Suppose that the following condition is satisfied*:

(W'') *for any $x, y \in A$, there exists $0 < L < 1$ such that*

$$\|T^2 x - T^2 y\| \leq L\|x - y\|. \tag{9}$$

Then, for every $x_0 \in A$, the sequence $\{x_m\} \subset A$, defined by

$$x_m = \left(1 - \sum_{j=1}^{n} \alpha_j\right) x_{m-1} + \sum_{j=1}^{n} \alpha_j T^j x_{m-1}, \quad m \in \mathbb{N},$$

where $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$, converges to a fixed point of T_{α_n}.

Proof. For $x, y \in A$, one has

$$\|T_{\alpha_n} x - T_{\alpha_n} y\| = \left\| \left(1 - \sum_{j=1}^{n} \alpha_j\right)(x - y) + \sum_{j=1}^{n} \alpha_j (T^j x - T^j y) \right\| \leq \left(1 - \sum_{j=1}^{n} \alpha_j\right) \|x - y\| + \sum_{j=1}^{n} \alpha_j \|T^j x - T^j y\|.$$

Taking $y = T_{\alpha_n} x$ in the aforementioned inequality, it follows from (9) that we have

Case 1. If $n = 2l$, $l \in \mathbb{N}$.

$$\begin{aligned}
\|T_{\alpha_n} x - T_{\alpha_n}^2 x\| &\leq \left(1 - \sum_{j=1}^{2l} \alpha_j\right) \|x - T_{\alpha_n} x\| + \sum_{j=1}^{2l} \alpha_j \|T^j x - T^j T_{\alpha_n} x\| \\
&\leq \left(1 - \sum_{j=1}^{2l} \alpha_j\right) \|x - T_{\alpha_n} x\| + \alpha_1 \beta \|x - T_{\alpha_n} x\| + \alpha_2 L \|x - T_{\alpha_n} x\| + \alpha_3 \beta L \|x - T_{\alpha_n} x\| + \alpha_4 L^2 \|x - T_{\alpha_n} x\| \\
&\quad + \ldots + \alpha_{2l} \|T^{2l-2} x - T^{2l-2} T_{\alpha_n} x\| \\
&\leq \left(1 - \sum_{j=1}^{2l} \alpha_j\right) \|x - T_{\alpha_n} x\| + \left(\sum_{r=1}^{l} \alpha_{2r-1} \beta L^{r-1}\right) \|x - T_{\alpha_n} x\| + \left(\sum_{r=1}^{l} \alpha_{2r} L^r\right) \|x - T_{\alpha_n} x\| \\
&= \left[\left(1 - \sum_{j=1}^{2l} \alpha_j\right) + \left(\sum_{r=1}^{l} \alpha_{2r-1} \beta L^{r-1}\right) + \left(\sum_{r=1}^{l} \alpha_{2r} L^r\right)\right] \|x - T_{\alpha_n} x\|.
\end{aligned}$$

Case 2. If $n = 2l + 1$, $l \in \mathbb{N}$.

$$\begin{aligned}
\|T_{\alpha_n} x - T_{\alpha_n}^2 x\| &\leq \left(1 - \sum_{j=1}^{2l+1} \alpha_j\right) \|x - T_{\alpha_n} x\| + \sum_{j=1}^{2l+1} \alpha_j \|T^i x - T^i T_{\alpha_n} x\| \\
&\leq \left(1 - \sum_{j=1}^{2l+1} \alpha_j\right) \|x - T_{\alpha_n} x\| + \alpha_1 \beta \|x - T_{\alpha_n} x\| + \alpha_2 L \|x - T_{\alpha_n} x\| + \alpha_3 \beta L \|x - T_{\alpha_n} x\| + \alpha_4 L^2 \|x - T_{\alpha_n} x\| \\
&\quad + \ldots + \alpha_{2l+1} \|T^{2l-1} x - T^{2l-1} T_{\alpha_n} x\| \\
&\leq \left(1 - \sum_{j=1}^{2l+1} \alpha_j\right) \|x - T_{\alpha_n} x\| + \left(\sum_{r=0}^{l} \alpha_{2r+1} \beta L^r\right) \|x - T_{\alpha_n} x\| + \left(\sum_{r=1}^{l} \alpha_{2r} L^r\right) \|x - T_{\alpha_n} x\| \\
&= \left[\left(1 - \sum_{j=1}^{2l+1} \alpha_j\right) + \left(\sum_{r=0}^{l} \alpha_{2r+1} \beta L^r\right) + \left(\sum_{r=1}^{l} \alpha_{2r} L^r\right)\right] \|x - T_{\alpha_n} x\|.
\end{aligned}$$

Since $\beta < 1$, if we denote

$$\gamma := \max\left\{\left(1 - \sum_{j=1}^{2l} \alpha_j\right) + \left(\sum_{r=1}^{l} \alpha_{2r-1} \beta L^{r-1}\right) + \left(\sum_{r=1}^{l} \alpha_{2r} L^r\right), \left(1 - \sum_{j=1}^{2l+1} \alpha_j\right) + \left(\sum_{r=0}^{l} \alpha_{2r+1} \beta L^r\right) + \left(\sum_{j=1}^{l} \alpha_{2j} L^j\right)\right\},$$

then we easily obtain $\gamma < 1$ and

$$\|T_{\alpha_n} x - T_{\alpha_n}^2 x\| \leq \gamma \|x - T_{\alpha_n} x\|,$$

for all $x \in A$. This implies that T_{α_n} is a graphic γ-contraction. From the graphic contraction principle it follows that T_{α_n} is a weakly Picard operator; hence, the sequence $\{x_m\}$ defined earlier converges to a fixed point of T_{α_n}. □

Theorem 2.4. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, A be a nonempty closed and convex subset of X, and $T : A \to A$ be an operator with a closed graph. Let n be an integer with $n \geq 2$. We suppose that*

(i) *T is a weak (α_n, β_n)-convex orbital Lipschitz operator of type II with $0 < \beta_i \leq 1$, $i = 1, 2, \ldots, n$;*

(ii) $\operatorname{Re}\langle x - y, T^j x - T^j y\rangle \leq \mu_j \|x - y\|^2$, *for any $1 \leq j \leq n$ and $x, y \in A$, where $\mu_j \in (0, \frac{1}{2})$.*

Then, for every $x_0 \in A$, the sequence $\{x_m\}_{n\in\mathbb{N}} \subset A$, defined by

$$x_m = \left(1 - \sum_{j=1}^{n} \alpha_j\right) x_{m-1} + \sum_{j=1}^{n} \alpha_j T^j x_{m-1}, \quad m \in \mathbb{N},$$

where $\alpha_1 > 0$, $\alpha_j \geq 0$, $2 \leq j \leq n$ with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$, converges to the unique fixed point of T_{α_n}.

Proof. For every $x, y \in A$, we have

$$\begin{aligned}\|T_{\alpha_n}x - T_{\alpha_n}y\|^2 &= \left\|\left(1 - \sum_{j=1}^{n} \alpha_j\right)(x - y) + \sum_{j=1}^{n} \alpha_j (T^j x - T^j y)\right\|^2 \\ &\leq \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - y\|^2 + \left\|\sum_{j=1}^{n} \alpha_j (T^j x - T^j y)\right\|^2 + \sum_{j=1}^{n} 2\alpha_j \left(1 - \sum_{j=1}^{n} \alpha_j\right) \operatorname{Re}\langle x - y, T^j x - T^j y\rangle \\ &\leq \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - y\|^2 + \left(\sum_{j=1}^{n} \alpha_j \|T^j x - T^j y\|\right)^2 + \sum_{j=1}^{n} 2\alpha_j \left(1 - \sum_{j=1}^{n} \alpha_j\right) \operatorname{Re}\langle x - y, T^j x - T^j y\rangle.\end{aligned}$$

Taking $y = T_{\alpha_n}x$ in the aforementioned inequality and using (ii) and the assumption imposed on T, we have

$$\begin{aligned}\|T_{\alpha_n}x - T_{\alpha_n}^2 x\|^2 &\leq \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - T_{\alpha_n}x\|^2 + \left(\sum_{j=1}^{n} \beta_j \alpha_j\right)^2 \|x - T_{\alpha_n}x\|^2 + \left(\sum_{j=1}^{n} 2\alpha_j \left(1 - \sum_{j=1}^{n} \alpha_j\right) \mu_j\right) \|x - T_{\alpha_n}x\|^2 \\ &= \left[\left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 + \left(\sum_{j=1}^{n} \beta_j \alpha_j\right)^2 + \sum_{j=1}^{n} 2\alpha_j \mu_j \left(1 - \sum_{j=1}^{n} \alpha_j\right)\right] \|x - T_{\alpha_n}x\|^2.\end{aligned}$$

Let denote $\xi = \sum_{j=1}^{n} \alpha_j$ and $\gamma = \sqrt{(1 - \xi)^2 + \left(\sum_{j=1}^{n} \beta_j \alpha_j\right)^2 + \sum_{j=1}^{n} 2\alpha_j \mu_j (1 - \xi)}$.

From hypothesis, one has $\sum_{j=1}^{n} \alpha_j \mu_j < \frac{\xi}{2}$ and $\left(\sum_{j=1}^{n} \beta_j \alpha_j\right)^2 \leq \xi^2 \leq \xi$. Hence,

$$\gamma^2 = (1 - \xi)^2 + \left(\sum_{j=1}^{n} \beta_j \alpha_j\right)^2 + \sum_{j=1}^{n} 2\alpha_j \mu_j (1 - \xi) < 1 - 2\xi + \xi^2 + \xi + 2(1 - \xi) \cdot \frac{\xi}{2} = 1.$$

Consequently, we have $\gamma < 1$ and

$$\|T_{\alpha_n}x - T_{\alpha_n}^2 x\| \leq \gamma \|x - T_{\alpha_n}x\|,$$

for all $x \in A$. Thus, by the graphic contraction principle, T_{α_n} is a weakly Picard operator and the sequence $\{T_{\alpha_n}^n x_0\}_{n\in\mathbb{N}}$ converges to $T_{\alpha_n}^{\infty} x_0 \coloneqq x^* \in F(T_{\alpha_n})$ for any $x_0 \in A$.

Now, assume that there exist $x^*, y^* \in F(T_{\alpha_n})$ with $x^* \neq y^*$. Then, we have $x^* = T_{\alpha_n}x^*$ and $y^* = T_{\alpha_n}y^*$. Taking $x = x^*, y = y^*$ in (ii), we obtain

$$\operatorname{Re}\langle x^* - y^*, \alpha_j (T^j x^* - T^j y^*)\rangle \leq \mu_j \alpha_j \|x^* - y^*\|^2, j = 1, 2, \ldots, n.$$

Hence,

$$\operatorname{Re}\left\langle x^* - y^*, \sum_{j=1}^{n} \alpha_j (T^j x^* - T^j y^*)\right\rangle \leq \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2$$

$$\Rightarrow \ \mathrm{Re}\left\langle x^* - y^*, \sum_{j=1}^{n} \alpha_j T^j x^* \right\rangle - \mathrm{Re}\left\langle x^* - y^*, \sum_{j=1}^{n} \alpha_j T^j y^* \right\rangle \le \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2$$

$$\Rightarrow \ \mathrm{Re}\left\langle x^* - y^*, T_{\alpha_n} x^* - \left(1 - \sum_{j=1}^{n} \alpha_j\right) x^* \right\rangle - \mathrm{Re}\left\langle x^* - y^*, T_{\alpha_n} x^* - \left(1 - \sum_{j=1}^{n} \alpha_j\right) y^* \right\rangle \le \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2$$

$$\Rightarrow \ \mathrm{Re}\left\langle x^* - y^*, \sum_{j=1}^{n} \alpha_j x^* \right\rangle - \mathrm{Re}\left\langle x^* - y^*, \sum_{j=1}^{n} \alpha_j y^* \right\rangle \le \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2$$

$$\Rightarrow \ \sum_{j=1}^{n} \alpha_j \mathrm{Re}\langle x^* - y^*, x^* - y^* \rangle \le \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2$$

$$\Rightarrow \ \sum_{j=1}^{n} \alpha_j \|x^* - y^*\|^2 \le \sum_{j=1}^{n} \alpha_j \mu_j \|x^* - y^*\|^2.$$

Since $0 < \mu_j < 1$, we have $\|x^* - y^*\| = 0$, which is a contradiction. Thus, $F(T_{\alpha_n})$ is a singleton and T_{α_n} is a Picard operator. □

Theorem 2.5. *Let $(X, \langle \cdot \rangle)$ be a Hilbert space, A be a nonempty closed and convex subset of X, and $T : A \to A$ be an operator with a closed graph. Let i, n be two positive integers with $i < n$, $2 \le n$. We suppose that:*

(*i*) *T is a weak (α_n, β_i)-convex orbital Lipschitz operator of type II with $\beta_k > 0$, $k = 1, 2, \ldots, i$;*

(*ii*) *the assumption (W') is satisfied;*

(*iii*) $\mathrm{Re}(x - y, T^j x - T^j y) \le \mu_j \|x - y\|^2$, *for any $1 \le j \le n$ and $x, y \in A$, where $\mu_j \in (0, \frac{1}{2})$.*

Then, for every $x_0 \in A$, the sequence $\{x_m\}_{m \in \mathbb{N}} \subset A$, defined by

$$x_m = \left(1 - \sum_{j=1}^{n} \alpha_j\right) x_{m-1} + \sum_{j=1}^{n} \alpha_j T^j x_{m-1}, \ m \in \mathbb{N},$$

where $\alpha_1 > 0$, $\alpha_j \ge 0$, $2 \le j \le n$ with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$, converges to the unique fixed point $x^* \in A$ of T_{α_n}.

Proof. Consider the operator $T_{\alpha_n} : A \to A$ defined by

$$T_{\alpha_n} x := \left(1 - \sum_{j=1}^{n} \alpha_j\right) x + \sum_{j=1}^{n} \alpha_j T^j x, \ x \in A,$$

where $\alpha_1 > 0$, $\alpha_j \ge 0$, $2 \le j \le n$ with $\sum_{j=1}^{n} \alpha_j \in (0, 1]$.

For every $x, y \in A$, by (*ii*), we have

$$\begin{aligned}
\|T_{\alpha_n} x - T_{\alpha_n} y\|^2 &= \left\| \left(1 - \sum_{j=1}^{n} \alpha_j\right)(x - y) + \sum_{j=1}^{n} \alpha_j (T^j x - T^j y) \right\|^2 \\
&\le \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - y\|^2 + \left\| \sum_{j=1}^{n} \alpha_j (T^j x - T^j y) \right\|^2 + \sum_{j=1}^{n} 2\alpha_j \left(1 - \sum_{j=1}^{n} \alpha_j\right) \mathrm{Re}(x - y, T^j x - T^j y) \\
&\le \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - y\|^2 + \left(\sum_{j=1}^{n} \alpha_j \|T^j x - T^j y\|\right)^2 + \sum_{j=1}^{n} 2\alpha_j \left(1 - \sum_{j=1}^{n} \alpha_j\right) \mathrm{Re}(x - y, T^j x - T^j y).
\end{aligned}$$

Taking $y = T_{\alpha_n} x$ in the aforementioned inequality and using (*iii*) and the assumption imposed on T, we have

$$\|T_{\alpha_n} x - T_{\alpha_n}^2 x\|^2 \le \left(1 - \sum_{j=1}^{n} \alpha_j\right)^2 \|x - T_{\alpha_n} x\|^2 + \left(\sum_{k=1}^{i} \beta_k \alpha_k\right)^2 \|x - T_{\alpha_n} x\|^2$$

$$+\left(\sum_{r=i+1}^{n}L_r\alpha_r\right)^2\|x-T_{\alpha_n}x\|^2+\sum_{j=1}^{n}2\alpha_j\mu_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\|x-T_{\alpha_n}x\|^2$$
$$=\left[\left(1-\sum_{i=1}^{n}\alpha_i\right)^2+\left(\sum_{k=1}^{i}\beta_k\alpha_k\right)^2+\left(\sum_{r=i+1}^{n}L_r\alpha_r\right)^2+\sum_{j=1}^{n}2\alpha_j\mu_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\right]\|x-T_{\alpha_n}x\|^2.$$

We denote $\gamma=\sqrt{\left(1-\sum_{j=1}^{n}\alpha_j\right)^2+\left(\sum_{k=1}^{i}\beta_k\alpha_k\right)^2+\left(\sum_{r=i+1}^{n}L_r\alpha_r\right)^2+\sum_{j=1}^{n}2\alpha_j\mu_j(1-\sum_{j=1}^{n}\alpha_j)}$ and $\xi=\sum_{j=1}^{n}\alpha_j$.

In the same manner as in the proof of Theorem 2.4, we deduce easily that $\gamma<1$. Also, we have by (*ii*) that $\sum_{j=1}^{n}\alpha_j\mu_j<\frac{\xi}{2}$.

So, we have $\gamma<1$ and

$$\|T_{\alpha_n}x-T_{\alpha_n}^2x\|\le\gamma\|x-T_{\alpha_n}x\|,$$

for all $x\in A$. Thus, by graphic contraction principle, T_{α_n} is a weakly Picard operator and the sequence $\{T_{\alpha_n}^m x_0\}_{m\in\mathbb{N}}$ converges to $T_{\alpha_n}^{\infty}x_0 \coloneqq x^*\in F(T_{\alpha_n})$ for every $x_0\in A$.

We can prove the uniqueness of the fixed point of T_{α_n}, similar to the preceding theorem.

Thus, $F(T_{\alpha_n})$ is a singleton and T_{α_n} is a Picard operator. □

Theorem 2.6. *Let $(X,\langle\cdot\rangle)$ be a Hilbert space, A be a nonempty closed and convex subset of X, and $T:A\to A$ be an operator with a closed graph. Let n be an integer with $n\ge 2$. We suppose that*
(*i*) *T is a weak (α_n,β)-convex orbital Lipschitz operator of type II*;
(*ii*) *the assumption (W'') is satisfied*;
(*iii*) $\mathrm{Re}\langle x-y,T^jx-T^jy\rangle\le\mu_j\|x-y\|^2$, *for any $1\le j\le n$ and $x,y\in A$, where $\mu_j\in(0,\frac{1}{2})$.*

Then, for every $x_0\in A$, the sequence $\{x_m\}\subset A$, defined by

$$x_m=\left(1-\sum_{j=1}^{n}\alpha_j\right)x_{m-1}+\sum_{j=1}^{n}\alpha_jT^jx_{m-1},\ m\in\mathbb{N},$$

where $\alpha_1>0$, $\alpha_j\ge 0$, $2\le j\le n$ with $\sum_{j=1}^{n}\alpha_j\in(0,1]$, converges to the unique fixed point $x^*\in A$ of T_{α_n}.

Proof. In much the same way as in the proof of Theorem 2.5, we have for any $x,y\in A$,

$$\|T_{\alpha_n}x-T_{\alpha_n}y\|^2\le\left(1-\sum_{j=1}^{n}\alpha_j\right)^2\|x-y\|^2+\left(\sum_{j=1}^{n}\alpha_j\|T^jx-T^jy\|\right)^2+\sum_{j=1}^{n}2\alpha_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\mathrm{Re}\langle x-y,T^jx-T^jy\rangle.$$

Taking $y=T_{\alpha_n}x$ in the aforementioned inequality and using (*ii*), (*iii*), and (2), we have:

Case 1. If $n=2l$, $l\in\mathbb{N}$.

$$\|T_{\alpha_n}x-T_{\alpha_n}^2x\|^2\le\left(1-\sum_{j=1}^{2l}\alpha_j\right)^2\|x-T_{\alpha_n}x\|^2+\left[\left(\sum_{r=1}^{l}\alpha_{2r-1}\beta L^{r-1}\right)+\left(\sum_{j=1}^{l}\alpha_{2j}L^j\right)\right]^2\|x-T_{\alpha_n}x\|^2$$
$$+\left(\sum_{j=1}^{n}2\alpha_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\mu_j\right)\|x-T_{\alpha_n}x\|^2$$
$$=\left[\left(1-\sum_{i=1}^{n}\alpha_i\right)^2+\left[\left(\sum_{r=1}^{l}\alpha_{2r-1}\beta L^{r-1}\right)+\left(\sum_{j=1}^{l}\alpha_{2j}L^j\right)\right]^2+\sum_{j=1}^{n}2\alpha_j\mu_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\right]\|x-T_{\alpha_n}x\|^2.$$

Case 2. If $n=2l+1$, $l\in\mathbb{N}$.

$$\|T_{\alpha_n}x-T_{\alpha_n}^2x\|^2\le\left(1-\sum_{j=1}^{2l}\alpha_j\right)^2\|x-T_{\alpha_n}x\|^2+\left[\left(\sum_{r=0}^{l}\alpha_{2r+1}\beta L^r\right)+\left(\sum_{j=1}^{l}\alpha_{2j}L^j\right)\right]^2\|x-T_{\alpha_n}x\|^2$$

$$+\left(\sum_{j=1}^{n} 2\alpha_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\mu_j\right)\|x-T_{\alpha_n}x\|^2$$
$$=\left[\left(1-\sum_{i=1}^{n}\alpha_i\right)^2+\left[\left(\sum_{r=0}^{l}\alpha_{2r+1}\beta L^r\right)+\left(\sum_{j=1}^{l}\alpha_{2j}L^j\right)\right]^2+\sum_{j=1}^{n}2\alpha_j\mu_j\left(1-\sum_{j=1}^{n}\alpha_j\right)\right]\|x-T_{\alpha_n}x\|^2.$$

We denote $\gamma_1 = \sqrt{(1-\sum_{i=1}^{n}\alpha_i)^2 + [(\sum_{r=1}^{l}\alpha_{2r-1}\beta L^{r-1}) + (\sum_{j=1}^{l}\alpha_{2j}L^j)]^2 + \sum_{j=1}^{n}2\alpha_j\mu_j(1-\sum_{j=1}^{n}\alpha_j)}$ in the first case and $\gamma_2 = \sqrt{(1-\sum_{i=1}^{n}\alpha_i)^2 + [(\sum_{r=0}^{l}\alpha_{2r+1}\beta L^{r}) + (\sum_{j=1}^{l}\alpha_{2j}L^j)]^2 + \sum_{j=1}^{n}2\alpha_j\mu_j(1-\sum_{j=1}^{n}\alpha_j)}$ in the second one. We also set $\gamma = \max\{\gamma_1, \gamma_2\}$.

In the same manner as in the proof of Theorem 2.4, we deduce easily that $\gamma < 1$ and thus,

$$\|T_{\alpha_n}x - T_{\alpha_n}^2x\| \le \gamma\|x - T_{\alpha_n}x\|,$$

for all $x \in A$. Consequently, by graphic contraction principle, T_{α_n} is a weakly Picard operator and the sequence $\{T_{\alpha_n}^m x_0\}_{m\in\mathbb{N}}$ converges to $T_{\alpha_n}^\infty x_0 \coloneqq x^* \in F(T_{\alpha_n})$ for every $x_0 \in A$.

We can prove the uniqueness of the fixed point of T_{α_n}, similar to those of Theorem 2.4. Thus, $F(T_{\alpha_n})$ is a singleton and T_{α_n} is a Picard operator.

Now, we present some additional properties of the fixed point equation $x = T_{\alpha_n}x$. □

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X, and n be an integer with $n \ge 2$. Let $T : A \to A$ be a weak (α_n, β_n)-convex orbital Lipschitz operator of type II with closed graph, where $0 < \beta_k < 1$, $k = 1, 2, \ldots, n$. Then, the following conclusions hold*:

(*i*) *T_{α_n} satisfies the following retraction-displacement condition:*

$$\|x - x^*(x)\| \le \frac{1}{1-\gamma}\|x - T_{\alpha_n}x\|, \quad \textit{for every } x \in A,$$

where $\gamma = (1 - \sum_{k=1}^{n}\alpha_i) + \sum_{k=1}^{n}\beta_k\alpha_k$ and $x^*(x) = T_{\alpha_n}^\infty x$ (the fixed point of T_{α_n} starting from x);

(*ii*) *the fixed point equation $T_{\alpha_n}x = x$ is Ulam-Hyers stable*;

(*iii*) *if $\min\{\beta_k\}_{k=1}^n < \frac{1}{3}$ and $\sum_{k=1}^{n}\alpha_k > \frac{2}{3(1-\min\{\beta_k\}_{k=1}^n)}$, then T_{α_n} has the Ostrowski stability property.*

Proof.

(i) By the proof of Theorem 2.1, the operator T_{α_n} is weakly Picard. By graphic contraction principle (Theorem 1.1), we have

$$\|x - x^*(x)\| \le \frac{1}{1-\gamma}\|x - T_{\alpha_n}x\|,$$

for every $x \in A$, where $\{T_{\alpha_n}^n x\}_{n\in\mathbb{N}}$ converges to $x^*(x)$ and $\gamma = \left(1 - \sum_{k=1}^{n}\alpha_k\right) + \sum_{k=1}^{n}\beta_k\alpha_k$.

(ii) Let $\varepsilon > 0$ and $v \in A$ such that $\|v - T_{\alpha_n}v\| \le \varepsilon$. Then, we have

$$\|v - x^*(v)\| \le \frac{1}{1-\gamma}\|v - T_{\alpha_n}v\| \le \frac{\varepsilon}{1-\gamma}.$$

Hence, the fixed point equation $T_{\alpha_n}x = x$ is Ulam-Hyers stable.

(iii) From the graphic contraction principle, we know that T_{α_n} has the Ostrowski stability property if $\gamma < \frac{1}{3}$. This implies that $(1 - \sum_{k=1}^{n}\alpha_k) + \sum_{k=1}^{n}\beta_k\alpha_k < \frac{1}{3}$. Thus,

$$\left(1-\sum_{k=1}^{n}\alpha_k\right) + \min\{\beta_k\}_{k=1}^n\sum_{k=1}^{n}\alpha_k$$
$$\le \left(1-\sum_{k=1}^{n}\alpha_k\right) + \sum_{k=1}^{n}\beta_k\alpha_k < \frac{1}{3}$$

$$\Rightarrow \frac{2}{3\left(\sum_{k=1}^{n}\alpha_k\right)} < 1 - \min\{\beta_k\}_{k=1}^{n}$$

$$\Rightarrow \frac{2}{3(1 - \min\{\beta_k\}_{k=1}^{n})} < \sum_{k=1}^{n}\alpha_k.$$

Since $\min\{\beta_k\}_{k=1}^{n} < \frac{1}{3}$, we have $\sum_{k=1}^{n}\alpha_k < 1$; therefore, there exist $\alpha_1 > 0, \alpha_k \geq 0, 1 \leq k \leq n$, such that $\frac{2}{3(1-\min\{\beta_k\}_{k=1}^{n})} < \sum_{k=1}^{n}\alpha_k$. Also, in this case, T_{α_n} is a $\frac{\gamma}{1-2\gamma}$-quasi-contraction. □

Theorem 2.8. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X and i, n be any positive integers with $i < n, 2 \leq n$. Let $T : A \to A$ be a weak (α_n, β_i)-convex orbital Lipschitz operator of type II with closed graph, where $\beta_k > 0, k = 1, 2, \ldots, i$. Suppose that the assumption (W') is satisfied. Then, the following conclusions hold*:

(*i*) *T_{α_n} satisfies the following retraction-displacement condition*:

$$\|x - x^*(x)\| \leq \frac{1}{1-\gamma}\|x - T_{\alpha_n}x\|, \quad \textit{for every } x \in A,$$

where $\gamma = (1 - \sum_{i=1}^{n}\alpha_i) + \sum_{k=i}^{n}\beta_k\alpha_k + \sum_{r=i+1}^{n}L_r\alpha_r$;

(*ii*) *the fixed point equation $T_{\alpha_n}x = x$ is Ulam-Hyers stable*;

(*iii*) *if* $\min\{\beta_k, L_r\} < \frac{1}{3}$ *and* $\sum_{j=1}^{n}\alpha_j > \frac{2}{3(1-\min\{\beta_k, L_r\})}$, *then T_{α_n} has the Ostrowski stability property.*

Proof. The conclusions follow using similar arguments stated in the proof of Theorem 2.7. □

Theorem 2.9. *Let $(X, \|\cdot\|)$ be a Banach space, A be a nonempty closed and convex subset of X, and n be any integer with $n \geq 2$. Let $T : A \to A$ be a weak (α_n, β)-convex orbital Lipschitz operator of type II with closed graph with $\beta > 0$. Suppose that the assumption (W'') is satisfied. Then, the following conclusions hold*:

(*i*) *T_{α_n} satisfies the following retraction-displacement condition:*

$$\|x - x^*(x)\| \leq \frac{1}{1-\gamma}\|x - T_{\alpha_n}x\|, \quad \textit{for every } x \in A,$$

where

$$\gamma \coloneqq \left(1 - \sum_{i=1}^{2l}\alpha_i\right) + \left(\sum_{r=1}^{l}\alpha_{2r-1}\beta L^{r-1}\right) + \left(\sum_{j=1}^{l}\alpha_{2j}L^j\right)$$

or

$$\gamma \coloneqq \left(1 - \sum_{i=1}^{2l+1}\alpha_i\right) + \left(\sum_{r=0}^{l}\alpha_{2r+1}\beta L^{r}\right) + \left(\sum_{j=1}^{l}\alpha_{2j}L^j\right),$$

with $l \in \mathbb{N}$;

(*ii*) *the fixed point equation $T_{\alpha_n}x = x$ is Ulam-Hyers stable*;

(*iii*) *if* $\min\{\beta, L\} < \frac{1}{3}$ *and* $\sum_{i=1}^{n}\alpha_i > \frac{2}{3(1-\min\{\beta, L\})}$, *then T_{α_n} has the Ostrowski stability property.*

Proof. The conclusions may be proved in much the same way as Theorem 2.8. □

3 Conclusion

This study introduces the concepts of (α_n, β_i)-convex orbital Lipschitz operators, weak (α_n, β_i)-convex orbital Lipschitz operators, which can be seen as generalizations of the concepts of convex orbital λ-Lipschitz operators, weak convex orbital Lipschitz operators, and convex orbital (λ, β)-Lipschitz operators previously introduced by Popescu [8] and Petrusel et al. [9]. Furthermore, many well-known classical contractions, such as Banach contractions, Kannan contractions, Ćirć-Reich-Rus contractions, Berinde contractions, non-expansive operators, enriched (b, θ)-contractions, and Lipschitz operators, can be viewed as special cases of our concept. A fixed point of a self-mapping T can be approximated using Krasnoselskii's iterates, which are equivalent to the Picard iteration of the averaged mapping associated with T defined by $x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n$ for every initial point x_0 and $\lambda \in [0, 1)$. It is well known that Krasnoselskii's iteration is a generalization of Picard's iteration. Recently, Popescu [8] proved a fixed point for a convex orbital (λ, β)-Lipschitz operator T using the graphic contraction principle and obtained an approximation of the fixed point with Krasnoselskii's iterates. On the other hand, the Kirk iteration [14] can also be seen as a generalization of Krasnoselskii's iteration, in which the iterates are generated by the k-fold averaged mapping, a generalization of classical averaged mappings and double-averaged mappings. This approximation method is full of great practical significance because the iterative formula not only contains Tx but also $T^k (k \geq 2)$. These higher-order items may represent many useful information in practical engineering and physical problems. Based on this, we extend Popescu's main results from Krasnoselskii's iterative scheme to Kirk's iterative scheme. This motivation leads to the introduction of new notions of operators and new fixed point theorems proved in this study.

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