Research Article

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Vanishing viscosity limit for a one-dimensional viscous conservation law in the presence of two noninteracting shocks

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Abstract: In this article, we study the inviscid limit of the solution to the Cauchy problem of a one-dimensional viscous conservation law, where the second-order term is nonlinear. Under the assumption that the inviscid equation admits a piecewise smooth solution with two noninteracting entropy shocks, we prove that the solution of the viscous equation converges uniformly to the piecewise smooth inviscid solution away from the shocks, even the strength of shocks is not small.

Keywords: shock layer, viscous shocks, matched asymptotic expansion, nonlinear stability, energy estimates

MSC 2020: 35L50, 35L60, 35L65, 35K59, 35K65

1 Introduction

In fluid mechanics, it is well known that a fluid is composed of a large number of molecules that continuously make thermal motion and have no fixed equilibriums. When there is a relative sliding among adjacent layers of a fluid, shear stress, which is known as the viscous stress, will occur due to the interaction of these molecules. Actual fluids in nature are all viscous fluids. The motion of one-dimensional (1D) viscous fluids is described by the viscous conservation laws:

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x (B(u^\varepsilon)\partial_x u^\varepsilon), \quad u^\varepsilon \in \mathbb{R}^n,\ x \in \mathbb{R}^1,\ t > 0, \tag{1.1}$$

where u^ε denotes the density, velocity, or other physical quantities, and B is called the viscosity matrix. In the case where the viscosity is small and the relative sliding velocity is not large, the viscous stress will be small, and the thermal conduction and diffusion effects can be neglected; thus, the fluid is considered to be an ideal fluid. In this way, ideal models are only approximations of real fluids. The motion of ideal fluids can be represented in the following system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad u \in \mathbb{R}^n,\ x \in \mathbb{R}^1,\ t > 0. \tag{1.2}$$

In many physical phenomena and their numerical computations, to study the asymptotic relation between the viscous parabolic system (1.1) and its associated inviscid hyperbolic equations (1.2) in the limit of small dissipations for various viscous terms is of considerable significance. In general, the motion of the fluid is always in a domain with boundaries, and solutions of conservation laws will produce singularities except for some special cases (see [1–4], such as shock waves). The viscous flow will display singular behavior in the limit

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of small viscosity [5]. So the topic of vanishing viscosity limit has attracted much attention especially for problems in the presence of shocks and boundaries (see [6–14] and references therein).

The structure of the viscosity matrix also plays an important role in the proof of vanishing viscosity limits. The case of identity viscosity matrix has been studied in [7–9,12,14,15] for both Cauchy problems and initial boundary value problems. Among these studies, the vanishing viscosity limit of solutions to a 1D quasilinear parabolic system in the presence of a single shock discontinuity is studied in [7], where the solution of the inviscid problem is piecewise smooth with a single shock satisfying the entropy condition. It is proved that if the strength of the single shock is small for the underlying inviscid flow, then the solution to the viscous system will converge to the solution of the inviscid system away from the shock as the viscosity coefficient ε tends to zero. When the viscosity term is nonlinear, the boundary-layer problems have been discussed in [10,11,13,16–19], where the nonlinear viscosity led to more complicated analyses and computations.

In this article, enlightened by [7], we consider the Cauchy problem of the scalar case of (1.2). Precisely speaking, we study the asymptotic equivalence between the viscous problem with the general viscous term

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon(x,t)) = \varepsilon\partial_x(b(u^\varepsilon)\partial_x u^\varepsilon), \quad t>0, x\in\mathbb{R}^1, u\in\mathbb{R}^1, \tag{1.3}$$

$$u^\varepsilon(x,t=0) = u_0^\varepsilon(x), \tag{1.4}$$

and the inviscid problem

$$\partial_t u + \partial_x f(u(x,t)) = 0, \quad t>0,\ x\in\mathbb{R}^1, \tag{1.5}$$

$$u(x,t=0) = u_0^0(x), \tag{1.6}$$

where f is smooth, and we require that

$$f'(u) < 0 \quad \text{and} \quad f''(u) > 0. \tag{1.7}$$

Moreover, $b(u)$ in the viscous term is a nonnegative smooth function, and there exists $g(u(x,t))$ satisfying

$$g'(u(x,t)) = b(u(x,t)). \tag{1.8}$$

We assume that (1.5)–(1.6) admits a piecewise smooth solution, which satisfies the following conditions:
- (i) $u(x,t)$ is a distributional solution of the hyperbolic equation (1.5) in $\mathbb{R}\times[0,T]$ with $T>0$;
- (ii) there are two disjoint smooth shock curves $x=s_i(t), i=1,2, 0\le t\le T$, so that $u(x,t)$ is sufficiently smooth at any point $x\ne s_i(t)$, $i=1,2$;
- (iii) the limits

$$\partial_x^k u(s_i(t)-0,t) = \lim_{x\to s_i(t)^-}\partial_x^k(u(x,t)), \quad \text{and} \tag{1.9}$$

$$\partial_x^k u(s_i(t)+0,t) = \lim_{x\to s_i(t)^+}\partial_x^k(u(x,t)), \tag{1.10}$$

  exist and are finite for $t\le T$ and $i=1,2$;
- (iv) the Lax entropy condition is satisfied at $x=s_i(t)$, $i=1,2$, i.e.,

$$f'(u(s_i(t)-0,t)) > \frac{\mathrm{d}s_i}{\mathrm{d}t} > f'(u(s_i(t)+0,t)). \tag{1.11}$$

Then, $u(x,t)$ is a unique piecewise smooth solution with two noninteracting shocks of (1.5) for $[0,T]$.

For $m(x)\in C_0^\infty(\mathbb{R})$ satisfying $0\le m(x)\le 1$,

$$m(x) = \begin{cases} 1, & |x|\le 1,\\ 0, & |x|\ge 2,\\ h(x), & 1<|x|<2, \end{cases} \tag{1.12}$$

with $0 < h(x) < 1$ being a smooth function. Let $m_i = m(\frac{x - s_i(t)}{\varepsilon^\gamma})$, $i = 1, 2$, $\gamma \in (\frac{6}{7}, 1)$. In [7], the initial data of the two types of equations are not involved. Here, to avoid the phenomenon of the initial layer, we assume that the initial data $u_0^\varepsilon(x)$ have the following asymptotic expansion:

$$\|u_0^\varepsilon - m_1 \sum_{i=0}^{3} \varepsilon^i u_s^i(\xi, 0) + m_2 \sum_{i=0}^{3} \varepsilon^i \bar{u}_s^i(\eta, 0) + (1 - m_1 - m_2) \sum_{i=0}^{3} \varepsilon^i u_0^i(x)\|_{L^1(\mathbb{R}) \cap H^2(\mathbb{R})} \le C\varepsilon^2, \tag{1.13}$$

where $u_s^i(\xi, 0)$, $\bar{u}_s^i(\eta, 0)$, and u_0^i are the known functions in the process of expansions near or away from the shocks. In this way, the initial data u_0^ε is a more general version than the so-called "well prepared" initial data in the previous research.

We study the evolution and structure of viscous shock layers, which are related to the viscous shock profiles [20], and their interaction with interior hyperbolic inviscid flow, and show that the uniform convergence of the viscous solutions to the piecewise smooth inviscid flow away from the shock discontinuities without the assumption that the strength of the shocks is small. We obtain the results of the inviscid limit for the Cauchy problem as follows:

Theorem 1.1. *Suppose that the inviscid equation* (1.5) *is strictly hyperbolic, and there exists a constant* $\varepsilon_0 > 0$ *such that* $u(x, t)$ *is a piecewise smooth solution with two noninteracting shocks of* (1.5)–(1.6) *up to time* $T > 0$, *then for each* $0 \le \varepsilon \le \varepsilon_0$, *the nonlinear viscous equations* (1.3)–(1.4) *with the initial data satisfying* (1.13) *have a unique smooth solution* $u^\varepsilon(x, t) \in C^1([0, T]; H^2(\mathbb{R}))$ *such that*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}} |u^\varepsilon(\cdot, t) - u(\cdot, t)|^2 \mathrm{d}x \le C\varepsilon^\sigma, \tag{1.14}$$

$$\sup_{\substack{0 \le t \le T \\ |x - s_i(t)| \ge \varepsilon^\gamma}} |u^\varepsilon(\cdot, t) - u(\cdot, t)| \le C\varepsilon, \quad i = 1, 2, \tag{1.15}$$

where $\frac{3}{4} \le \sigma \le 1$, $\frac{6}{7} \le \gamma \le 1$, *and* C *is a positive constant.*

By the method of multiple-scale asymptotic expansions, we first construct the four-term approximate solutions of the viscous equations (1.3) in different regions, and then match them up to obtain the approximate solution in the whole space. For the leading order functions with fast variables, we derive the explicit solution formulas of the shock profiles from the ordinary differential equations, so that we can clearly obtain the exponential decay property. Then, similar to [14], we decompose the viscous solution into two parts with the term $\varepsilon^{1/2+\delta}\varphi$, with $0 < \delta < 1/2$, to carry out energy estimates of the error equation of φ, and we prove that the approximate solution converges uniformly to the viscous solution $u^\varepsilon(x, t)$. Consequently, the vanishing viscosity limit can be verified away from the shocks. Moreover, the method here can also be applied to the case of finite noninteracting shocks.

2 Construction of the approximate solution

In this section, we use the method of matched asymptotic expansions to give a detailed construction of the approximate solution $\bar{u}_a(x, t)$ to (1.3)–(1.4).

2.1 Outer expansion

Away from the shocks, we expand the viscous solution as

$$u^\varepsilon \sim u^0(x, t) + \varepsilon u^1(x, t) + \varepsilon^2 u^2(x, t) + \varepsilon^3 u^3(x, t) + \dots . \tag{2.1}$$

Substituting it into (1.3) and equating the coefficients of different orders of ε give

$$O(1):\partial_t u^0+\partial_x f(u^0)=0, \tag{2.2}$$

$$O(\varepsilon):\partial_t u^1+\partial_x(f'(u^0)u^1)=\partial_x(b(u^0)\partial_x u^0), \tag{2.3}$$

$$O(\varepsilon^2):\partial_t u^2+\partial_x(f'(u^0)u^2)=\partial_x^2(b(u^0)u^1)-\frac{1}{2}\partial_x(f''(u^0)(u^1)^2), \tag{2.4}$$

$$\begin{aligned}O(\varepsilon^3):\partial_t u^3+\partial_x(f'(u^0)u^3)&=\partial_x^2(b(u^0)u^2)+\frac{1}{2}\partial_x^2(b'(u^0)(u^1)^2)\\&-\partial_x(f''(u^0)(u^1\cdot u^2))-\frac{1}{6}\partial_x(f'''(u^0)(u^1)^3).\end{aligned} \tag{2.5}$$

We now require the initial data of the outer functions u^i $(i=0,\dots,3)$, which are as follows:

$$u^0(x,0)=u_0^0(x), \tag{2.6}$$

$$u^i(x,0)=u_0^i(x),\quad i=1,2,3. \tag{2.7}$$

Note that problems (2.2)–(2.6) for the leading order outer function are exactly the nonlinear problems (1.5)–(1.6). Therefore, we take u^0 to be the given unique piecewise smooth solution with two noninteracting shocks of (1.5)–(1.6). We assume the initial data $u_0^i(x)$, $i=1,2,3$ are smooth, then the functions u^1, u^2, and u^3 determined by the linear equations are generally discontinuous at shock curves $x=s_i(t)$, $i=1,2$, but smooth up to the shocks [2,4].

2.2 Expansions near shocks

Near the shock $x=s_1(t)$, we approximate u^ε as

$$u^\varepsilon(x,t)\sim u_s^0(\xi,t)+\varepsilon u_s^1(\xi,t)+\varepsilon^2u_s^2(\xi,t)+\varepsilon^3u_s^3(\xi,t)+\dots\ ,$$

where

$$\xi=\frac{x-s_1(t)}{\varepsilon}+\delta_1(t,\varepsilon),$$

and δ_1 is a disturbance of the shock position, which is to be determined. Assume that the expansion of $\delta_1(t,\varepsilon)$ is

$$\delta_1(t,\varepsilon)=\delta_1^0(t)+\varepsilon\delta_1^1(t)+\varepsilon^2\delta_1^2(t)+\dots\ .$$

Substituting it into (1.3) and equating the coefficients of different orders of ε give

$$\begin{aligned}
&O\left(\frac{1}{\varepsilon}\right):\partial_\xi f(u_s^0)-\dot{s}_1(t)\partial_\xi u_s^0-\partial_\xi(b(u_s^0)\partial_\xi u_s^0)=0,\\
&O(1):\partial_\xi^2(b(u_s^0)u_s^1)-\partial_\xi(f'(u_s^0)u_s^1)+\dot{s}_1(t)\partial_\xi u_s^1=\dot{\delta}_1^0\partial_\xi u_s^0+\partial_t u_s^0,\\
&O(\varepsilon):\partial_\xi^2(b(u_s^0)u_s^2(\xi,t)-\partial_\xi(f'(u_s^0)u_s^2)+\dot{s}_1(t)\partial_\xi u_s^2\\
&\qquad=\ \dot{\delta}_1^1\partial_\xi u_s^0+\dot{\delta}_1^0\partial_\xi u_s^1+\partial_t u_s^1+\frac{1}{2}\partial_\eta(f''(u_s^0)(u_s^1)^2)-\frac{1}{2}\partial_\eta^2(b'(u_s^0)(u_s^1)^2),\\
&O(\varepsilon^2):\partial_\xi^2(b(u_s^0)u_s^3)-\partial_\xi(f'(u_s^0)u_s^3)+\dot{s}_1(t)\partial_\xi u_s^3\\
&\qquad=\ \dot{\delta}_1^2\partial_\xi u_s^0+\dot{\delta}_1^1\partial_\xi u_s^1+\dot{\delta}_1^0\partial_\xi u_s^2+\partial_t u_s^2+\partial_\xi(f''(u_s^0)(u_s^1\cdot u_s^2))\\
&\qquad\quad+\ \frac{1}{6}\partial_\xi(f'''(u_s^0)(u_s^1)^3)-\partial_\xi[b'(u_s^0)\partial_\xi(u_s^1\cdot u_s^2)]-\frac{1}{6}\partial_\xi^2(b''(u_s^0)(u_s^1)^3),
\end{aligned}$$

where $\dot{s}_1(t)=\frac{\mathrm{d}s_1}{\mathrm{d}t}$, $\dot{\delta}_1^i=\frac{\mathrm{d}\delta_1^i}{\mathrm{d}t}$, $i=1,2,3$.

In the matching region, we expect both the expansion near the shock and the expansion outside to be effective, so they must agree with each other in the matching region. Then, the following relations, the so-called matching conditions hold if $\xi \to \pm\infty$:

$$u_s^0(\xi,t) = u^0(s_1(t)\pm 0,t) + o(1), \tag{2.8}$$

$$u_s^1(\xi,t) = u^1(s_1(t)\pm 0,t) + (\xi-\delta_1^0)\partial_x u^0(s_1(t)\pm 0,t) + o(1), \tag{2.9}$$

$$\begin{aligned} u_s^2(\xi,t) = {} & u^2(s_1(t)\pm 0,t) + (\xi-\delta_1^0)\partial_x u^1(s_1(t)\pm 0,t) - \delta_1^1\partial_x u^0(s_1(t)\pm 0,t) \\ & + \frac{1}{2}(\xi-\delta_1^0)^2\partial_x^2 u^0(s_1(t)\pm 0,t) + o(1), \end{aligned} \tag{2.10}$$

$$\begin{aligned} u_s^3(\xi,t) = {} & u^3(s_1(t)\pm 0,t) + (\xi-\delta_1^0)\partial_x u^2(s_1(t)\pm 0,t) - \delta_1^1\partial_x u^1(s_1(t)\pm 0,t) \\ & + \frac{1}{2}(\xi-\delta_1^0)^2\partial_x^2 u^1(s_1(t)\pm 0,t) - \delta_1^2\partial_x u^0(s_1(t)\pm 0,t) - (\xi-\delta_1^0)\delta_1^1\partial_x^2 u^0(s_1(t)\pm 0,t) \\ & + \frac{1}{6}(\xi-\delta_1^0)^3\partial_x^3 u^0(s_1(t)\pm 0,t) + o(1). \end{aligned} \tag{2.11}$$

Similarly, near the shock $x = s_2(t)$, we approximate u^ε as

$$u^\varepsilon(x,t) \sim \bar{u}_s^0(\eta,t) + \varepsilon\bar{u}_s^1(\eta,t) + \varepsilon^2\bar{u}_s^2(\eta,t) + \varepsilon^3\bar{u}_s^3(\eta,t) + \dots\,,$$

where

$$\eta = \frac{x-s_2(t)}{\varepsilon} + \delta_2(t,\varepsilon),$$

and δ_2 is a disturbance of the shock position, which is to be determined. Assume that the expansion of $\delta_1(t,\varepsilon)$ is

$$\delta_2(t,\varepsilon) = \delta_2^0(t) + \varepsilon\delta_2^1(t) + \varepsilon^2\delta_2^2(t) + \dots\,.$$

Plugging it into (1.3) and equating the coefficients of different orders of ε, we have

$$\begin{aligned} & O\left(\frac{1}{\varepsilon}\right): \partial_\eta f(\bar{u}_s^0) - \dot{s}_2(t)\partial_\eta\bar{u}_s^0 - \partial_\eta(b(\bar{u}_s^0)\partial_\eta\bar{u}_s^0) = 0, \\ & O(1): \partial_\eta^2(b(\bar{u}_s^0)\bar{u}_s^1) - \partial_\eta(f'(\bar{u}_s^0)u_s^1) + \dot{s}_2(t)\partial_\eta\bar{u}_s^1 = \dot{\delta}_2^0\partial_\eta\bar{u}_s^0 + \partial_t\bar{u}_s^0, \\ & O(\varepsilon): \partial_\eta^2(b(\bar{u}_s^0)\bar{u}_s^2(\eta,t)) - \partial_\eta(f'(\bar{u}_s^0)\bar{u}_s^2) + \dot{s}_2(t)\partial_\eta\bar{u}_s^2 \\ & \qquad = \dot{\delta}_2^1\partial_\eta\bar{u}_s^0 + \dot{\delta}_2^0\partial_\eta\bar{u}_s^1 + \partial_t\bar{u}_s^1 + \frac{1}{2}\partial_\eta(f''(\bar{u}_s^0)(\bar{u}_s^1)^2) - \frac{1}{2}\partial_\eta^2(b'(\bar{u}_s^0)(\bar{u}_s^1)^2), \\ & O(\varepsilon^2): \partial_\eta^2(b(\bar{u}_s^0)\bar{u}_s^3) - \partial_\eta(f'(\bar{u}_s^0)u_s^3) + \dot{s}_2(t)\partial_\eta u_s^3 \\ & \qquad = \dot{\delta}_2^2\partial_\eta\bar{u}_s^0 + \dot{\delta}_2^1\partial_\eta\bar{u}_s^1 + \dot{\delta}_2^0\partial_\eta\bar{u}_s^2 + \partial_t\bar{u}_s^2 \\ & \qquad\quad + \partial_\eta(f''(\bar{u}_s^0)(\bar{u}_s^1\cdot\bar{u}_s^2)) + \frac{1}{6}\partial_\eta[f'''(\bar{u}_s^0)(\bar{u}_s^1)^3] - \partial_\eta[b'(\bar{u}_s^0)\partial_\eta(\bar{u}_s^1\cdot\bar{u}_s^2)] - \frac{1}{6}\partial_\eta^2(b''(\bar{u}_s^0)(\bar{u}_s^1)^3), \end{aligned}$$

where $\dot{s}_2(t) = \frac{\mathrm{d}s_2}{\mathrm{d}t}$, $\dot{\delta}_2^i = \frac{\mathrm{d}\delta_2^i}{\mathrm{d}t}$, $i = 1,2,3$. Also, in the matching region, when $\eta \to \pm\infty$, we have

$$\bar{u}_s^0(\eta,t) = u^0(s_2(t)\pm 0,t) + o(1), \tag{2.12}$$

$$\bar{u}_s^1(\eta,t) = u^1(s_2(t)\pm 0,t) + (\eta-\delta_2^0)\partial_x u^0(s_2(t)\pm 0,t) + o(1), \tag{2.13}$$

$$\begin{aligned} \bar{u}_s^2(\eta,t) = {} & u^2(s_2(t)\pm 0,t) + (\eta-\delta_2^0)\partial_x u^1(s_2(t)\pm 0,t) - \delta_2^1\partial_x u^0(s_2(t)\pm 0,t) \\ & + \frac{1}{2}(\eta-\delta_2^0)^2\partial_x^2 u^2(s_2(t)\pm 0,t) + o(1), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \bar{u}_s^3(\eta,t) = {} & u^3(s_2(t)\pm 0,t) + (\eta-\delta_2^0)\partial_x\bar{u}^2(s_2(t)\pm 0,t) - \delta_2^1\partial_x\bar{u}^1(s_2(t)\pm 0,t) \\ & + \frac{1}{2}(\eta-\delta_2^0)^2\partial_x^2\bar{u}^1(s_2(t)\pm 0,t) - \delta_2^2\partial_x\bar{u}^0(s_2(t)\pm 0,t) - (\eta-\delta_2^0)\delta_2^1\partial_x^2\bar{u}^0(s_2(t)\pm 0,t) \\ & + \frac{1}{6}(\eta-\delta_2^0)^3\partial_x^3\bar{u}^0(s_2(t)\pm 0,t) + o(1). \end{aligned} \tag{2.15}$$

Now, let us discuss the solvability of each order functions u_s^i, $\bar{u}_s^i$ with $i = 0, \ldots, 3$ in the expansions near the shocks. Note that near $x = s_1(t)$, $u_s^0(\xi, t)$ satisfies

$$\partial_\xi f(u_s^0(\xi,t)) - \dot{s}_1(t)\partial_\xi u_s^0(\xi,t) = \partial_\xi(b(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t)), \tag{2.16}$$

$$u_s^0(\xi,t) \to u_l = u^0(s_1(t)-0,t), \quad \xi \to -\infty, \tag{2.17}$$

$$u_s^0(\xi,t) \to u_m = u^0(s_1(t)+0,t), \quad \xi \to +\infty. \tag{2.18}$$

Near $x = s_2(t)$, it follows that

$$\partial_\eta f(\bar{u}_s^0(\eta,t)) - \dot{s}_2(t)\partial_\eta \bar{u}_s^0(\eta,t) = \partial_\eta(b(\bar{u}_s^0(\eta,t))\partial_\eta \bar{u}_s^0(\eta,t)), \tag{2.19}$$

$$\bar{u}_s^0(\eta,t) \to \bar{u}_m = u^0(s_2(t)-0,t), \quad \eta \to -\infty, \tag{2.20}$$

$$\bar{u}_s^0(\eta,t) \to u_r = u^0(s_2(t)+0,t), \quad \eta \to +\infty. \tag{2.21}$$

We have the following existence and properties of u_s^0 and $\bar{u}_s^0$.

Lemma 2.1. *There exists a unique smooth solution $u_s^0(\xi,t)$ to the boundary value problems* (2.16)–(2.18), *such that*

$$|\partial_\xi u_s^0(\eta,t)| \le Ce^{-\delta_0|\xi|}, \tag{2.22}$$

where $\delta_0 > 0$ is a constant. Similarly, problems (2.19)–(2.21) *also have a smooth solution $\bar{u}_s^0(\eta,t)$ satisfying*

$$|\partial_\eta \bar{u}_s^0(\eta,t)| \le Ce^{-\bar{\delta}_0|\eta|}, \tag{2.23}$$

where $\bar{\delta}_0 > 0$ is a constant.

Proof. It follows from (2.16) that

$$f'(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t) - \dot{s}(t)\partial_\xi u_s^0(\xi,t) = \partial_\xi b(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t) + b(u_s^0(\xi,t))\partial_\xi^2 u_s^0(\xi,t).$$

By setting $P = \partial_\xi u_s^0$, we have

$$\partial_\xi P = \frac{f'(u_s^0(\xi,t)) - \dot{s}(t) - \partial_\xi b(u_s^0(\xi,t))}{b(u_s^0(\xi,t))}P. \tag{2.24}$$

Integrating (2.24) from 0 to ξ shows that

$$\ln\frac{P(\xi)}{P(0)} = \int_0^\xi \frac{f'(u_s^0(\rho,t)) - \dot{s}(t) - \partial_\rho b(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho,$$

i.e.,

$$\partial_\xi u_s^0(\xi,t) = \frac{\partial_\xi(u_s^0(0,t))b(u_l)}{b(u_s^0(\xi,t))}\exp\left\{\int_0^\xi \frac{f'(u_s^0(\rho,t)) - \dot{s}(t)}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}. \tag{2.25}$$

Again integrating (2.25) from $-\infty$ to $+\infty$, we have

$$u_r - u_l = \partial_\xi u_s^0(0,t)b(u_l)\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\xi,t))}\exp\left\{\int_0^\xi \frac{f'(u_s^0(\rho,t)) - \dot{s}(t)}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\xi.$$

Then,

$$\partial_\xi u_s^0(0,t)b(u_l) = \frac{u_r - u_l}{\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\xi,t))}\exp\left\{\int_0^\xi \frac{f'(u_s^0(\rho,t)) - \dot{s}(t)}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\xi}.$$

Combining it with (2.25) gives

$$\partial_\xi u_s^0(\xi,t)=\frac{(u_r-u_l)\exp\left\{\int_0^\xi\frac{f'(u_s^0(\rho,t))-\dot s(t)}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}}{b(u_s^0(\xi,t))\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\xi,t))}\exp\left\{\int_0^\xi\frac{f'(u_s^0(\rho,t))-\dot s(t)}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\xi}.$$

Similarly, the solution $\bar u_s^0(\eta,t)$ of (2.19) and (2.21) satisfies

$$\partial_\eta\bar u_s^0(\eta,t)=\frac{(\bar u_r-\bar u_l)\exp\left\{\int_0^\eta\frac{f'(\bar u_s^0(\rho,t))-\dot s(t)}{b(\bar u_s^0(\rho,t))}\mathrm{d}\rho\right\}}{b(\bar u_s^0(\eta,t))\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\eta,t))}\exp\left\{\int_0^\eta\frac{1}{b(\bar u_s^0(\eta,t))}\frac{f'(\bar u_s^0(\rho,t))-\dot s(t)}{b(\bar u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\eta}.$$

Thus, the entropy condition (1.11) and the boundedness of $b(\cdot)$ imply (2.22) and (2.23). □

Next, according to (2.9), we expect u_s^1 to be

$$u_s^1=\xi\cdot\partial_x u^0(s(t)\pm0,t)+O(1),\quad \xi\to\pm\infty.$$

This suggests us to write

$$u_s^1(\xi,t)=V_1(\eta,t)+D_1(\xi,t),\tag{2.26}$$

where $D_1(\xi,t)$ is a smooth function and satisfies

$$D_1(\xi,t)=\begin{cases}\xi\cdot\partial_x u^0(s(t)-0,t), & \xi<-1,\\ \xi\cdot\partial_x u^0(s(t)+0,t), & \xi>1.\end{cases}$$

It follows from (2.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}u^0(s(t)\pm0,t)-(\partial_u f(u^0(s(t)\pm0,t))-\dot s(t))\partial_x u^0(s(t)\pm0,t)=0.$$

We substitute (2.26) into (2.2) to obtain

$$\partial_\xi^2(b(u_s^0(\xi,t))V_1(\xi,t))-\partial_\xi(\partial_u f(u_s^0(\xi,t))V_1(\xi,t))+\dot s(t)\partial_\xi V_1(\xi,t)=\dot\delta_1^0\partial_\xi u_s^0(\xi,t)+h(\xi,t),\tag{2.27}$$

where

$$h(\xi,t)=\partial_t u_s^0(\xi,t)-\partial_\xi^2(b(u_s^0(\xi,t))D_1(\xi,t))-\dot s(t)\partial_\xi D_1(\xi,t)+\partial_\xi(\partial_u f(u_s^0(\xi,t))D_1(\xi,t)).$$

Therefore,

$$\begin{aligned}h(\xi,t)&=\partial_u^2 f(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t)\cdot\xi\partial_x u^0(s(t)\pm0,t)+[\partial_u f(u_s^0(\xi,t))-\dot s(t)]\partial_x u_0(s(t)\pm0,t)+\partial_t u_s^0(\xi,t)\\&=\partial_u^2 f(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t)\cdot\xi\partial_x u^0(s(t)\pm0,t)+\int_0^\xi\partial_t\partial_\xi u_s^0(\sigma,t)\mathrm{d}\sigma-\int_0^\xi\partial_t\partial_\xi u_s^0(\sigma,t)\mathrm{d}\sigma.\end{aligned}$$

In view of Lemma 2.1, there exists a positive constant $\delta_0>0$, such that

$$|h(\xi,t)|\le Ce^{-\delta_0|\xi|}.$$

Define $H(\xi,t)=\int_0^\xi h(\sigma,t)\mathrm{d}\sigma$; substituting it into (2.27), we obtain

$$\partial_\xi[b(u_s^0(\xi,t))V_1(\xi,t)]=[f'(u_s^0(\xi,t))-\dot s(t)]V_1(\xi,t)+\dot\delta_0 u_s^0(\xi,t)+H(\xi,t)+c(t),\tag{2.28}$$

where $c(t)$ is a function to be determined.

Lemma 2.2. *There exists a unique smooth solution $V_1(\xi,t)$ to equation* (2.28) *and satisfies*

$$V_1(\xi,t)=\begin{cases}(\dot{s}(t)-f'(u_l))^{-1}[u_l\dot{\delta}_1^0+H_-+c(t)]+O(1)e^{-\sigma|\xi|}, & \xi\to-\infty,\\ (\dot{s}(t)-f'(u_r))^{-1}[u_r\dot{\delta}_1^0+H_++c(t)]+O(1)e^{-\sigma|\xi|}, & \xi\to+\infty,\end{cases} \tag{2.29}$$

where $H_\pm=\lim_{\xi\to\pm\infty}H(\xi,t)$, $\sigma>0$.

Proof. It follows from (2.28) that

$$\partial_\xi(b(u_s^0(\xi,t))V_1(\xi,t))+\frac{\dot{s}(t)-f'(u_s^0(\xi,t))}{b(u_s^0(\xi,t))}b(u_s^0(\xi,t))V_1(\xi,t)=\dot{\delta}_1^0u_s^0(\xi,t)+H(\xi,t)+c(t),$$

which is equivalent to

$$\begin{aligned}&\partial_\xi\left[b(u_s^0(\xi,t))V_1(\xi,t)\exp\left\{\int_a^\xi\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\right]\\&\quad=[\dot{\delta}_1^0u_s^0(\xi,t)+H(\xi,t)+c(t)]\exp\left\{\int_a^\xi\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}.\end{aligned} \tag{2.30}$$

Integrating (2.30) from a to ξ shows that

$$\begin{aligned}&b(u_s^0(\xi,t))V_1(\xi,t)\exp\left\{\int_a^\xi\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}-b(u_s^0(a,t))V_1(a,t)\\&\quad=\int_a^\xi[\dot{\delta}_1^0u_s^0(\tau,t)+H(\tau,t)+c(t)]\exp\left\{\int_a^\tau\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\tau.\end{aligned}$$

Then,

$$\begin{aligned}V_1(\xi,t)={}&\frac{b(u_s^0(a,t))V_1(a,t)}{b(u_s^0(\xi,t))}\exp\left\{-\int_a^\xi\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\\&+\frac{1}{b(u_s^0(\xi,t))}\int_a^\xi[\dot{\delta}_1^0u_s^0(\tau,t)+H(\tau,t)+c(t)]\exp\left\{\int_\xi^\tau\frac{\dot{s}(t)-f'(u_s^0(\rho,t))}{b(u_s^0(\rho,t))}\mathrm{d}\rho\right\}\mathrm{d}\tau.\end{aligned}$$

It follows from (1.11) that

$$\lim_{\xi\to\pm\infty}\partial_\xi V_1(\xi,t)\sim O(1)\exp\{-\alpha_1|\xi|\},\quad \alpha_1>0.$$ □

Now, we can select $c(t)$ and δ_1^0 so that they satisfy the matching conditions (2.9) and (2.28). We obtain from (2.9) that

$$\lim_{\xi\to\pm\infty}V_1(\xi,t)=u^1(s(t)\pm0,t)-\delta_1^0u^0(s(t)\pm0,t)$$

and gain

$$[u_r\dot{\delta}_1^0+H_++c(t)]=[\dot{s}(t)-f'(u_r)][u^1(s(t)+0,t)-\delta_1^0\partial_xu^0(s(t)+0,t)], \tag{2.31}$$

$$[u_l\dot{\delta}_1^0+H_-+c(t)]=[\dot{s}(t)-f'(u_l)][u^1(s(t)-0,t)-\delta_1^0\partial_xu^0(s(t)-0,t)]. \tag{2.32}$$

Subtracting (2.31) from (2.32) gives

$$\begin{aligned}\dot{\delta}_1^0(u_l-u_r)+H_+-H_-={}&[\dot{s}(t)-f'(u_l)]u^1(s(t)-0,t)-[\dot{s}(t)-f'(u_r)]u^1(s(t)+0,t)\\&+\delta_1^0\{[\dot{s}(t)-f'(u_r)]\partial_xu^0(s(t)+0,t)-[\dot{s}(t)-f'(u_l)]\partial_xu^0(s(t)-0,t)\}.\end{aligned}$$

Thus, we obtain a first-order linear ordinary differential equation for δ_1^0 as follows:

$$\dot{\delta}_1^0 + W_1(t)\delta_1^0 - W_2(t) - H_1(t) = 0,$$

where $W_1(t)$, W_2, and $H_1(t)$ are the known smooth functions. Combining with (2.31) gives the expression of δ_1^0 that

$$\delta_1^0(t) = \delta_1^0(t)\exp\left\{-\int_0^t W_1(\rho)\mathrm{d}\rho\right\} + \int_0^t (W_2(\tau) + H_1(\tau)\exp\left\{\int_t^\tau W_1(\rho)\right\}\mathrm{d}\rho)\mathrm{d}\tau,$$

which is a smooth function. Substitute δ_1^0 back into (2.31) or (2.32) to obtain $c(t)$. Again, substitute δ_1^0 and $c(t)$ into (2.29) to solve $V_1(\eta, t)$.

Proposition 2.1. *u_s^1 and δ_1^0 are the smooth functions, there exists $\sigma > 0$ such that*

$$u_s^1 = u^1(s_1(t) \pm 0, t) + (\xi - \delta_1^0)\partial_x u^0(s_1(t) \pm 0, t) + O(1)e^{-\sigma|\xi|}, \quad \xi \to \pm\infty. \tag{2.33}$$

And $\bar{u}_s^1$ and δ_2^0 are the smooth functions, there exists $\bar{\sigma} > 0$ such that

$$\bar{u}_s^1 = \bar{u}^1(s_2(t) \pm 0, t) + (\eta - \delta_2^0)\partial_x u^0(s_2(t) \pm 0, t) + O(1)e^{-\bar{\sigma}|\eta|}, \quad \eta \to \pm\infty. \tag{2.34}$$

Similarly, we can determine δ_0^2, u_s^2, $\bar{u}_s^2$, δ_2^1, u_s^3, δ_1^2, $\bar{u}_s^3$, and δ_2^2.

Lemma 2.3. *Assume that the convexity condition $\partial_u^2 f(x, t) > 0$ holds for $\leq t \leq T$, then*

$$\partial_\xi \partial_u f(u_s^0(\xi, t)) < 0, \tag{2.35}$$

$$\partial_\eta \partial_u f(\bar{u}_s^0(\eta, t)) < 0. \tag{2.36}$$

Proof. Since

$$\partial_\xi u_s^0(\xi, t) = \frac{(u^0(s_1(t) + 0, t) - u^0(s_1(t) - 0, t))\exp\left\{\int_a^\xi \frac{f'(u_s^0(\rho, t)) - \dot{s}(t)}{b(u_s^0(\rho, t))}\mathrm{d}\rho\right\}}{b(u_s^0(\xi, t))\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\xi, t))}\exp\left\{\int_a^\xi \frac{f'(u_s^0(\rho, t)) - \dot{s}(t)}{b(u_s^0(\rho, t))}\mathrm{d}\rho\right\}\mathrm{d}\xi},$$

$$\partial_\eta \bar{u}_s^0(\eta, t) = \frac{(u^0(s_2(t) + 0, t) - u^0(s_2(t) - 0, t))\exp\left\{\int_a^\eta \frac{f'(\bar{u}_s^0(\rho, t)) - \dot{s}(t)}{b(\bar{u}_s^0(\rho, t))}\mathrm{d}\rho\right\}}{b(\bar{u}_s^0(\eta, t))\int_{-\infty}^{+\infty}\frac{1}{b(u_s^0(\eta, t))}\exp\left\{\int_a^\eta \frac{f'(\bar{u}_s^0(\rho, t)) - \dot{s}(t)}{b(\bar{u}_s^0(\rho, t))}\mathrm{d}\rho\right\}\mathrm{d}\eta}.$$

Then, it follows from the entropy condition (1.11) that $\partial_\xi u_s^0 < 0$ and $\partial_\eta \bar{u}_s^0 < 0$. Furthermore, we have

$$\partial_\xi \partial_u f(u_s^0(\xi, t)) = \partial_u^2 f(u_s^0(\xi, t))\partial_\xi u_s^0(\xi, t),$$

$$\partial_\eta \partial_u f(\bar{u}_s^0(\eta, t)) = \partial_u^2 f(\bar{u}_s^0(\eta, t))\partial_\eta \bar{u}_s^0(\eta, t).$$

□

2.3 Approximate solutions

We now construct an approximate solution to (1.3) by patching the truncated outer and shock-layer solutions as

$$O(x, t) = u^0(x, t) + \varepsilon u^1(x, t) + \varepsilon^2 u^2(x, t) + \varepsilon^3 u^3(x, t), \quad x \neq s_i(t),\ i = 1, 2,$$

$$I_1(x, t) = u_s^0\left(\frac{x - s_1(t)}{\varepsilon} + \delta_1^0 + \varepsilon\delta_1^1 + \varepsilon^2\delta_1^2, t\right) + \varepsilon u_s^1\left(\frac{x - s_1(t)}{\varepsilon} + \delta_1^0 + \varepsilon\delta_1^1 + \varepsilon^2\delta_1^2, t\right)$$

$$+ \varepsilon^2 u_s^2\left(\frac{x - s_1(t)}{\varepsilon} + \delta_1^0 + \varepsilon\delta_1^1 + \varepsilon^2\delta_1^2, t\right) + \varepsilon^3 u_s^3\left(\frac{x - s_1(t)}{\varepsilon} + \delta_1^0 + \varepsilon\delta_1^1 + \varepsilon^2\delta_1^2, t\right),$$

$$I_2(x,t) = \bar{u}_s^0\left(\frac{x - s_2(t)}{\varepsilon} + \delta_2^0 + \varepsilon\delta_2^1 + \varepsilon^2\delta_2^2, t\right) + \varepsilon\bar{u}_s^1\left(\frac{x - s_2(t)}{\varepsilon} + \delta_2^0 + \varepsilon\delta_2^1 + \varepsilon^2\delta_2^2, t\right)$$
$$+ \varepsilon^2\bar{u}_s^2\left(\frac{x - s_2(t)}{\varepsilon} + \delta_2^0 + \varepsilon\delta_2^1 + \varepsilon^2\delta_2^2, t\right) + \varepsilon^3\bar{u}_s^3\left(\frac{x - s_2(t)}{\varepsilon} + \delta_2^0 + \varepsilon\delta_2^1 + \varepsilon^2\delta_2^2, t\right).$$

The functions u_i, u_s^i, $\bar{u}_s^i$, δ_1^i, and δ_2^i, $i = 0, 1, 2, 3$ were given in the previous section. Then, the approximate solution to (1.3) is defined as

$$\bar{u}_a(x,t) = m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)O + d(x,t) = u_a(x,t) + d(x,t), \tag{2.37}$$

where $m_i = m(\frac{x - s_i(t)}{\varepsilon^\gamma})$, $i = 1, 2$, $\gamma \in (\frac{6}{7}, 1)$ with m being defined in (1.12), and $d(x,t)$ is a higher-order correction term to be determined. Due to the structure of the various orders of inner and outer solutions, $\bar{u}_a$ solves

$$\partial_t\bar{u}_a + \partial_x f(\bar{u}_a) - \varepsilon\partial_x(b(\bar{u}_a)\partial_x\bar{u}_a) = \sum_{i=1}^{5} q_i(x,t), \tag{2.38}$$

$$\bar{u}_a(x, t = 0) = m_1\sum_{i=0}^{3}\varepsilon^i u_s^i(\xi, 0) + m_2\sum_{i=0}^{3}\varepsilon^i\bar{u}_s^i(\eta, 0) + (1 - m_1 - m_2)\sum_{i=0}^{3}\varepsilon^i u_0^i(x), \tag{2.39}$$

where $q_i(x,t)$ are the smooth functions as follows:

$$\begin{aligned}
q_1(x,t) = (1 - m_1 - m_2)\Bigg\{\Bigg[&f(O) - f(u^0) - \varepsilon f'(u^0)u^1 - \varepsilon^2 f'(u^0)u^2 - \frac{\varepsilon^2}{2}f''(u^0)(u^1)^2 \\
&- \varepsilon^3 f''(u^0)(u^1\cdot u^2) - \frac{\varepsilon^3}{6}f'''(u^0)(u^1)^3 - \varepsilon^4 f''(u^0)(u^1\cdot u^3) - \frac{\varepsilon^4}{2}f''(u^0)(u^2)^2 \\
&- \frac{\varepsilon^4}{2}f'''(u^0)((u^1)^2\cdot u^2) - \frac{\varepsilon^4}{24}f^{(4)}(u^0)(u^1)^4\Bigg]_x - \varepsilon^5\partial_x^2(b(u^0)u^4) \\
&- \varepsilon^5\Bigg[\partial_x(b'(u^0)\partial_x(u^1\cdot u^3)) + \frac{1}{2}\partial_x(b''(u^0)(u^1)^2\partial_x u^2) + \frac{1}{2}\partial_x^2(b'(u^0)(u^2)^2) + \frac{1}{24}\partial_x^2(b'''(u^0)(u^1)^4)\Bigg]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
q_2(x,t) = m_1\Bigg\{\Bigg(&f(I_1) - f(u_s^0) - \varepsilon f'(u_s^0)u_s^1 - \varepsilon^2 f'(u_s^0)u_s^2 - \varepsilon^3 f'(u_s^0)u_s^3 - \frac{\varepsilon^2}{2}f'(u_s^0)(u_s^1)^2 \\
&- \varepsilon^3 f''(u_s^0)(u_1^1\cdot u_s^2) - \frac{\varepsilon^3}{6}f'''(u_s^0)(u_s^1)^3\Bigg)_x + \varepsilon^3\partial_t u_s^3 + \varepsilon^4(\dot{\delta}_1^0 u_s^3 + \dot{\delta}_1^1 u_s^2 + \dot{\delta}_1^2 u_s^1 + \dot{\delta}_1^3 u_s^0 - \dot{s}_1 u_s^3 + \varepsilon\delta_1^2 u_s^2)_x\Bigg\},
\end{aligned}$$

$$\begin{aligned}
q_3(x,t) = m_2\Bigg\{\Bigg(&f(I_2) - f(\bar{u}_s^0) - \varepsilon f'(\bar{u}_s^0)\bar{u}_s^1 - \varepsilon^2 f'(\bar{u}_s^0)\bar{u}_s^2 - \varepsilon^3 f'(\bar{u}_s^0)\bar{u}_s^3 - \frac{\varepsilon^2}{2}f'(\bar{u}_s^0)(\bar{u}_s^1)^2 \\
&- \varepsilon^3 f''(\bar{u}_s^0)(\bar{u}_1^1\cdot\bar{u}_s^2) - \frac{\varepsilon^3}{6}f'''(\bar{u}_s^0)(\bar{u}_s^1)^3\Bigg)_x + \varepsilon^3\partial_t\bar{u}_s^3 + \varepsilon^4(\dot{\delta}_2^0\bar{u}_s^3 + \dot{\delta}_2^1\bar{u}_s^2 + \dot{\delta}_2^2\bar{u}_s^1 + \dot{\delta}_2^3\bar{u}_s^0 - \dot{s}_2\bar{u}_s^3 + \varepsilon\delta_2^2\bar{u}_s^2)_x\Bigg\},
\end{aligned}$$

$$\begin{aligned}
q_4(x,t) = {} &\partial_t m_1(I_1 - O) + \partial_t m_2(I_2 - O) + f(m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)O)_x - \{m_1 f(I_1) + m_2 f(I_2) \\
&+ (1 - m_1 - m_2)f(O)\}_x + \varepsilon(b(O)O_x)_x + \partial_x m_1(f(I_1) - f(O)) + \partial_x m_2(f(I_2) - f(O)) \\
&+ \varepsilon\{m_1(b(I_1) - b(O))(I_1 - O)_x + m_2(b(I_2) - b(O))(I_2 - O)_x\}_x - \varepsilon\partial_x m_1(b(I_1) - b(O))(I_1 - O)_x \\
&- \varepsilon\partial_x m_2(b(I_2) - b(O))(I_2 - O)_x + \varepsilon\{m_1 b(O)(I_1 - O)_x + m_2 b(O)(I_2 - O)_x\}_x \\
&- \varepsilon\partial_x m_1 b(O)(I_1 - O)_x - \varepsilon\partial_x m_2 b(O)(I_2 - O)_x + \varepsilon\{m_1(b(I_1) - b(O))O_x + m_2(b(I_2) - b(O))O_x\}_x \\
&- \varepsilon\partial_x m_1[b(I_1) - b(O)]O_x - \varepsilon\partial_x m_2[b(I_2) - b(O)]O_x - \varepsilon\{g(m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)O)\}_{xx},
\end{aligned}$$

$$q_5(x,t) = d_t - \varepsilon(g(\bar{u}_a) - g(\bar{u}_a - d))_{xx} + (f(\bar{u}_a) - f(\bar{u}_a - d))_x.$$

In view of our construction, we have (i) $\operatorname{supp} q_1 \subseteq \{(x,t) : s_i(t) + \varepsilon^\gamma < x \text{ or } x < s_i(t) - \varepsilon^\gamma,\ 0 \le t \le T,\ i = 1,2\}$,

$$\partial_x^k q_1(x,t) = O(1)\varepsilon^{5-k\gamma}, \quad \int_0^T (\|\partial_x^k q_1(\cdot,t)\|^2 \mathrm{d}t)^{\frac{1}{2}} \le O(1)\varepsilon^{5-(k-\frac{1}{2})\gamma}, \quad k = 0,1,2, \tag{2.40}$$

(ii) $\operatorname{supp} q_2 \subseteq \{(x,t) : s_1(t) - 2\varepsilon^\gamma < x < s_1(t) + 2\varepsilon^\gamma,\ 0 \le t \le T\}$,

$$\partial_x^k q_2(x,t) = O(1)\varepsilon^{(3-l)\gamma}, \quad k = 0,1,2, \tag{2.41}$$

(iii) $\operatorname{supp} q_3 \subseteq \{(x,t) : s_1(t) - 2\varepsilon^\gamma < x < s_2(t) + 2\varepsilon^\gamma,\ 0 \le t \le T\}$,

$$\partial_x^k q_3(x,t) = O(1)\varepsilon^{(3-l)\gamma}, \quad k = 0,1,2, \tag{2.42}$$

(iv) $\operatorname{supp} q_4 \subseteq \{(x,t) : \varepsilon^\gamma \le |x - s_i(t)| \le 2\varepsilon^\gamma,\ 0 \le t \le T\}$,

$$\partial_x^k q_4(x,t) = O(1)\varepsilon^{(3-l)\gamma}, \quad k = 0,1,2. \tag{2.43}$$

Here, we have used the estimate

$$\partial_x^l (I_1 - O) = O(1)\varepsilon^{(4-l)\gamma}, \quad l = 0,1,2, \tag{2.44}$$

on $\{(x,t) : \varepsilon^\gamma \le |x - s_1(t)| \le 2\varepsilon^\gamma,\ 0 \le t \le T\}$. Similarly,

$$\partial_x^l (I_2 - O) = O(1)\varepsilon^{(4-l)\gamma}, \quad l = 0,1,2, \tag{2.45}$$

on $\{(x,t) : \varepsilon^\gamma \le |x - s_2(t)| \le 2\varepsilon^\gamma,\ 0 \le t \le T\}$, which can be obtained by the matching conditions (2.8)–(2.11), (2.12)–(2.15), and $O(1) = \exp\{-\alpha_0|\xi|\}$. Set $R^\varepsilon = \sum_{i=1}^4 q_i(x,t)$, then $R^\varepsilon = O(1)\varepsilon^{3\gamma}$. We now choose $d(x,t)$ to be the solution of the diffusion problem

$$\begin{cases} d_t = \varepsilon(b(u_a)d_x)_x - \sum\limits_{i=1}^4 q_i(x,t), \\ d(x,0) = 0. \end{cases} \tag{2.46}$$

Consequently, for $\bar{u}_a$, we obtain the conservative form

$$\partial_t \bar{u}_a + \partial_f(\bar{u}_a) - \varepsilon\partial_x(b(\bar{u}_a)\partial_x\bar{u}_a) = \varepsilon(b(u_a)d_x)_x - \varepsilon[g(\bar{u}_a) - g(u_a)]_{xx} + (f(\bar{u}_a) - f(u_a))_x, \tag{2.47}$$

with the initial data (2.39). It then remains to estimate the linear diffusion wave $d(x,t)$.

Lemma 2.4. *Let $d(x,t)$ be the solution of* (2.46). *Then, the following estimates*

$$\sup_{0\le t\le T} \|d\|^2_{L^2(\mathbb{R})} + \varepsilon\int_0^T \|\partial_x d\|^2_{L^2(\mathbb{R})} \mathrm{d}t \le C\varepsilon^{7\gamma}, \tag{2.48}$$

$$\sup_{0\le t\le T} \|\partial_x d\|^2_{L^2(\mathbb{R})} + \varepsilon\int_0^T \|\partial_x^2 d\|^2_{L^2(\mathbb{R})} \mathrm{d}t \le C\varepsilon^{7\gamma-2}, \tag{2.49}$$

$$\sup_{0\le t\le T} \|\partial_x^2 d\|^2_{L^2(\mathbb{R})} + \varepsilon\int_0^T \|\partial_x^3 d\|^2_{L^2(\mathbb{R})} \mathrm{d}t \le C\varepsilon^{7\gamma-4}, \tag{2.50}$$

$$\sup_{0\le t\le T} \|d(x,t)\|_{L^\infty(\mathbb{R})} \le C\varepsilon^{\frac{7\gamma-1}{2}}, \tag{2.51}$$

$$\sup_{0\le t\le T} \|\partial_x d(x,t)\|_{L^\infty(\mathbb{R})} \le C\varepsilon^{\frac{7\gamma-3}{2}}, \tag{2.52}$$

hold for all $t \in [0,T]$.

Proof. Multiplying (2.46) by d, integrating on $\mathbb{R}$, and using integration by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} d^2\mathrm{d}x + \varepsilon\int_{\mathbb{R}} b(u_a)|\partial_x d|^2\mathrm{d}x + \int_{\mathbb{R}} d\cdot R^\varepsilon\mathrm{d}x = 0.$$

By the uniform parabolic condition and Gronwall's inequality, we obtain (2.48). Let $D = \partial_x d$, and D satisfies

$$\begin{cases} D_t = \varepsilon(b(u_a)D)_{xx} - \partial_x R^\varepsilon, \\ D(x,0) = 0. \end{cases} \tag{2.53}$$

Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} D^2\mathrm{d}x + (C_0-\beta)\varepsilon\int_{\mathbb{R}}|\partial_x D|^2\mathrm{d}x \le \varepsilon^{-1}\int_{\mathbb{R}} D^2\mathrm{d}x + C\int_{\mathbb{R}} D^2\mathrm{d}x + C\varepsilon^{5\gamma},$$

where β is small enough satisfying $C_0 - \beta > 0$. Then, it is easy to obtain (2.49) after integration over $[0,T]$. Next, let $\bar{D} = \partial_x D$, and $\bar{D}$ satisfies

$$\begin{cases} \bar{D}_t = \varepsilon(b(u_a)D)_{xxx} - \partial_x^2 R^\varepsilon, \\ \bar{D}(x,0) = 0. \end{cases} \tag{2.54}$$

By the same procedure, we obtain (2.49). It follows from (2.48) and (2.49) that

$$\sup_{0\le t\le T}\|d(x,t)\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\,\|d(x,t)\|^{\frac{1}{2}}_{L^2_{\mathbb{R}}}\|\partial_x d(x,t)\|^{\frac{1}{2}}_{L^2_{\mathbb{R}}} \le C\varepsilon^{\frac{7\gamma-1}{2}}. \tag{2.55}$$

Similarly, (2.52) follows from (2.49) and (2.50). □

Lemma 2.5. *Let $\bar{u}_a(x,t)$ be defined as in* (2.37). *Then,*

$$\bar{u}_a(x,t) = \begin{cases} u^0(x,t) + O(1)\varepsilon, & |x - s_i(t)| \ge \varepsilon^\gamma,\ i = 1,2, \\ u_s^0(\xi,t) + O(1)\varepsilon^\gamma, & |x - s_1(t)| \le 2\varepsilon^\gamma, \\ \bar{u}_s^0(\eta,t) + O(1)\varepsilon^\gamma, & |x - s_2(t)| \le 2\varepsilon^\gamma. \end{cases} \tag{2.56}$$

Proof. According to our construction process, we have

$$\bar{u}_a(x,t) = \begin{cases} I_1 + d, & |x - s_1(t)| \le \varepsilon^\gamma, \\ O + m_1(I_1 - O) + d, & \varepsilon^\gamma \le |x - s_1(t)| \le 2\varepsilon^\gamma, \\ O + d, & |x - s_i(t)| \ge 2\varepsilon^\gamma,\ i = 1,2, \\ O + m_2(I_2 - O) + d, & \varepsilon^\gamma \le |x - s_2(t)| \le 2\varepsilon^\gamma, \\ I_2 + d, & |x - s_2(t)| \le \varepsilon^\gamma, \end{cases}$$

where $O(x,t) = u^0(x,t) + O(1)\varepsilon$ on $|x - s_i(t)| > \varepsilon^\gamma$, $i = 1,2$, and $I_1(x,t) = u_s^0(\xi,t) + O(1)\varepsilon^\gamma$ on $s_1(t) - 2\varepsilon^\gamma < x < s_1(t) + 2\varepsilon^\gamma$. Similarly, $I_2(x,t) = \bar{u}_s^0(\eta,t) + O(1)\varepsilon^\gamma$ on $s_2(t) - 2\varepsilon^\gamma < x < s_2(t) + 2\varepsilon^\gamma$. These, together with the structure of the approximate solution, result in (2.56). □

3 Vanishing viscosity limit

In this section, we prove that there exists an exact solution of (1.3) in the vicinity of the approximate solution constructed and establish the asymptotic equivalence between the two viscous equations and the inviscid equation. We first derive the error equation, and using the structure of the approximate solution constructed in the previous section to obtain the H^1-estimates on the error equation, which consequently implies the vanishing viscosity limit away from the shocks.

3.1 Error equation

We decompose the exact solution $u^\varepsilon(x,t)$ as the sum of the approximate solution $\bar{u}_a(x,t)$ and the error term

$$u^\varepsilon(x,t) = \bar{u}_a(x,t) + \varepsilon^{1/2+\delta} v(x,t), \quad \delta \in \left(0, \frac{1}{2}\right], \; x \in \mathbb{R}, \; t \in [0,T].$$

It follows from (1.13), (2.39), and (2.47) that

$$\begin{aligned} &\partial_t v - \varepsilon^{1/2-\delta}\partial_x^2[g(\bar{u}_a + \varepsilon^{1/2+\delta}v) - g(\bar{u}_a - d)] + \varepsilon^{1/2-\delta}\partial_x[b(\bar{u}_a - d)\partial_x d] \\ &\quad + \varepsilon^{-(1/2+\delta)}\partial_x[f(\bar{u}_a + \varepsilon^{1/2+\delta}v) - f(\bar{u}_a - d)] = 0, \end{aligned} \tag{3.1}$$

$$v(x,0) = O(\varepsilon^2).$$

To exploit the compressibility of the shocks, we set

$$\varphi(x,t) = \int_{-\infty}^{x} v(z,t)\mathrm{d}z, \quad \forall x \in \mathbb{R}.$$

Then, together with (1.13), it follows

$$\partial_t\varphi - \varepsilon^{1/2-\delta}\partial_x[g(\bar{u}_a + \varepsilon^{1/2+\delta}v) - g(\bar{u}_a - d)] + \varepsilon^{1/2-\delta}b(\bar{u}_a - d)\partial_x d + \varepsilon^{-(1/2+\delta)}[f(\bar{u}_a + \varepsilon^{1/2+\delta}v) - f(\bar{u}_a - d)] = 0. \tag{3.2}$$

$$\varphi(x,0) = O(\varepsilon^2). \tag{3.3}$$

Our purpose is to show the Cauchy problems (3.2)–(3.3) have a unique solution $v \in C^1([0,T]; H^2(\mathbb{R}))$ with the property that

$$\|v\|_{L^\infty([0,T]\times\mathbb{R})} \le C\varepsilon^{7\gamma/2-\delta-2}. \tag{3.4}$$

So (3.4) is a consequence of the following priori estimates.

Proposition 3.1. *Suppose that for* $\forall\, \varepsilon > 0$, *there exists a unique solution* $\varphi \in C^1([0,T]; H^2(\mathbb{R}))$ *to the Cauchy problems* (3.2)–(3.3). *There exist positive constants C and γ, which are independent of ε, such that if*

$$\sup_{0\le t\le T} \|\partial_x\varphi\|_{L^\infty(\mathbb{R})} \le C, \tag{3.5}$$

then

$$\sup_{0\le t\le T} \|\varphi\|^2_{H^2(\mathbb{R})} + \varepsilon\int_0^T \|\partial_x\varphi\|^2_{H^2(\mathbb{R})}\mathrm{d}t \le C\varepsilon^{7\gamma-2\delta-5}, \tag{3.6}$$

where $\frac{6}{7} < \gamma < 1$.

To prove this proposition, we need the H^2-estimate of the error term φ, which is presented in the following three subsections.

3.2 Basic L^2 estimate

Lemma 3.1. *Under the assumptions of Proposition* 3.1, *there exists a positive constant C such that* $\varphi \in C^1([0,T]; H^2(\mathbb{R}))$ *is a solution to problems* (3.2)–(3.3), *satisfying*

$$\sup_{0\le t\le T} \|\varphi\|^2_{L^2(\mathbb{R})} + \varepsilon\int_0^T \|\partial_x\varphi\|^2_{L^2(\mathbb{R})}\mathrm{d}t \le C\varepsilon^{7\gamma-2\delta-1}. \tag{3.7}$$

Proof. Multiplying (3.2) by φ and integrating over $\mathbb{R}$ yield after integration by parts that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\varphi|^2\mathrm{d}x + H + I + J = 0,$$

where

$$\begin{aligned}
H &= -\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x[g(\bar{u}_a + \varepsilon^{1/2+\delta}v) - g(\bar{u}_a - d)]\cdot\varphi\mathrm{d}x,\\
I &= \varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}[f(\bar{u}_a + \varepsilon^{1/2+\delta}v) - f(\bar{u}_a - d)]\cdot\varphi\mathrm{d}x,\\
J &= \varepsilon^{1/2-\delta}\int_{\mathbb{R}}b(\bar{u}_a - d)\partial_x d\cdot\varphi\mathrm{d}x.
\end{aligned}$$

In view of the structure of $\bar{u}_a$, we have

$$\begin{aligned}
H &= \varepsilon^{1/2-\delta}\int_{\mathbb{R}}[g'(u_a)(\varepsilon^{1/2+\delta}\varphi_x + d) + O(1)(\varepsilon^{1/2+\delta}\varphi_x + d)^2]\cdot\varphi_x\mathrm{d}x\\
&= \varepsilon\int_{\mathbb{R}}b(u_a)|\varphi_x|^2\mathrm{d}x + \varepsilon^{1/2-\delta}\int_{\mathbb{R}}b(u_a)d\cdot\varphi_x\mathrm{d}x + \varepsilon^{1/2-\delta}\int_{\mathbb{R}}O(1)(\varepsilon^{1/2+\delta}\varphi_x + d)^2\cdot\varphi_x\mathrm{d}x\\
&= \varepsilon\int_{\mathbb{R}}b(u_a)|\varphi_x|^2\mathrm{d}x + A_1 + A_2.
\end{aligned}$$

It follows from the uniform parabolic condition that there exists $C_0 > 0$ such that

$$C_0\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x \le \varepsilon\int_{\mathbb{R}}b(u_a)|\varphi_x|^2\mathrm{d}x.$$

Based on the previous estimation for $\|d\|^2_{L^2(\mathbb{R})}$, we have

$$\begin{aligned}
|A_1| &= |\varepsilon^{1/2-\delta}\int_{\mathbb{R}}b(u_a)d\cdot\varphi_x\mathrm{d}x| \le C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}|d\cdot\varphi_x|\mathrm{d}x\\
&\le C\varepsilon^{-2\delta}\int_{\mathbb{R}}|d|^2\mathrm{d}x + \beta\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x \le C\varepsilon^{7\gamma-2\delta-1} + \beta\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x.
\end{aligned}$$

In view of estimate (2.51), there exist positive constants β_1 and β_2 such that

$$\begin{aligned}
|A_2| &= |\varepsilon^{1/2-\delta}\int_{\mathbb{R}}O(1)(\varepsilon^{1/2+\delta}\varphi_x + d)^2\cdot\varphi_x\mathrm{d}x| \le C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}|d^2\cdot\varphi_x|\mathrm{d}x + C\varepsilon^{3/2+\delta}\int_{\mathbb{R}}|(\varphi_x)^2\cdot\varphi_x|\mathrm{d}x\\
&\le C\varepsilon^{7\gamma/2-\delta}\int_{\mathbb{R}}|d|^2\mathrm{d}x + \beta_1\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x + \beta_2\,\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x \le C\varepsilon^{21\gamma/2-\delta} + (\beta_1 + \beta_2)\,\varepsilon\int_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x,
\end{aligned}$$

where we have used assumption (3.5). Next, we have

$$\begin{aligned}
I &= \varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(\partial_u f(u_a)(\varepsilon^{1/2+\delta}\varphi_x + d) + O(1)(\varepsilon^{1/2+\delta}\varphi_x + d)^2)\cdot\varphi\mathrm{d}x\\
&= \varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}\partial_u f(u_a)d\cdot\varphi\mathrm{d}x + \int_{\mathbb{R}}\partial_u f(u_a)\varphi_x\cdot\varphi\mathrm{d}x + O(1)\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(\varepsilon^{1/2+\delta}\varphi_x + d)^2\cdot\varphi\mathrm{d}x\\
&= \sum_{i=1}^{3}B_i,
\end{aligned}$$

where

$$B_1 = \varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}\partial_u f(u_a)d\cdot\varphi\mathrm{d}x \le C\varepsilon^{-1-2\delta}\int_{\mathbb{R}}|d|^2\mathrm{d}x + C\int_{\mathbb{R}}|\varphi|^2\mathrm{d}x \le C\int_{\mathbb{R}}|\varphi|^2\mathrm{d}x + C\varepsilon^{7\gamma-2\delta-1}.$$

In view of the structure of the approximate solutions, we obtain

$$
\begin{aligned}
B_2 &= \int_{\mathbb{R}} \partial_u f(u_a)\varphi \cdot \partial_x \varphi \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \partial_u f(u_a)\partial_x \varphi^2 \mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x + \frac{1}{2}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x + \frac{1}{2}\int_{s_1(t)+\varepsilon^\gamma}^{s_1(t)+2\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{s_1(t)+2\varepsilon^\gamma}^{s_2(t)-2\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x + \frac{1}{2}\int_{s_2(t)-2\varepsilon^\gamma}^{s_2(t)-\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x + \frac{1}{2}\int_{s_2(t)+\varepsilon^\gamma}^{s_2(t)+2\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{s_2(t)+2\varepsilon^\gamma}^{+\infty} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x = \sum_{i=1}^{9} K_i,
\end{aligned}
$$

with

$$
K_1 = \frac{1}{2}\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x = -\frac{1}{2}\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma} \partial_x \partial_u f(u_a)\cdot\ \varphi^2 \mathrm{d}x = O(1)\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma} \varphi^2 \mathrm{d}x,
$$

and

$$
\begin{aligned}
K_2 &= \frac{1}{2}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x \\
&= -\frac{1}{2}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma} \partial_u^2 f(u_a)(\partial_x m_1(I_1 - O) + m_1 \partial_x (I_1 - O) + \partial_x O)\cdot \varphi^2 \mathrm{d}x \\
&= O(1)\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma} \varphi^2 \mathrm{d}x,
\end{aligned}
$$

where we have used

$$
\partial_x^l (I_1 - O) = O(1)\varepsilon^{(4-l)\gamma}.
$$

It follows from (2.35) that

$$
\begin{aligned}
K_3 &= \frac{1}{2}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma} \partial_u f(u_a)\cdot \partial_x \varphi^2 \mathrm{d}x = -\frac{1}{2}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma} \frac{1}{\varepsilon}\partial_\xi \partial_u f(I_1)\cdot \varphi^2 \mathrm{d}x \\
&= \frac{1}{2\varepsilon}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma} |\partial_\xi \partial_u f(u_s^0)\cdot \varphi^2| \mathrm{d}x + O(1)\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma} \varphi^2 \mathrm{d}x,
\end{aligned}
$$

where we have used (2.35). Similarly, we have

$$
K_4 + K_5 + K_6 = O(1)\left(\int_{s_1(t)+\varepsilon^\gamma}^{s_1(t)+2\varepsilon^\gamma} \varphi^2 \mathrm{d}x + \int_{s_1(t)+2\varepsilon^\gamma}^{s_2(t)-2\varepsilon^\gamma} \varphi^2 \mathrm{d}x + \int_{s_2(t)-2\varepsilon^\gamma}^{s_2(t)-\varepsilon^\gamma} \varphi^2 \mathrm{d}x\right),
$$

$$K_7 = \frac{1}{2\varepsilon}\int\limits_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma} |\partial_\eta\partial_u f(\bar{u}_s^0)\varphi^2|\mathrm{d}x + O(1)\int\limits_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}\varphi^2\mathrm{d}x,$$

$$K_8 + K_9 = O(1)\left(\int\limits_{s_2(t)+\varepsilon^\gamma}^{s_2(t)+2\varepsilon^\gamma}\varphi^2\mathrm{d}x + \int\limits_{s_2(t)+2\varepsilon^\gamma}^{+\infty}\varphi^2\mathrm{d}x\right).$$

Then, in view of (2.51) and (3.3), the following estimates are obtained:

$$\begin{aligned} B_3 &\le C\varepsilon^{-(1/2+\delta)}\int\limits_{\mathbb{R}}|d^2\cdot\varphi|\mathrm{d}x + \varepsilon^{1+2\delta}|(\varphi_x)^2\cdot\varphi|\mathrm{d}x \\ &\le C\varepsilon^{7\gamma/2-1-\delta}\int\limits_{\mathbb{R}}|d|^2\mathrm{d}x + C\varepsilon^{2\delta}\cdot\varepsilon\int\limits_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x + C\int\limits_{\mathbb{R}}|\varphi|^2\mathrm{d}x \\ &\le C\varepsilon^{21\gamma/2-\delta-1} + \beta_3\varepsilon\int\limits_{\mathbb{R}}|\varphi_x|^2\mathrm{d}x + C\int\limits_{\mathbb{R}}|\varphi|^2\mathrm{d}x, \end{aligned}$$

for some $\beta_3 > 0$. Then, combining (2.49), we have

$$|J| \le \varepsilon^{1/2-\delta}\int\limits_{\mathbb{R}}|b(u_a)\partial_x d\cdot\varphi|\mathrm{d}x \le C\varepsilon^{1-2\delta}\int\limits_{\mathbb{R}}|d_x|^2\mathrm{d}x + C\int\limits_{\mathbb{R}}|\varphi|^2\mathrm{d}x.$$

Then, collecting all the estimates to obtain after choosing β_i, $i = 1, 2, 3$ small enough that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}}\varphi^2\mathrm{d}x + \varepsilon\int\limits_{\mathbb{R}}|\partial_x\varphi|\mathrm{d}x + \frac{1}{2\varepsilon}\int\limits_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}|\partial_\xi\partial_u f(u_s^0(x,t))\varphi^2|\mathrm{d}x \\ &+ \frac{1}{2\varepsilon}\int\limits_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}|\partial_\eta\partial_u f(\bar{u}_s^0)\varphi^2|\mathrm{d}x \le C\int\limits_{\mathbb{R}}\varphi^2\mathrm{d}x + C\varepsilon^{7\gamma-2\delta-1}. \end{aligned}$$

Then, Gronwall's inequality implies (3.7). Thus, the proof of Lemma 3.1 is complete. □

3.3 Estimate of the first-order derivative

Lemma 3.2. *Under the same assumptions as in Proposition* 3.1, *there is a positive constant C such that*

$$\sup_{0\le t\le T}\|\partial_x\varphi\|^2_{L^2(\mathbb{R})} + \varepsilon\int\limits_0^T\|\partial_x^2\varphi\|^2_{L^2(\mathbb{R})}\mathrm{d}t \le C\varepsilon^{7\gamma-2\delta-3}. \tag{3.8}$$

Proof. Multiplying (3.1) by v and integrating over $\mathbb{R}$ yield after integration by parts that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}}|v|^2\mathrm{d}x - \varepsilon^{1/2-\delta}\int\limits_{R}\partial_x^2[g(\bar{u}_a + \varepsilon^{1/2+\delta}v) - g(\bar{u}_a - d)]v\mathrm{d}x \\ &+ \varepsilon^{-(1/2+\delta)}\int\limits_{\mathbb{R}}\partial_x[f(\bar{u}_a + \varepsilon^{1/2+\delta}v) - f(\bar{u}_a - d)]v\mathrm{d}x + \varepsilon^{1/2-\delta}\int\limits_{\mathbb{R}}\partial_x[b(\bar{u}_a - d)\partial_x d]v\mathrm{d}x = 0. \end{aligned} \tag{3.9}$$

The same estimate as in Lemma 3.1, the second term can be written as

$$\begin{aligned} &-\varepsilon^{1/2-\delta}\int\limits_{\mathbb{R}}\partial_x^2[g(\bar{u}_a + \varepsilon^{1/2+\delta}v) - g(\bar{u}_a - d)]\cdot v\mathrm{d}x \\ &= \varepsilon^{1/2-\delta}\int\limits_{\mathbb{R}}g''(u_\xi)\partial_x u_\xi(\varepsilon^{1/2+\delta}v + d)\cdot\partial_x v\mathrm{d}x + \varepsilon\int\limits_{\mathbb{R}}g'(u_\xi)|\partial_x v|^2\mathrm{d}x + \varepsilon^{1/2-\delta}\int\limits_{\mathbb{R}}g'(u_\xi)\partial_x d\cdot\partial_x v\mathrm{d}x \\ &= D_1 + \varepsilon\int\limits_{\mathbb{R}}g'(u_\xi)|\partial_x v|^2\mathrm{d}x + D_2, \end{aligned}$$

where $u_\xi \in (\bar{u}_a - d, \bar{u}_a + \varepsilon^{5/8}v)$, and

$$\begin{aligned}
D_1 &= \varepsilon^{1/2-\delta}\int_{\mathbb{R}}[g''(u_\xi)\partial_x u_\xi(\varepsilon^{1/2+\delta}v + d)]\cdot v_x\mathrm{d}x \\
&\le O(1)\varepsilon\int_{\mathbb{R}}\frac{1}{\varepsilon}\cdot v\cdot v_x\mathrm{d}x + O(1)\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\frac{1}{\varepsilon}\cdot d\cdot v_x\mathrm{d}x \\
&\le C\varepsilon^{-1}\int_{\mathbb{R}}|v|^2\mathrm{d}x + \beta\varepsilon\int_{\mathbb{R}}|v_x|^2\mathrm{d}x + C\varepsilon^{-2\delta}\int_{\mathbb{R}}|d|^2\mathrm{d}x
\end{aligned}$$

and

$$D_2 \le C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}d_x\cdot v_x\mathrm{d}x \le C\varepsilon^{-2\delta}\int_{\mathbb{R}}|d_x|^2\mathrm{d}x + \beta\varepsilon\int_{\mathbb{R}}|v_x|^2\mathrm{d}x.$$

By (2.37), we obtain

$$\begin{aligned}
&\varepsilon^{-(1/2+\delta)}\Big|\int_{\mathbb{R}}(f(\bar{u}_a + \varepsilon^{1/2+\delta}v) - f(\bar{u}_a - d))\cdot\partial_x v\mathrm{d}x\Big| \\
\le\ & O(1)\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|\partial_u f(u_\xi)d\ \cdot\partial_x v|\mathrm{d}x + \int_{\mathbb{R}}|\partial_u f(u_\xi)v\cdot\partial_x v|\mathrm{d}x \\
\le\ & C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|d|^2\mathrm{d}x + 2\beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x + C\varepsilon^{-1}\int_{\mathbb{R}}|v|^2\mathrm{d}x \\
\le\ & C\varepsilon^{7\gamma-2\delta-2} + 2\beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x + C\varepsilon^{-1}\int_{\mathbb{R}}|v|^2\mathrm{d}x,
\end{aligned}$$

where $u_\xi \in (\bar{u}_a - d, \bar{u}_a + \varepsilon^{5/8}v)$, which is probably different from that mentioned earlier, but for simplicity of notations, we still denote it by u_ξ, and we will not mention it later. Next, we turn to estimate the fourth term of (3.9), and we have

$$\begin{aligned}
\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x[b(\bar{u}_a - d)\partial_x d]\cdot v\mathrm{d}x &= -\varepsilon^{1/2-\delta}\int_{\mathbb{R}}b(u_a)\partial_x d\cdot\partial_x v\mathrm{d}x \\
&\le C\varepsilon^{-2\delta}\int_{\mathbb{R}}|\partial_x d|^2\mathrm{d}x + \beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x.
\end{aligned}$$

Combining all the aforementioned estimates, and choosing β such that $C_0 - 4\beta > 0$ yield

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|v|^2\mathrm{d}x + (C_0 - 4\beta)\varepsilon\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x &\le C\varepsilon^{7\gamma-2\delta-2} + C\varepsilon^{-2\delta}\int_{\mathbb{R}}|d_x|^2\mathrm{d}x + C\varepsilon^{-1}\int_{\mathbb{R}}|v|^2\mathrm{d}x \\
&\le C\varepsilon^{7\gamma-2\delta-2} + C\varepsilon^{7\gamma-2\delta-2} + C\varepsilon^{-1}\int_{\mathbb{R}}|v|^2\mathrm{d}x.
\end{aligned}$$

Gronwall's inequality implies that

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|v|^2\mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x \le C\varepsilon^{7\gamma-2\delta-3}, \tag{3.10}$$

which is exactly (3.8). □

3.4 Estimate of the second derivative

To give the L^∞ bound of the error term $\partial_x\varphi$, we still need to estimate the second derivative $\partial_x^2\varphi$.

Lemma 3.3. *Under the same assumptions as in Proposition* 3.1, *there is a positive constant* C *such that*

$$\sup_{0\le t\le T}\|\partial_x^2\varphi\|^2_{L^2(\mathbb{R})}+\varepsilon\int_0^T\|\partial_x^3\varphi\|^2_{L^2(\mathbb{R})}\mathrm{d}t\le C\varepsilon^{7\gamma-2\delta-5}. \tag{3.11}$$

Proof. Let $\theta(x,t)=\partial_x v(x,t)=\partial_x^2\varphi(x,t)$, then θ satisfies

$$\begin{aligned}&\partial_t\theta-\varepsilon^{1/2-\delta}\partial_x^3[g(\bar u_a+\varepsilon^{1/2+\delta}v)-g(\bar u_a-d)]+\varepsilon^{-(1/2+\delta)}\partial_x^2[f(\bar u_a+\varepsilon^{1/2+\delta}v)-f(\bar u_a-d)]\\&\quad+\varepsilon^{1/2-\delta}\partial_x^2[b(\bar u_a-d)\partial_x d]=0.\end{aligned} \tag{3.12}$$

Multiplying (3.12) by θ and integrating over $\mathbb{R}$ yield after integration by parts that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\theta|^2\mathrm{d}x-\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x^3[g(\bar u_a+\varepsilon^{1/2+\delta}v)-g(\bar u_a-d)]\theta\mathrm{d}x\\&\quad+\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}\partial_x^2[f(\bar u_a+\varepsilon^{1/2+\delta}v)-f(\bar u_a-d)]\theta\mathrm{d}x+\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x^2[b(\bar u_a-d)\partial_x d]\theta\mathrm{d}x=0,\end{aligned} \tag{3.13}$$

with

$$\theta(x,0)=O(\varepsilon).$$

First, we estimate the second term of (3.13), which is written as

$$\begin{aligned}&-\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x^3[g(\bar u_a+\varepsilon^{1/2+\delta}v)-g(\bar u_a-d)]\cdot\theta\mathrm{d}x\\&=\ \varepsilon^{1/2-\delta}\int_{\mathbb{R}}[\partial_x^2g'(u_\xi)(\varepsilon^{1/2+\delta}v+d)+2\partial_xg'(u_\xi)(\varepsilon^{1/2+\delta}\partial_xv+\partial_xd)+g'(u_\xi)(\varepsilon^{1/2+\delta}\partial_x^2v+\partial_x^2d)]\cdot\partial_x\theta\mathrm{d}x\\&=\ F_1+F_2+F_3,\end{aligned}$$

where $u_\xi\in(\bar u_a-d,\bar u_a+\varepsilon^{5/8}v)$, and

$$\begin{aligned}F_1&=\varepsilon^{1/2-\delta}\int_{\mathbb{R}}(g'''(u_\xi)(\partial_xu_\xi)^2+g''(u_\xi)\partial_x^2u_\xi)(\varepsilon^{1/2+\delta}v+d)\cdot\partial_x\theta\mathrm{d}x\\&\le O(1)\varepsilon^{1/2-\delta}\int_{\mathbb{R}}(O(1)+\frac{O(1)}{\varepsilon^2})(\varepsilon^{1/2+\delta}v+d)\cdot\partial_x\theta\mathrm{d}x\\&\le C\varepsilon^{-3}\int_{\mathbb{R}}|v|^2\mathrm{d}x+2\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x+C\varepsilon^{7\gamma-2\delta-4},\end{aligned} \tag{3.14}$$

$$\begin{aligned}F_2&\le C\varepsilon\int_{\mathbb{R}}\partial_xu_\xi\partial_xv\cdot\partial_x\theta\mathrm{d}x+C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_xu_\xi\partial_xd\cdot\partial_x\theta\mathrm{d}x\\&\le C\varepsilon^{-1}\int_{\mathbb{R}}|\partial_xv|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x+C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_xd|^2\mathrm{d}x,\end{aligned}$$

and

$$F_3=\varepsilon\int_{\mathbb{R}}g'(u_\xi)|\partial_x\theta|^2\mathrm{d}x+\varepsilon^{1/2-\delta}\int_{\mathbb{R}}g'(u_\xi)\partial_x^2d\cdot\partial_x\theta\mathrm{d}x,$$

where

$$\varepsilon^{1/2-\delta}\int_{\mathbb{R}}g'(u_\xi)\partial_x^2d\cdot\partial_x\theta\mathrm{d}x\le C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x^2d\cdot\partial_x\theta\mathrm{d}x\le C\varepsilon^{-2\delta}\int_{\mathbb{R}}|\partial_x^2d|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x.$$

We estimate the third term of (3.13) and the detailed calculation process is as follows:

$$
\begin{aligned}
&\varepsilon^{-(1/2+\delta)}\left|\int_{\mathbb{R}}\partial_x(f(\bar{u}_a+\varepsilon^{1/2+\delta}v)-f(\bar{u}_a-d))\cdot\partial_x\theta\mathrm{d}x\right|\\
&\le \varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|\partial_u^2 f(u_\xi)\partial_x(u_\xi)(\varepsilon^{1/2+\delta}v+d)\cdot\partial_x\theta|\mathrm{d}x+\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|\partial_u f(u_\xi)(\varepsilon^{1/2+\delta}\partial_x v+\partial_x d)\cdot\partial_x\theta|\mathrm{d}x\\
&\le C\varepsilon^{-(3/2+\delta)}\int_{\mathbb{R}}|(\varepsilon^{1/2+\delta}v+d)\cdot\partial_x\theta|\mathrm{d}x+C\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|(\varepsilon^{1/2+\delta}\partial_x v+\partial_x d)\cdot\partial_x\theta|\mathrm{d}x\\
&\le C\varepsilon^{-1}\int_{\mathbb{R}}|v\cdot\partial_x\theta|\mathrm{d}x+C\varepsilon^{-(3/2+\delta)}\int_{\mathbb{R}}|d\cdot\partial_x\theta|\mathrm{d}x+C\int_{\mathbb{R}}|\partial_x v\cdot\partial_x\theta|\mathrm{d}x+C\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|\partial_x d\cdot\partial_x\theta|\mathrm{d}x\\
&\le C\varepsilon^{-3}\int_{\mathbb{R}}|v|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x+O(1)\varepsilon^{-4-2\delta}\int_{\mathbb{R}}d^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x\\
&\quad+C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_x d|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x+C\varepsilon^{-1}\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x\\
&\le C\varepsilon^{-3}\int_{\mathbb{R}}|v|^2\mathrm{d}x+C\varepsilon^{-1}\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x+4\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x+C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_x d|^2\mathrm{d}x+C\varepsilon^{7\gamma-2\delta-4},
\end{aligned}
$$

where $u_\xi\in(\bar{u}_a-d,\bar{u}_a+\varepsilon^{5/8}v)$. For the fourth term of (3.13), we have

$$
\begin{aligned}
&\varepsilon^{1/2-\delta}\int_{\mathbb{R}}\partial_x^2(b(\bar{u}_a-d)\partial_x d)\cdot\theta\mathrm{d}x\\
&\le C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}|\partial_x b(u_a)\partial_x d\cdot\partial_x\theta|\mathrm{d}x+C\varepsilon^{1/2-\delta}\int_{\mathbb{R}}|b(u_a)\partial_x^2 d\cdot\partial_x\theta|\mathrm{d}x\\
&\le C\varepsilon^{-1/2-\delta}\int_{\mathbb{R}}|\partial_x d\cdot\theta_x|\mathrm{d}x+C\varepsilon^{-2\delta}\int_{\mathbb{R}}|\partial_x^2 d|^2\mathrm{d}x+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x\\
&\le C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_x d|^2\mathrm{d}x+2\beta\varepsilon\int_{\mathbb{R}}|\theta_x|^2\mathrm{d}x+C\varepsilon^{-2\delta}\int_{\mathbb{R}}|\partial_x^2 d|^2\mathrm{d}x.
\end{aligned}
$$

Collecting all of the aforementioned estimates, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\theta|^2\mathrm{d}x+(C_0-10\beta)\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2\mathrm{d}x\\
&\le C\varepsilon^{7\gamma-2\delta-4}+C\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_x d|^2\mathrm{d}x+C\varepsilon^{-2\delta}\int_{\mathbb{R}}|\partial_x^2 d|^2\mathrm{d}x+C\varepsilon^{-3}\int_{\mathbb{R}}|v|^2\mathrm{d}x+C\varepsilon^{-1}\int_{\mathbb{R}}|\partial_x v|^2\mathrm{d}x\\
&\le C\varepsilon^{7\gamma-2\delta-4}+C\varepsilon^{-1}\int_{\mathbb{R}}|\theta|^2\mathrm{d}x+C\varepsilon^{-3}\int_{\mathbb{R}}|\partial_x\varphi|^2\mathrm{d}x,
\end{aligned}
$$

where choosing appropriate β such that $(C_0-10\beta)>0$ and using Gronwall's inequality to obtain (3.11). □

Now, we are in the position to verify that hypothesis (3.5) is correct. It follows from (3.8) and (3.11) that

$$
\sup_{0\le t\le T}\|\partial_x\varphi(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\sqrt{2}\sup_{0\le t\le T}\|\partial_x\varphi(\cdot,t)\|_{L^2(\mathbb{R})}^{1/2}\cdot\sup_{0\le t\le T}\|\partial_x^2\varphi(\cdot,t)\|_{L^2(\mathbb{R})}^{1/2}\le C\varepsilon^{7\gamma/2-\delta-2}.
$$

Thus, we obtain a unique solution $u^\varepsilon\in C^1([0,T];H^2(\mathbb{R}))$ and corresponding estimate of the solution for the Cauchy problems (3.2)–(3.3), which yield

$$
\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-\bar{u}_a(\cdot,t)\|_{L^\infty(\mathbb{R})}=\sup_{0\le t\le T}\|\varepsilon^{1/2+\delta}\partial_x\varphi(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C\varepsilon^{7\gamma/2-3/2},\tag{3.15}
$$

which complete the proof of Proposition 3.1.

3.5 Proof of Theorem 2.1

Combining the Proposition 3.1 and all of the aforementioned estimates, we finally give the proof of Theorem 1.1. It follows from (3.8) that

$$\sup_{0\le t\le T} \|u^\varepsilon(\cdot,t) - \bar{u}_a(\cdot,t)\|_{L^2(\mathbb{R})} = \sup_{0\le t\le T} \|\varepsilon^{1/2+\delta}\partial_x\varphi(\cdot,t)\|_{L^2(\mathbb{R})} \le C\varepsilon^{7\gamma/2-1}.$$

Based on Lemma 2.5 and (3.15), we obtain

$$\sup_{\substack{0\le t\le T\\ |x-s_i(t)|\ge\varepsilon^\gamma}} |u^\varepsilon(\cdot,t) - u^0(\cdot,t)| \le \sup_{0\le t\le T} |u^\varepsilon(\cdot,t) - \bar{u}_a(\cdot,t)| + \sup_{\substack{0\le t\le T\\ |x-s_i(t)|\ge\varepsilon^\gamma}} |\bar{u}_a(\cdot,t) - u^0(\cdot,t)| \le C\varepsilon,$$

where $|x - s_i(t)| \ge \varepsilon^\gamma$, $i = 1, 2$, $\frac{6}{7} < \gamma < 1$. Thus, we have completed the proof of Theorem 1.1.

4 Conclusion

To understand the asymptotic relationship between viscous systems and their corresponding inviscid equations is crucial in physics and mathematical analysis, especially in fluid mechanics and numerical computations. This article addresses the inviscid limit problem for the solution of a 1D viscous conservation law. Assuming that the inviscid equation has a piecewise smooth solution with two non-interacting entropy shocks, it is proven that the solutions of the viscous equation uniformly converge to the piecewise smooth inviscid solution away from the shocks, even if the shocks are strong. The mathematical tools in this article such as the multiple-scale asymptotic expansions and energy estimates can also be applied to nonlinear systems to explore the asymptotic behavior of these equations under different conditions.

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