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Research Article

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Solving nonlinear fractional differential equations by common fixed point results for a pair of (α, Θ)-type contractions in metric spaces

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Abstract: The problem of common solutions for nonlinear equations has significant theoretical and practical value. In this article, we first introduce a new concept of a pair of (α, Θ)-type contractions, and then, we present some common fixed point results for the contractions in complete metric spaces. Finally, our results are applied to consider the existence, uniqueness and approximation of common solutions for two classes of nonlinear fractional differential equations.

Keywords: common fixed point, a pair of (α, Θ)-type contractions, existence and uniqueness, nonlinear fractional differential equations

MSC 2020: 54H25, 47H09, 47H10

1 Introduction

The famous Banach contraction principle plays a milestone role in the research of fixed point theory, and it is also widely applied in the study of the existence and convergence of solutions for nonlinear differential equations and integral equations, control theory, fractal generation, and so on. For the purpose of theoretical and practical application, many scholars have extended the Banach contraction principle, which mainly focuses on more general nonlinear operators in various types of spaces, such as Kannan contraction [1], Chatterjea contraction [2], Reich contraction [3], Ciric contraction [4], and Moosaei contraction [5,6].

In 2012, Samet et al. [7] first introduced a new concept of α-ψ-contractive type mappings, and if the mappings are continuous and α-admissible, then they established some fixed point results in complete metric spaces and considered some applications to ordinary differential equations. Based on the notions of α-ψ-contractive mappings and α-admissible mappings, Dumitru et al. [8] presented some existence theorems for some nonlinear fractional differential equations with various boundary conditions. Jleli and Samet [9] constructed a new variant of Θ-contractive mappings and gave a new fixed point theorem for such mappings in generalized metric spaces. Hussain et al. [10] developed a generalized contraction called the JS-contraction and proved a fixed point result using the continuity of the mapping. Later, researchers [11,12] gave the notion of weak Θ-contraction by changing or omitting some conditions posed on the function set Θ, and they also established some related fixed point theorems in complete metric spaces. Motived by [7–11], Abdou [13] introduced the notions of orthogonal Θ-contraction and orthogonal (α, Θ)-contraction in orthogonally

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complete metric spaces (which can be considered as Hilbert spaces) and obtained two generalied fixed point results, and as an application, he also investigated the solution of a nonlinear fractional differential equation.

Common fixed point results are of great significance in discussing the existence, uniqueness, and convergence of common solutions for nonlinear equations. In recent years, many authors [14–18] have discussed common fixed point theorems of nonlinear mappings and their related applications in solving nonlinear equations. Recently, Iqbal et al. [19] derived some common fixed point results for $\mathfrak{I}$-type mappings satisfying certain contractive conditions and the existence results of common solutions of nonlinear fractional differential equation were developed.

In this article, we first introduce the concept of a pair of (α, Θ)-type contractions in complete metric spaces and prove some common fixed point theorems for the mappings. And then, we provide an example to illustrate our main conclusion. Moreover, we apply our results to discuss the common solutions for two classes of nonlinear fractional differential equations. Our results extend the main results in [9–13] and many existing results.

2 Preliminaries

Throughout this article, let (X, d) be a complete metric space and $\Gamma_1, \Gamma_2 : X \to X$ be two self-mappings. $z^* \in X$ is a common fixed point of Γ_1 and Γ_2 if $z^* = \Gamma_1 z^* = \Gamma_2 z^*$. Consistent with [9–13], we denote Θ by the set of all functions $\theta : [0, +\infty) \to [1, +\infty)$ and give the following five conditions for the function θ:

(θ1): θ is nondecreasing and $\theta(t) = 1 \Leftrightarrow t = 0$.

(θ2): for any $\{t_n\} \subseteq (0, +\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0.$$

(θ3): θ is continuous.

(θ4): there exist $r \in (0, 1)$ and $l \in (0, +\infty)$ such that

$$\lim_{t\to 0+} \frac{\theta(t) - 1}{t^r} = l.$$

(θ5): $\theta(a + b) \leq \theta(a)\cdot \theta(b)$, for any $a, b > 0$.

We know that condition (θ3) and condition (θ4) are independent of each other (see some examples in [11,12]). Particularly if any $\theta \in \Theta$ satisfies condition (θ4), then some useful functions do not belong to the function set Θ, for example $\theta(t) = e^t$. (We will use this function to demonstrate that our mappings is a natural extension of many existing mappings.) Now, we give the following new concept.

Definition 2.1. The two mappings Γ_1 and Γ_2 are called a pair of (α, Θ)-type contractions if there exist the function $\alpha : X \times X \to [0, +\infty)$, $\theta \in \Theta$ and nonnegative real numbers a, b, c such that for any $x, y \in X$, $d(\Gamma_1 x, \Gamma_2 y) \neq 0$ implies

$$\alpha(x, y)\theta[d(\Gamma_1 x, \Gamma_2 y)] \leq [\theta(d(x, y))]^a \cdot [\theta(d(x, \Gamma_1 x) + d(y, \Gamma_2 y))]^b \cdot [\theta(d(x, \Gamma_2 y) + d(y, \Gamma_1 x))]^c, \tag{2.1}$$

where $a + 2b + 2c < 1$ and θ satisfies conditions (θ1 − θ3, θ5).

Remark 2.2.

(i) If $\Gamma_1 = \Gamma_2$, and θ satisfies conditions (θ1, θ2, θ4, θ5), by the symmetry, then (2.1) reduces to the (α, Θ)-type contraction in [13].

(ii) If $\alpha(x, y) = 1$, $\Gamma_1 = \Gamma_2$ is continuous and θ satisfies conditions (θ1, θ2, θ4, θ5), by the symmetry, then (2.1) reduces to the continuous JS-contraction in [10].

(iii) If $\alpha(x, y) = 1$, $\Gamma_1 = \Gamma_2$, $b = c = 0$, and θ satisfies conditions (θ2, θ3) or (θ2, θ4), then (2.1) reduces to the weak Θ-contraction in [12].

(iv) If $\alpha(x,y) = 1, \Gamma_1 = \Gamma_2, b = c = 0$ and θ satisfies conditions $(\theta 1, \theta 2, \theta 3)$, then (2.1) reduces to the Θ-contraction in [11, 13].
(v) If $\alpha(x,y) = 1, \Gamma_1 = \Gamma_2, b = c = 0$ and θ satisfies conditions $(\theta 1, \theta 2, \theta 4)$, then (2.1) reduces to the Θ-contractive mappings in [9].

According to the definition of α-admissible in [7], we introduce the following concept for a pair of mappings.

Definition 2.3. Let $\Gamma_1, \Gamma_2, : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then, Γ_1 and Γ_2 are said to be a pair of α-admissible if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \text{ implies } \alpha(\Gamma_1 x, \Gamma_2 y) \geq 1.$$

Example 2.1. Let $X = [0, +\infty)$. Define $\Gamma_1, \Gamma_2 : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$\Gamma_1(x) = x \quad \text{and} \quad \Gamma_2(x) = \ln(x+1)$$

and

$$\alpha(x,y) = \begin{cases} 1, & x \geq y, \\ 0, & x < y. \end{cases}$$

Then, Γ_1 and Γ_2 are a pair of α-admissible mappings.

3 Main results

Theorem 3.1. *Suppose* $\Gamma_1, \Gamma_2 : X \to X$ *are a pair of* (α, Θ)*-type contractions and* α*-admissible mappings. If there exists* z_0 *such that*

$$\alpha(z_0, \Gamma_1 z_0) \geq 1 \quad \text{and} \quad \alpha(z_1, \Gamma_2 z_1) \geq 1,$$

where $z_1 = \Gamma_1 z_0$, *then* Γ_1 *and* Γ_2 *have a unique common fixed point.*

Proof. Define the sequence $\{z_n\}$ $(n \geq 0)$ by

$$\{x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots\}, \tag{3.1}$$

where $x_0 = z_0, y_n = z_{2n+1} = \Gamma_1 x_n$ and $x_n = z_{2n} = \Gamma_2 y_n$. Since $\alpha(z_0, \Gamma_1 z_0) \geq 1$, $\alpha(z_1, \Gamma_2 z_1) \geq 1$, and Γ_1 and Γ_2 are a pair of α-admissible mappings, we obtain $\alpha(\Gamma_1 z_n, \Gamma_2 z_{n+1}) \geq 1$ for any $n \geq 0$. If $z_n = z_{n+1}$, then

$$z_n = \Gamma_1 z_n = \Gamma_2 z_{n+1} \quad \text{or} \quad z_n = \Gamma_2 z_n = \Gamma_1 z_{n+1}.$$

It is obvious that z_n is a common fixed point of Γ_1 and Γ_2. Without loss of generality, we assume that $z_n \neq z_{n+1}$ for any $n \geq 0$. We give the following four Claims to prove our result.

Claim I. $\lim_{n\to\infty} d(z_{n+1}, z_n) = 0$ for any $n \geq 0$. There are two cases.

Case 1: If n is an odd number, then by (3.1), conditions $(\theta 1, \theta 5)$, we have

$$\begin{aligned}
\theta[d(z_{n+1}, z_n)] &= \theta[d(\Gamma_2 z_n, \Gamma_1 z_{n-1})] \\
&\leq [\theta(d(z_n, z_{n-1}))]^a \cdot [\theta(d(z_{n-1}, \Gamma_1 z_{n-1}) + d(z_n, \Gamma_2 z_n))]^b \cdot [\theta(d(z_n, \Gamma_1 z_{n-1}) + d(z_{n-1}, \Gamma_2 z_n))]^c \\
&\leq [\theta(d(z_n, z_{n-1}))]^a \cdot [\theta(d(z_{n-1}, \Gamma_1 z_{n-1}))]^b \cdot [\theta(d(z_n, \Gamma_2 z_n))]^b \cdot [\theta(d(z_n, \Gamma_1 z_{n-1}))]^c \cdot [\theta(d(z_{n-1}, \Gamma_2 z_n))]^c \\
&\leq [\theta(d(z_n, z_{n-1}))]^a \cdot [\theta(d(z_{n-1}, z_n))]^b \cdot [\theta(d(z_n, z_{n+1}))]^b \cdot [\theta(d(z_n, z_n))]^c \cdot [\theta(d(z_{n-1}, z_{n+1}))]^c \\
&\leq [\theta(d(z_n, z_{n-1}))]^a \cdot [\theta(d(z_{n-1}, z_n))]^b \cdot [\theta(d(z_n, z_{n+1}))]^b \cdot [\theta(d(z_{n-1}, z_n))]^c \cdot [\theta(d(z_n, z_{n+1}))]^c,
\end{aligned}$$

which implies

$$1 < \theta[d(z_{n+1}, z_n)] \leq [\theta(d(z_n, z_{n-1}))]^{\frac{a+b+c}{1-b-c}} \leq [\theta(d(z_1, z_0))]^{(\frac{a+b+c}{1-b-c})^n}.$$

Since $0 \leq \frac{a+b+c}{1-b-c} < 1$, we know that $\lim_{n\to\infty}\theta(d(z_{n+1}, z_n)) = 1$. By condition ($\theta2$), we have

$$\lim_{n\to\infty} d(z_{n+1}, z_n) = 0.$$

Case 2: If n is an even number, then by the similar method as Case 1, we can also obtain $\lim_{n\to\infty} d(z_{n+1}, z_n) = 0$. So, for any $n \geq 0$, we have

$$\lim_{n\to\infty} d(z_{n+1}, z_n) = 0.$$

Claim II. There exists $z^* \in X$ such that $\lim_{n\to\infty} z_n = z^*$. We first show that $\{x_n = z_{2n}\}$ is a Cauchy sequence in X. Suppose that $\{z_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon_0 > 0$, for any $2k$ ($k \geq 0$), there exist $2n(k)$, $2m(k)$ with $2k < 2n(k) < 2m(k)$ such that

$$d(z_{2n(k)}, z_{2m(k)}) > \varepsilon_0. \tag{3.2}$$

Without loss of generality, for each integer $2k$, suppose $2m(k)$ is the least exceeding $2n(k)$ satisfying (3.2), that is

$$d(z_{2n(k)}, z_{2m(k)-2}) \leq \varepsilon_0. \tag{3.3}$$

Then

$$\varepsilon_0 < d(z_{2n(k)}, z_{2m(k)}) \leq d(z_{2n(k)}, z_{2m(k)-2}) + d(z_{2m(k)-2}, z_{2m(k)-1}) + d(z_{2m(k)-1}, z_{2m(k)}).$$

By (3.3) and Claim I, we obtain

$$\lim_{k\to\infty} d(z_{2n(k)}, z_{2m(k)}) = \varepsilon_0. \tag{3.4}$$

Note that

$$|d(z_{2n(k)}, z_{2m(k)-1}) - d(z_{2n(k)}, z_{2m(k)})| \leq d(z_{2m(k)-1}, z_{2m(k)}).$$

Then

$$\lim_{k\to\infty} d(z_{2n(k)}, z_{2m(k)-1}) = \varepsilon_0. \tag{3.5}$$

Since

$$\begin{aligned}
&\theta[d(\Gamma_1 z_{2n(k)}, \Gamma_2 z_{2m(k)-1})] \\
&\quad\leq [\theta(d(z_{2n(k)}, z_{2m(k)-1}))]^a \cdot [\theta(d(z_{2m(k)-1}, \Gamma_2 z_{2m(k)-1}))]^b \\
&\qquad\cdot [\theta(d(z_{2n(k)}, \Gamma_1 z_{2n(k)}))]^b \cdot [\theta(d(z_{2m(k)-1}, \Gamma_1 z_{2n(k)}))]^c \cdot [\theta(d(z_{2n(k)}, \Gamma_2 z_{2m(k)-1}))]^c \\
&\quad\leq [\theta(d(z_{2n(k)}, z_{2m(k)-1}))]^a \cdot [\theta(d(z_{2m(k)-1}, z_{2m(k)}))]^b \cdot [\theta(d(z_{2n(k)}, z_{2n(k)+1}))]^b \\
&\qquad\cdot [\theta(d(z_{2m(k)-1}, z_{2n(k)}))]^c \cdot [\theta(d(z_{2n(k)}, z_{2n(k)+1}))]^c \cdot [\theta(d(z_{2n(k)}, z_{2m(k)}))]^c
\end{aligned}$$

and

$$d(z_{2n(k)}, z_{2m(k)-1}) - d(z_{2n(k)}, \Gamma_1 z_{2n(k)}) - d(z_{2m(k)-1}, \Gamma_2 z_{2m(k)-1}) \leq d(\Gamma_1 z_{2n(k)}, \Gamma_2 z_{2m(k)-1}),$$

by the condition ($\theta1 - \theta3$), (3.4), (3.5) and Claim I, taking $k \to \infty$, we have

$$\theta(\varepsilon_0) \leq [\theta(\varepsilon_0)]^{a+2c} < \theta(\varepsilon_0),$$

which yields a contradiction. So, $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $z^* \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_{2n} = z^*.$$

Similarly, we can prove that the sequence $\{y_n = z_{2n+1}\}$ is also a Cauchy sequence. So, there exists $y^* \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_{2n+1} = y^*.$$

Since $\lim_{n\to\infty} d(z_{n+1}, z_n) = 0$, we have $d(y^*, z^*) = 0$, i.e. $y^* = z^*$. So, $\lim_{n\to\infty} z_n = z^*$.

Claim III. z^* is a common fixed point of Γ_1 and Γ_2, that is $z^* = \Gamma_1 z^* = \Gamma_2 z^*$. Since $z_{2n+1} = \Gamma_1 x_n$, we have

$$\lim_{n\to\infty} \Gamma_1 x_n = z^*. \tag{3.6}$$

Suppose $z^* \neq \Gamma_2 z^*$. Let $x = x_n$ and $y = z^*$ in (2.1), we have

$$\begin{aligned}1 &< \theta[d(\Gamma_1 x_n, \Gamma_2 z^*)]\\ &\leq [\theta(d(x_n, z^*))]^a \cdot [\theta(d(x_n, \Gamma_1 x_n))]^b \cdot [\theta(d(z^*, \Gamma_2 z^*))]^b \cdot [\theta(d(x_n, \Gamma_2 z^*))]^c \cdot [\theta(d(z^*, \Gamma_1 x_n))]^c.\end{aligned}$$

By (3.6), the condition ($\theta 2 - \theta 3$) and Claim II, we have

$$\begin{aligned}1 &< \lim_{n\to\infty} \theta[d(\Gamma_1 x_n, \Gamma_2 z^*)]\\ &= \theta(d(z^*, \Gamma_2 z^*))\\ &\leq 1^a \cdot 1^b \cdot [\theta(d(z^*, \Gamma_2 z^*))]^b \cdot \left[\lim_{n\to\infty} \theta(d(x_n, \Gamma_2 z^*))\right]^c \cdot 1^c\\ &= [\theta(d(z^*, \Gamma_2 z^*))]^{b+c},\end{aligned}$$

which yields a contradiction. So, $z^* = \Gamma_2 z^*$. On the other hand, by the same method, it follows from $\lim_{n\to\infty} \Gamma_2 y_n = z^*$ that $z^* = \Gamma_1 z^*$. So, z^* is a common fixed point of Γ_1 and Γ_2.

Claim IV. z^* is a unique common fixed point of Γ_1 and Γ_2. If there exists another point $\tilde{z}$ such that

$$\tilde{z} \neq z^* \quad \text{and} \quad \tilde{z} = \Gamma_1 \tilde{z} = \Gamma_2 \tilde{z}.$$

Then by substituting x with z^* and y with $\tilde{z}$ in (2.1), we have

$$1 < \theta(d(z^*, \tilde{z})) \leq [\theta(d(z^*, \tilde{z}))]^{a+2c} < \theta(d(z^*, \tilde{z})),$$

which yields a contradiction. Thus, Γ_1 and Γ_2 have the unique common fixed point z^*. □

Remark 3.2. By the proof of Theorem 3.1 and the symmetry, we can change (2.1) into the following type:

$$\alpha(x,y)\theta[d(\Gamma_1 x, \Gamma_2 y)] \leq [\theta(d(x,y))]^a \cdot [\theta(d(x, \Gamma_1 x))]^b \cdot [\theta(d(y, \Gamma_1 y))]^c \cdot [\theta(d(x, \Gamma_2 y) + d(y, \Gamma_1 x))]^d,$$

with $a + b + c + 2d < 1$, or

$$\alpha(x,y)\theta[d(\Gamma_1 x, \Gamma_2 y)] \leq [\theta(d(x,y))]^a \cdot [\theta(d(x, \Gamma_1 x))]^b \cdot [\theta(d(y, \Gamma_2 y))]^c \cdot [\theta(d(x, \Gamma_2 y))]^d \cdot [\theta(d(y, \Gamma_1 x))]^e,$$

with $a + b + c + d + e < 1$.

Remark 3.3. Let $\alpha(x,y) = 1$ and $\theta(t) = e^t$ in (2.1). Then θ satisfies the conditions ($\theta 1 - \theta 3, \theta 5$) and we have

$$d(\Gamma_1 x, \Gamma_2 y) \leq ad(x,y) + b[d(x, \Gamma_1 x) + d(y, \Gamma_2 y)] + c[d(x, \Gamma_2 y) + d(y, \Gamma_1 x)],$$

which is exactly Ciric-type contraction [4] for a pair of mappings. Furthermore, if $c = 0$, then we have

$$d(\Gamma_1 x, \Gamma_2 y) \leq ad(x,y) + b[d(x, \Gamma_1 x) + d(y, \Gamma_2 y)],$$

which is exactly Reich-type contraction [3] for a pair of mappings; if $a = c = 0$, then we have

$$d(\Gamma_1 x, \Gamma_2 y) \leq b[d(x, \Gamma_1 x) + d(y, \Gamma_2 y)],$$

which is exactly Kannan-type contraction [1] for a pair of mappings, if $a = b = 0$, then we have

$$d(\Gamma_1 x, \Gamma_2 y) \leq c[d(x, \Gamma_2 y) + d(y, \Gamma_1 x)],$$

which is exactly Chatterjea-type contraction [2] for a pair of mappings.

Remark 3.4. In Theorem 3.1, we do not need to suppose Γ_1 and Γ_2 are continuous functions.

Corollary 3.5. *Suppose* $\Gamma_1, \Gamma_2 : X \to X$ *are a pair of* Θ*-type contractions, i.e., there exist* $\theta \in \Theta$ *and* $k \in (0,1)$ *such that for any* $x, y \in X$, $d(\Gamma_1 x, \Gamma_2 y) \neq 0$ *implies*

$$\theta[d(\Gamma_1 x, \Gamma_2 y)] \leq [\theta(d(x,y))]^k,$$

where θ *satisfies condition* $(\theta 1 - \theta 3)$. *Then,* Γ_1 *and* Γ_2 *have a unique common fixed point.*

Proof. Let $\alpha(x,y) = 1$ and $a = k, b = c = 0$ in (2.1), by the proof of Theorem 3.1, we obtain the result. ☐

Example 3.1. Let $X = \mathbb{R}$. Define the standard metric d on X by

$$d(x,y) = |x - y|,$$

for any $x, y \in X$. Clearly, (X, d) is a complete metric space. Define $\Gamma_1, \Gamma_2 : X \to X$ by

$$\Gamma_1(x) = \begin{cases} \frac{1}{3}x, & x \in [0,1], \\ \frac{1}{4}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \Gamma_2(x) = \begin{cases} \frac{1}{4}x, & x \in [0,1], \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

We observe that Γ_1 and Γ_2 are not continuous functions. Meanwhile, the mapping $\alpha : X \times X \to [0, +\infty)$ is given by

$$\alpha(x,y) = \begin{cases} 1, & x, y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

In this case, there exists z_0 such that $\alpha(z_0, \Gamma_1 z_0) \geq 1$ and $\alpha(z_1, \Gamma_2 z_1) \geq 1$, In fact, for $z_0 = 1$, we have

$$\alpha(1, \Gamma_1(1)) = \alpha\left(1, \frac{1}{3}\right) = 1 \quad \text{and} \quad \alpha\left(\frac{1}{3}, \Gamma_2\left(\frac{1}{3}\right)\right) = \alpha\left(\frac{1}{3}, \frac{1}{12}\right) = 1.$$

Let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies $x, y \in [0,1]$ and by the definitions of Γ_1, Γ_2 and α, we have

$$\Gamma_1 x = \frac{1}{3}x \in [0,1], \quad \Gamma_2 y = \frac{1}{4}y \in [0,1], \quad \text{and} \quad \alpha(\Gamma_1 x, \Gamma_2 y) = 1.$$

So, $\Gamma_1, \Gamma_2 : X \to X$ are a pair of α-admissible mappings.

Moreover, $\Gamma_1, \Gamma_2 : X \to X$ are a pair of (α, Θ)-type contractions with $\theta(t) = e^t$ for $t > 0$. In fact, there exist

$$a = \frac{1}{3}, \quad b = \frac{1}{9}, \quad \text{and} \quad c = \frac{1}{9},$$

such that

$$\begin{aligned}\alpha(x,y)\theta[d(\Gamma_1 x, \Gamma_2 y)] &= e^{d(\frac{1}{3}x, \frac{1}{4}y)} = e^{|\frac{1}{3}x - \frac{1}{4}y|} \\ &\leq (e^{|x-y|})^{\frac{1}{3}} \cdot \left(e^{|x - \frac{1}{4}x| + |y - \frac{1}{4}y|}\right)^{\frac{1}{9}} \left(e^{|x - \frac{1}{4}y| + |y - \frac{1}{3}x|}\right)^{\frac{1}{9}} \\ &\leq [\theta(d(x,y))]^a \cdot [\theta(d(x, \Gamma_1 x) + d(y, \Gamma_2 y))]^b \cdot [\theta(d(x, \Gamma_2 y) + d(y, \Gamma_1 x))]^c\end{aligned}$$

for any $x, y \in [0,1]$, where $a + 2b + 2c = \frac{7}{9} < 1$.

Now, all conditions of Theorem 3.1 are satisfied. Consequently, Γ_1 and Γ_2 have a unique common fixed point. Here, $x = 0$ is the unique common fixed point.

4 Applications

In this section, by using Theorem 3.1, we discuss the common solutions of two classes of nonlinear fractional differential equations [8,13]:

Let $X = C([0,1], \mathbb{R})$ be a set of real continuous functions defined on $[0, 1]$ with the Bielecki metric

$$d(u, v) = \max_{t\in[0,1]}\{|u(t) - v(t)|e^{-\lambda t}\}, \tag{4.1}$$

where $u, v \in X$ and $\lambda > 0$ is a constant. It is well-known that (X, d) is a complete metric space.

Case I. Considering the following two nonlinear fractional differential equations:

$$D^{\alpha}x(t) + D^{\beta}x(t) = f(t, x(t)), \quad t \in [0,1], 0 < \beta < \alpha < 1, \tag{4.2}$$

with boundary value condition $x(0) = x(1) = 1$, and

$$D^{\alpha}y(t) + D^{\beta}y(t) = g(t, y(t)), \quad t \in [0,1], 0 < \beta < \alpha < 1, \tag{4.3}$$

with boundary value condition $y(0) = y(1) = 1$, where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and D^{ρ} denotes the Caputo fractional derivative of order ρ, which is defined by

$$D^{\rho}h(t) = \frac{1}{\Gamma(n-\rho)}\int_0^t (t-s)^{n-\rho-1}h^{(n)}(s)\mathrm{d}s, \tag{4.4}$$

where $\Gamma(z) = \int_0^{+\infty} s^{z-1}e^{-s}\mathrm{d}s$ is the Gamma function, $h : [0, +\infty) \to \mathbb{R}$ is a continuous function and $n - 1 < \rho < n, n = [\rho] + 1$. The Green function of (4.2) or (4.3) is given by [8]

$$G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}),$$

where $E_{p,q}(z) := \Sigma_{m=0}^{\infty}\frac{z^m}{\Gamma(mp+q)}, \quad p, q > 0$. We know that $x(t) \in X$ is a solution of (4.2) is equivalent to $x(t) \in X$ is a solution of the integral equation

$$x(t) = \int_0^t G(t-s)f(s, x(s))\mathrm{d}s,$$

for any $t \in [0,1]$. Similarly, there exists also an integral equation with the same structure for a solution $y(t) \in X$ of (4.3). Now, we introduce the following two self-mappings on X:

$$(\Gamma_1 x)(t) = \int_0^t G(t-s)f(s, x(s))\mathrm{d}s$$

and

$$(\Gamma_2 y)(t) = \int_0^t G(t-s)g(s, y(s))\mathrm{d}s.$$

Theorem 4.1. *Suppose that*

(i) *there exist a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ and three nonnegative real constants A, B, C such that for any $t \in [0,1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}$*

$$|f(t,\varphi_1) - g(t,\varphi_2)| \le A|\varphi_1 - \varphi_2| + B[|\varphi_1 - \Gamma_1\varphi_1| + |\varphi_2 - \Gamma_2\varphi_2|] + C[|\varphi_1 - \Gamma_2\varphi_2| + |\varphi_2 - \Gamma_1\varphi_1|] \tag{4.5}$$

with $A + 2B + 2C < \lambda\alpha$ and $\xi(\varphi_1, \varphi_2) \ge 0$;

(ii) *there exists $z_0 \in X$ such that $\xi(z_0(t), (\Gamma_1 z_0)(t)) \ge 0$ for any $t \in [0,1]$;*

(iii) *for any $t \in [0,1]$ and $x, y \in X$, $\xi(x(t), y(t)) \ge 0$ implies*

$$\xi(\Gamma_1 x(t), \Gamma_2 y(t)) \ge 0.$$

Then, the fractional differential equations (4.2) *and* (4.3) *have a unique common solution in X and the Picard-type algorithm* (3.1) *converges uniformly to the unqiue common solution.*

Proof. By (4.5), for any $x(t), y(t) \in X$, we obtain

$$
\begin{aligned}
&|\Gamma_1 x(t) - \Gamma_2 y(t)| \\
&\quad \le \int_0^t G(t-s)\cdot |f(s,x(s)) - g(s,y(s))| \mathrm{d}s \\
&\quad \le \int_0^t G(t-s)\cdot \{A|x(t)-y(t)| + B[|x(t)-\Gamma_1 x(t)| + |y(t)-\Gamma_2 y(t)|] \\
&\qquad + C[|x(t)-\Gamma_2 y(t)| + |y(t)-\Gamma_1 x(t)|]\}\mathrm{d}s \\
&\quad \le \int_0^t G(t-s)\cdot \{A|x(t)-y(t)|e^{-\lambda s} + B[|x(t)-\Gamma_1 x(t)|e^{-\lambda s} + |y(t)-\Gamma_2 y(t)|\cdot e^{-\lambda s}] \\
&\qquad + C[|x(t)-\Gamma_2 y(t)|e^{-\lambda s} + |y(t)-\Gamma_1 x(t)|e^{-\lambda s}]\}\cdot e^{\lambda s}\mathrm{d}s.
\end{aligned} \tag{4.6}
$$

Note that $\sup_{t\in[0,1]}\int_0^t G(t-s)\mathrm{d}s \le \frac{1}{\alpha}$ [8]. By (4.1) and (4.6), we have

$$
\begin{aligned}
|\Gamma_1 x - \Gamma_2 y| &\le \frac{1}{\alpha}\cdot\{Ad(x,y) + B[d(x,\Gamma_1 x) + d(y,\Gamma_2 y)] + C[d(x,\Gamma_2 y) + d(y,\Gamma_1 x)]\}\cdot \int_0^t e^{\lambda s}\mathrm{d}s \\
&\le \frac{1}{\alpha}\cdot\{Ad(x,y) + B[d(x,\Gamma_1 x) + d(y,\Gamma_2 y)] + C[d(x,\Gamma_2 y) + d(y,\Gamma_1 x)]\}\cdot \frac{e^{\lambda t}}{\lambda},
\end{aligned}
$$

which implies

$$
d(\Gamma_1 x, \Gamma_2 y) \le \frac{1}{\lambda\alpha}\cdot\{Ad(x,y) + B[d(x,\Gamma_1 x) + d(y,\Gamma_2 y)] + C[d(x,\Gamma_2 y) + d(y,\Gamma_1 x)]\}. \tag{4.7}
$$

Define $\theta(t) = e^t$ $(t > 0)$ and $\alpha : X \times X \to [0, +\infty)$ by

$$
\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \ge 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Then, by (4.7) and condition (i), we have

$$
\alpha(x,y)\theta[d(\Gamma_1 x, \Gamma_2 y)] \le [\theta(d(x,y))]^{\frac{A}{\lambda\alpha}}\cdot[\theta(d(x,\Gamma_1 x) + d(y,\Gamma_2 y))]^{\frac{B}{\lambda\alpha}}\cdot[\theta(d(x,\Gamma_2 y) + d(y,\Gamma_1 x))]^{\frac{C}{\lambda\alpha}},
$$

where

$$
\frac{A}{\lambda\alpha} + \frac{2B}{\lambda\alpha} + \frac{2C}{\lambda\alpha} < 1.
$$

So, Γ_1 and Γ_2 are a pair of (α, Θ)-type contractions.

For $x, y \in X$, if $\alpha(x,y) \ge 1$, then $\xi(x,y) \ge 0$. From condition (iii), we obtain $\alpha(\Gamma_1 x, \Gamma_2 y) \ge 1$. Therefore, Γ_1 and Γ_2 are α-admissible mappings.

It follows from condition (ii) and Theorem 3.1 that Γ_1 and Γ_2 have a unique common fixed point z^*; i.e., the fractional differential equations (4.2) and (4.3) have a unique common solution $z^* \in X$ and the Picard-type algorithm (3.1) converges uniformly to the solution. □

Remark 4.2. Based on the determined values of A, B, C and α, we can select the appropriate real number $\lambda > 0$ to satisfy the parameter condition

$$
\lambda > \frac{A + 2B + 2C}{\alpha}
$$

in Theorem 4.1.

By Corollary 3.5, we can also give the following result.

Corollary 4.3. *Suppose f and g are a pair of L-Lipschitz functions with respect to the second variable, i.e.,*

$$|f(t,\varphi_1) - g(t,\varphi_2)| \leq L|\varphi_1 - \varphi_2|$$

for any $t \in [0,1]$ *and* $\varphi_1, \varphi_2 \in \mathbb{R}$, *where* $0 < L < \lambda\alpha$. *Then, the Picard-type algorithm* (3.1) *converges uniformly to a unqiue common solution of the fractional differential equations* (4.2) *and* (4.3).

Case II. Considering the following two nonlinear fractional differential equations:

$$D^\eta x(t) = f(t, x(t)), \quad t \in (0,1),\ 1 < \eta \leq 2 \tag{4.8}$$

with the integral boundary conditions $x(0) = 0, x(1) = \int_0^\beta x(s)\mathrm{d}s,\ (0 < \beta < 1)$, and

$$D^\eta y(t) = g(t, y(t)), \quad t \in (0,1),\ 1 < \eta \leq 2 \tag{4.9}$$

with the integral boundary conditions $y(0) = 0, y(1) = \int_0^\beta y(s)\mathrm{d}s,\ (0 < \beta < 1)$, where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. We know that $x(t) \in X$ is a solution of (4.8) is equivalent to $x(t) \in X$ is a solution of the integral equation [8]

$$x(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s, x(s))\mathrm{d}s - \frac{2t}{2-\beta^2\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1}$$
$$\cdot f(s, x(s))\mathrm{d}s + \frac{2t}{2-\beta^2\Gamma(\eta)}\int_0^\beta\left(\int_0^s (s-m)^{\eta-1} f(m, x(m))\mathrm{d}m\right)\mathrm{d}s$$

for any $t \in [0,1]$. Similarly, there exists also an integral equation with the same structure for a solution $y(t) \in X$ of (4.9). Define two self-mappings by $(\Gamma_1 x)(t) = x(t)$ and $(\Gamma_2 y)(t) = y(t)$.

By Theorem 7 in [13] and the proof of Theorem 4.1 (let $\alpha(x,y) = 1$), we obtain the following result.

Theorem 4.4. *Suppose there exist nonnegative real constants A, B, and C such that*

$$|f(t,\varphi_1) - g(t,\varphi_2)| \leq A|\varphi_1 - \varphi_2| + B[|\varphi_1 - \Gamma_1\varphi_1| + |\varphi_2 - \Gamma_2\varphi_2|] + C[|\varphi_1 - \Gamma_2\varphi_2| + |\varphi_2 - \Gamma_1\varphi_1|] \tag{4.10}$$

for any $t \in [0,1]$ *and* $\varphi_1, \varphi_2 \in \mathbb{R}$, *where*

$$A + 2B + 2C < \frac{\lambda\alpha}{\vartheta} \quad \text{and} \quad \vartheta = \frac{t^\eta(2-\beta^2)(\eta+1) + 2t(\eta+\beta+1)}{(2-\beta^2)\eta(\eta+1)\Gamma(\eta)}.$$

Then, the fractional differential equations (4.8) *and* (4.9) *have a unique common solution in X and the Picard-type algorithm* (3.1) *converges uniformly to the unqiue common solution.*

Remark 4.5. We omit the condition "$\varphi_1 \cdot \varphi_2 \geq 0$" in Theorem 7 of [13] and give the more general contractive condition (4.10) to discuss the existence and uniqueness of the common solution of (4.8) and (4.9).

5 Conclusions

The purpose of this article is to discuss the existence and uniqueness of a pair of (α, Θ)-type contractions in general complete metric spaces. We mainly use the Picard-type algorithm (3.1) to approximate a unique common fixed point for the mappings, our results are more general than several previous results. Moreover, as some applications of our fixed point results, we consider the existence and uniqueness of the common solutions for two classes of nonlinear fractional differential equations. Our results can also be used to study the common solution problem of other types of fractional differential equations.

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